Journal of Computer Science & Computational Mathematics, Volume 7, Issue 1, March 2017
DOI: 10.20967/jcscm.2017.01.003

Ulam-Hyers Stability of Euler-Lagrange-Jensen-(a,b)-Sextic Functional Equations in Quasi-β-Normed Spaces

John Michael Rassias[1*], K. Ravi[2], B.V. Senthil Kumar[3]

[1]Pedagogical Department E.E., Section of Mathematics and Informatics,
National and Capodistrian University of Athens,4, Agamemnonos Str., Aghia Paraskevi, Athens, Attikis 15342, Greece
[2]Department of Mathematics, Sacred Heart College, Tirupattur-635 601, Tamil Nadu, India
[3]Department of Mathematics, C. Abdul Hakeem College of Engineering & Technology, Melvisharam-632 509, Tamil Nadu, India
[*]Corresponding author email: jrassias@primedu.uoa.gr

Abstract: The purpose of this paper is to prove various stabilities of the following Euler-Lagrange-Jensen-(a,b)-sextic functional equation

$$f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right]=64(ab)^2(a^2+b^2)\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)\right]+2(a^2-b^2)(a^4-b^4)[f(x)+f(y)]$$

where $a\neq b$, such that $\mu\in\mathbb{R}$; $\mu=a+b\neq 0,\pm 1$ and $\lambda=1+(a-b)^6-2(a^6+b^6)-62(ab)^2(a^2+b^2)\neq 0$, in quasi-$\beta$-normed spaces by considering 'control function $\phi(x,y)$', a constant 'θ', 'sum of powers of norms', 'product of powers of norms' and 'mixed product-sum of different powers of norms' as upper bounds using direct method.

Keywords: Quasi-β-normed spaces, Sextic mapping, (β,p)-Banach spaces, Generalized Ulam-Hyers stabilities.

1. Introduction

A fascinating and renowned talk delivered by Ulam [1] in 1940, enthused to study the investigation of stability of functional equations. The foremost answer to the question of Ulam was provided by Hyers [2]. Hyers' theorem was generalized by Aoki [3] in 1950 for additive mappings. In 1978, Th.M. Rassias [4] tried to weaken the stipulation for the Cauchy difference and thrived in proving what is now known to be the Hyers-Ulam-Rassias stability for the additive Cauchy equation. During 1982-1989, J.M. Rassias [5-7] provided a further generalization of the result of Hyers and established a theorem using weaker conditions controlled by a product of different powers of norms. This type of stability involving a product of powers of norms is recognized as Ulam-Gavruta-Rassias stability by Bouikhalene and Elquorachi [8], Nakmahachalasint [9, 10], Park and Najati [11], Pietrzyk [12] and Sibaha *et al.* [13].

In 1994, a further generalization of the Th.M. Rassias' theorem was obtained by Gavruta [14] who replaced the bounds $\varepsilon(\|x\|^p+\|y\|^p)$ and $\varepsilon\|x\|^p\|y\|^p$ by a general control function $\varphi(x,y)$. This type of stability is celebrated as generalized Hyers-Ulam stability.

In 2008, Ravi *et al.* [15] investigated the stability of a new quadratic functional equation

$$Q(lx+y)+Q(lx-y)=2Q(x+y)+2Q(x-y)+2(l^2-2)Q(x)-2Q(y)$$

for any arbitrary but fixed real constant l with $l\neq 0; l\neq \pm 1; l\neq \pm\sqrt{2}$ using mixed product-sum of powers. The above mentioned stability is acknowledged as J.M. Rassias stability involving mixed product-sum of powers of norms by Ravi *et al.* [16, 17].

The Hyers-Ulam-Rassias stability theory has lots of applications in various type of mathematical problems such as non-linear analysis, fixed point theory, and asymptotic derivative of some non-linear operators. Jung [18] proved the Hyers-Ulam stability for Jensen's equation on a restricted domain and his result is applied for studying an interesting property of additive mapping. Zhou [19] applied the stability result of the functional equation $g(x+y)+g(x-y)=2g(x)$ to show a conjecture of Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the related Bernstein polynomials. These stability results are applied in stochastic analysis [20], financial and actuarial mathematics, psychology and sociology.

Several mathematicians have remarkably investigated Hyers-Ulam stability of various functional equations in modern spaces like intuitionistic fuzzy normed spaces, random normed spaces, probabilistic normed spaces, non-Archimedean intuitionistic fuzzy normed spaces, non-Archimedean spaces, paranormed spaces, and random normed spaces, that can be referred to in a number of studies [21-31]. There are many monographs and textbooks available in the literature of some other studies [32-36].

In this paper, we prove various stabilities associated with Hyers, Th.M. Rassias, J.M. Rassias and Gavruta of the following Euler-Lagrange-Jensen-(a,b)-sextic functional equation

$$f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right]=64(ab)^2(a^2+b^2)\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)\right]+2(a^2-b^2)(a^4-b^4)[f(x)+f(y)] \quad (1)$$

where $a \neq b$, such that $\mu \in \mathbb{R}$; $\mu = a + b \neq 0, \pm 1$ and $\lambda = 1 + (a-b)^6 - 2(a^6 + b^6) - 62(ab)^2(a^2 + b^2) \neq 0$, in quasi-$\beta$-normed spaces using direct method. It is easy to verify that the function $f(x) = cx^6$ is a solution of the equation (1.1). Hence we say that it is a sextic functional equation.

2. Preliminaries

Here, we will present some basic facts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{X}$ be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on $\mathcal{X}$ satisfying the following:

Let $\mathcal{X}$ be a linear space. A quasi-norm $\|\cdot\|$ is real-valued function on $\mathcal{X}$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\mu x\| = |\mu|^\beta \|x\|$ for all $\mu \in \mathbb{K}$ and all $x \in \mathcal{X}$.

(iii) There is a constant $K \geq 1$ such that

$$\|x + y\| \leq K(\|x\| + \|y\|), \text{ for all } x, y \in \mathcal{X}.$$

The pair $(\mathcal{X}, \|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on $\mathcal{X}$. The smallest possible K is called the modulus concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1 \leq)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

Given a p-norm, the formula $d(x, y) = \|x + y\|^p$ gives us a translation invariant metric on $\mathcal{X}$. By the Aoki-Rolewicz theorem [37] (see also [38]), each quasi-norm is equivalent to some p-norm, since it is much easier to work with p-norms than quasi-norms. Henceforth we restrict our attention mainly to p-norms.

3. Various stabilities of equation (1)

Throughout this section, we assume that $\mathcal{A}$ is a linear space and $\mathcal{B}$ is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_{\mathcal{B}}$. Let K be the modulus of concavity of $\|\cdot\|_{\mathcal{B}}$. For notational convenience, we define the difference operator for a given mapping $f: \mathcal{A} \to \mathcal{B}$ as

$$\begin{aligned} D_s f(x,y) &= f(ax+by) + f(bx+ay) \\ &\quad +(a-b)^6\left[f\left(\tfrac{ax-by}{a-b}\right) + f\left(\tfrac{bx-ay}{b-a}\right)\right] \\ &\quad -64(ab)^2(a^2+b^2)\left[f\left(\tfrac{x+y}{2}\right) + f\left(\tfrac{x-y}{2}\right)\right] \\ &\quad -2(a^2-b^2)(a^4-b^4)[f(x)+f(y)] \end{aligned}$$

for all $x, y \in \mathcal{X}$.

In this section, we prove various stabilities connected with Hyers, Th.M. Rassias, J.M. Rassias and Gavruta of the sextic functional equation (1) in quasi-β-normed spaces using direct method.

Theorem 1. *Let $\phi: \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping satisfying*

$$\sum_{j=0}^{\infty} \left(\frac{K}{\mu^{6\beta}}\right)^j \phi(\mu^j x, \mu^j y) < \infty \qquad (2)$$

for all $x, y \in \mathcal{A}$. Let $f: \mathcal{A} \to \mathcal{B}$ be a mapping with the condition $f(0) = 0$ such that

$$\|D_s f(x,y)\|_{\mathcal{B}} \leq \phi(x,y) \qquad (3)$$

for all $x, y \in \mathcal{A}$. Then there exists a unique sextic mapping $S: \mathcal{A} \to \mathcal{B}$ satisfying (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{K}{2^\beta \mu^{6\beta}} \sum_{j=0}^{\infty} \left(\frac{K}{\mu^{6\beta}}\right)^j \phi(\mu^j x, \mu^j x) \qquad (4)$$

for all $x \in \mathcal{A}$. The mapping $S(x)$ is defined by

$$S(x) = \lim_{n\to\infty} \frac{1}{\mu^{6n}} f(\mu^n x) \qquad (5)$$

for all $x \in \mathcal{A}$.

Proof. Switching (x, y) to (x, x) in (3) and simplifying further, we obtain

$$\left\|\frac{1}{\mu^6} f(\mu x) - f(x)\right\|_{\mathcal{B}} \leq \frac{1}{2^\beta \mu^{6\beta}} \phi(x,x) \qquad (6)$$

for all $x \in \mathcal{A}$. Now, replacing x by μx, dividing by $\mu^{6\beta}$ in (6), we find

$$\left\|\frac{1}{\mu^{12}} f(\mu^2 x) - \frac{1}{\mu^6} f(\mu x)\right\|_{\mathcal{B}} \leq \frac{1}{2^\beta \mu^{12\beta}} \phi(\mu x, \mu x) \qquad (7)$$

for all $x \in \mathcal{A}$. Combining (3.5) and (3.6) and using triangle inequality and since $K \geq 1$,

$$\left\|\frac{1}{\mu^{12}} f(\mu^2 x) - f(x)\right\|_{\mathcal{B}} \leq \frac{K}{2^\beta \mu^{6\beta}} \sum_{j=0}^{1} \left(\frac{K}{\mu^{6\beta}}\right)^j \phi(\mu^j x, \mu^j x) \qquad (8)$$

for all $x \in \mathcal{A}$. Using induction arguments on a positive integer n, we arrive

$$\left\|\frac{1}{\mu^{6n}} f(\mu^n x) - f(x)\right\|_{\mathcal{B}} \leq \frac{K}{2^\beta \mu^{6\beta}} \sum_{j=0}^{n-1} \left(\frac{K}{\mu^{6\beta}}\right)^j \phi(\mu^j x, \mu^j x) \qquad (9)$$

for all $x \in \mathcal{A}$. From (6), we obtain

$$\left\|\frac{1}{\mu^{6(j+1)}} f(\mu^{j+1} x) - \frac{1}{\mu^{6j}} f(\mu^j x)\right\|_{\mathcal{B}} \leq \frac{1}{\mu^{6j\beta} 2^\beta \mu^{6\beta}} \phi(\mu^j x, \mu^j x) \qquad (10)$$

for all $x \in \mathcal{A}$. For $n > m$

$$\begin{aligned} \left\|\frac{1}{\mu^{6n}} f(\mu^n x) - \frac{1}{\mu^{6m}} f(\mu^m x)\right\|_{\mathcal{B}} &\leq \sum_{j=m}^{n-1} \left\|\frac{1}{\mu^{6(j+1)}} f(\mu^{j+1} x) - \frac{1}{\mu^{6j}} f(\mu^j x)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{2^\beta \mu^{6\beta}} \sum_{j=m}^{n-1} \frac{1}{\mu^{6j\beta}} \phi(\mu^j x, \mu^j x) \end{aligned} \qquad (11)$$

for all $x \in \mathcal{A}$. The right-hand side of the above inequality (2) tends to 0 as $n \to \infty$. Hence $\left\{\frac{1}{\mu^{6n}} f(\mu^n x)\right\}$ is a Cauchy sequence in $\mathcal{B}$. Hence, we may define

$$S(x) = \lim_{n\to\infty} \frac{1}{\mu^{6n}} f(\mu^n x)$$

for all $x \in \mathcal{A}$. Since $K \geq 1$, replacing (x, y) by $(\mu^n x, \mu^n y)$ and dividing by $\mu^{6n\beta}$ in (3), we have

$$\frac{1}{\mu^{6n\beta}} \|D_s f(\mu^n x, \mu^n y)\|_{\mathcal{B}} \leq \frac{1}{\mu^{6n\beta}} K^n \phi(\mu^n x, \mu^n y) \qquad (12)$$

for all $x, y \in \mathcal{A}$. By taking $n \to \infty$, the definition of S implies that S satisfies (1) for all $x, y \in \mathcal{A}$. Thus S is a sextic mapping. Also, the inequality (9) implies the inequality (4). Now, it remains to show the uniqueness of S. Assume that there exists $S': \mathcal{A} \to \mathcal{B}$ satisfying (1.1) and (3.3). It is easy to show that for all $x \in \mathcal{A}$, $S'(\mu^n x) = \mu^{6n} S'(x)$ and $S(\mu^n x) = \mu^{6n} S(x)$. Then

$$\begin{aligned} \|S'(x) - S(x)\|_{\mathcal{B}} &= \left\|\frac{1}{\mu^{6n}} S'(\mu^n x) - \frac{1}{\mu^{6n}} S(\mu^n x)\right\|_{\mathcal{B}} \\ &= \frac{1}{\mu^{6n\beta}} \|S'(\mu^n x) - S(\mu^n x)\|_{\mathcal{B}} \end{aligned}$$

$$\leq \frac{K}{\mu^{6n\beta}}(\|S'(\mu^n x) - f(\mu^n x)\|_{\mathcal{B}} + \|f(\mu^n x) - S(\mu^n x)\|_{\mathcal{B}})$$
$$\leq 2K\sum_{j=0}^{\infty}\left(\frac{K}{\mu^{6\beta}}\right)^{n+j}\phi(\mu^{n+j}x, \mu^{n+j}x)$$

for all $x \in \mathcal{A}$. By letting $n \to \infty$, we immediately have the uniqueness of S. □

Theorem 2. *Let $\phi: \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping satisfying*

$$\sum_{j=0}^{\infty}\left(K\mu^{6\beta}\right)^j \phi\left(\frac{x}{\mu^{j+1}}, \frac{y}{\mu^{j+1}}\right) < \infty \tag{13}$$

for all $x, y \in \mathcal{A}$. Let $f: \mathcal{A} \to \mathcal{B}$ be a mapping with the condition $f(0) = 0$ satisfying (3) *for all $x, y \in \mathcal{A}$. Then there exists a unique sextic mapping $S: \mathcal{A} \to \mathcal{B}$ satisfying* (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{K}{2^\beta}\sum_{j=0}^{\infty}\left(K\mu^{6\beta}\right)^j \phi\left(\frac{x}{\mu^{j+1}}, \frac{x}{\mu^{j+1}}\right) \tag{14}$$

for all $x \in \mathcal{A}$. The mapping $S(x)$ is defined by

$$S(x) = \lim_{n\to\infty} \mu^{6n} f\left(\frac{x}{\mu^n}\right) \tag{15}$$

for all $x \in \mathcal{A}$.

Proof. Plugging (x, y) into $\left(\frac{x}{\mu}, \frac{x}{\mu}\right)$ in (3), we obtain

$$\left\|f(x) - \mu^6 f\left(\frac{x}{\mu}\right)\right\|_{\mathcal{B}} \leq \frac{1}{2^\beta}\phi\left(\frac{x}{\mu}, \frac{x}{\mu}\right) \tag{16}$$

for all $x \in \mathcal{A}$. Now, substituting x as $\frac{x}{\mu}$, multiplying by $\mu^{6\beta}$ in (16) and summing the resulting inequality with (16), we have

$$\left\|f(x) - \mu^{12} f\left(\frac{x}{\mu^2}\right)\right\|_{\mathcal{B}} \leq \frac{K}{2^\beta}\sum_{j=0}^{1}\left(K\mu^{6\beta}\right)^j \phi\left(\frac{x}{\mu^{j+1}}, \frac{x}{\mu^{j+1}}\right)$$

for all $x \in \mathcal{A}$. Using induction arguments on a positive integer n, we conclude that

$$\left\|f(x) - \mu^{6n} f\left(\frac{x}{\mu^n}\right)\right\|_{\mathcal{B}} \leq \frac{K}{2^\beta}\sum_{j=0}^{n-1}\left(K\mu^{6\beta}\right)^j \phi\left(\frac{x}{\mu^{j+1}}, \frac{x}{\mu^{j+1}}\right)$$

for all $x \in \mathcal{A}$. The rest of the proof is obtained by similar arguments as in Theorem 1. □

Corollary 1. *Let $\theta \geq 0$ be fixed. If a mapping $f: \mathcal{A} \to \mathcal{B}$ satisfies the inequality*

$$\|D_s f(x, y)\|_{\mathcal{B}} \leq \theta$$

for all $x, y \in \mathcal{A}$, then there exists a unique sextic mapping $S: \mathcal{A} \to \mathcal{B}$ satisfying (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{K\theta}{2^\beta(\mu^{6\beta} - K)}$$

for all $x \in \mathcal{A}$.

Proof. Considering $\phi(x, y) = \theta$, for all $x, y \in \mathcal{A}$ in Theorem 1, we have

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{K}{2^\beta\mu^{6\beta}}\sum_{j=0}^{\infty}\left(\frac{K}{\mu^{6\beta}}\right)^j \theta$$
$$\leq \frac{K\theta}{2^\beta\mu^{6\beta}}\left(1 - \frac{K}{\mu^{6\beta}}\right)^{-1}$$
$$\leq \frac{K\theta}{2^\beta(\mu^{6\beta} - K)}$$

for all $x \in \mathcal{A}$. □

Corollary 2. *Let $\theta_1 \geq 0$ be fixed and $r \neq 6\beta$. If a mapping $f: \mathcal{A} \to \mathcal{B}$ satisfies the inequality*

$$\|D_s f(x, y)\|_{\mathcal{B}} \leq \theta_1(\|x\|_{\mathcal{A}}^r + \|y\|_{\mathcal{A}}^r)$$

for all $x, y \in \mathcal{A}$, then there exists a unique sextic mapping $S: \mathcal{A} \to \mathcal{B}$ satisfying (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{2\theta_1 K}{2^\beta(\mu^{6\beta} - K\mu^r)}\|x\|_{\mathcal{A}}^r, & for\ r < 6\beta, \\ \dfrac{2\theta_1 K}{2^\beta(\mu^r - K\mu^{6\beta})}\|x\|_{\mathcal{A}}^r, & for\ r > 6\beta \end{cases}$$

for all $x \in \mathcal{A}$.

Proof. By choosing $\phi(x, y) = \theta_1(\|x\|_{\mathcal{A}}^r + \|y\|_{\mathcal{A}}^r)$, for all $x, y \in \mathcal{A}$ and $r < 6\beta$ in Theorem 1, we obtain

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{K}{2^\beta\mu^{6\beta}}\sum_{j=0}^{\infty}\frac{2\theta_1 K^j}{\mu^{6j\beta}}\mu^{jr}\|x\|_{\mathcal{A}}^r$$
$$\leq \frac{2\theta_1 K}{2^\beta\mu^{6\beta}}\sum_{j=0}^{\infty}\left(K\mu^{r-6\beta}\right)^j\|x\|_{\mathcal{A}}^r$$
$$\leq \frac{2\theta_1 K}{2^\beta(\mu^{6\beta} - K\mu^r)}\|x\|_{\mathcal{A}}^r \tag{17}$$

for all $x \in \mathcal{A}$ and $r > 6\beta$ in Theorem 2, we have

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{2\theta_1 K}{2^\beta\mu^r}\sum_{j=0}^{\infty}\left(K\mu^{6\beta-r}\right)^j\|x\|_{\mathcal{A}}^r$$
$$\leq \frac{2\theta_1 K}{2^\beta\mu^r}\left(1 - K\mu^{6\beta-r}\right)^{-1}\|x\|_{\mathcal{A}}^r$$
$$\leq \frac{2\theta_1 K}{2^\beta(\mu^r - K\mu^{6\beta})}\|x\|_{\mathcal{A}}^r \tag{18}$$

for all $x \in \mathcal{A}$. Combining (17) and (18), we arrive at the required results. □

Corollary 3. *Let $\theta_2 \geq 0$ be fixed and r, s such that $\gamma = r + s \neq 6\beta$. If a mapping $f: \mathcal{A} \to \mathcal{B}$ satisfies the inequality*

$$\|D_s f(x, y)\|_{\mathcal{B}} \leq \theta_2\|x\|_{\mathcal{A}}^r\|y\|_{\mathcal{A}}^s$$

for all $x, y \in \mathcal{A}$, then there exists a unique sextic mapping $S: \mathcal{A} \to \mathcal{B}$ satisfying (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\theta_2 K}{2^\beta(\mu^{6\beta} - K\mu^r)}\|x\|_{\mathcal{A}}^\gamma, & for\ \gamma < 6\beta, \\ \dfrac{\theta_2 K}{2^\beta(\mu^r - K\mu^{6\beta})}\|x\|_{\mathcal{A}}^\gamma, & for\ \gamma > 6\beta \end{cases}$$

for all $x \in \mathcal{A}$.

Proof. By replacing $\phi(x, y) = \theta_2\|x\|_{\mathcal{A}}^r\|y\|_{\mathcal{A}}^s$, for all $x, y \in \mathcal{A}$ and considering $\gamma < 6\beta$ in Theorem 1, one can have

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{\theta_2 K}{2^\beta\mu^{6\beta}}\sum_{j=0}^{\infty}\left(K\mu^{\gamma-6\beta}\right)^j\|x\|_{\mathcal{A}}^\gamma$$
$$\leq \frac{\theta_2 K}{2^\beta\mu^{6\beta}}\left(1 - K\mu^{\gamma-6\beta}\right)^{-1}\|x\|_{\mathcal{A}}^\gamma$$
$$\leq \frac{2\theta_2 K}{2^\beta(\mu^{6\beta} - K\mu^r)}\|x\|_{\mathcal{A}}^\gamma \tag{19}$$

for all $x \in \mathcal{A}$ and assuming $\gamma > 6\beta$ in Theorem 2, we arrive

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \frac{\theta_2 K}{2^\beta}\sum_{j=0}^{\infty}\left(K\mu^{6\beta-\gamma}\right)^j\|x\|_{\mathcal{A}}^\gamma$$
$$\leq \frac{\theta_2 K}{2^\beta\mu^\gamma}\left(1 - K\mu^{6\beta-\gamma}\right)^{-1}\|x\|_{\mathcal{A}}^\gamma$$
$$\leq \frac{\theta_2 K}{2^\beta(\mu^r - K\mu^{6\beta})}\|x\|_{\mathcal{A}}^\gamma \tag{20}$$

for all $x \in \mathcal{A}$. From (19) and (20), we obtain the desired results. □

Corollary 4. *Let $\theta_3 \geq 0$ be fixed and $\gamma \neq 6\beta$. If a mapping $f: \mathcal{A} \to \mathcal{B}$ satisfies the inequality*

$$\|D_s f(x, y)\|_{\mathcal{B}} \leq \theta_3\left(\|x\|_{\mathcal{A}}^{\frac{\gamma}{2}}\|y\|_{\mathcal{A}}^{\frac{\gamma}{2}} + \left(\|x\|_{\mathcal{A}}^\gamma + \|x\|_{\mathcal{A}}^\gamma\right)\right)$$

for all $x, y \in \mathcal{A}$, *then there exists a unique sextic mapping* $S: \mathcal{A} \to \mathcal{B}$ *satisfying* (1) *and*

$$\|f(x) - S(x)\|_{\mathcal{B}} \leq \begin{cases} \frac{3\theta_3 K}{2^\beta(\mu^{6\beta} - K\mu^r)} \|x\|_{\mathcal{A}}^\gamma, & for\ \gamma < 6\beta, \\ \frac{3\theta_3 K}{2^\beta(\mu^r - K\mu^{6\beta})} \|x\|_{\mathcal{A}}^\gamma, & for\ \gamma > 6\beta \end{cases}$$

for all $x \in \mathcal{A}$.

Proof. By putting $\phi(x,y) = \theta_3\left(\|x\|_{\mathcal{A}}^{\frac{\gamma}{2}}\|y\|_{\mathcal{A}}^{\frac{\gamma}{2}} + \left(\|x\|_{\mathcal{A}}^\gamma + \|x\|_{\mathcal{A}}^\gamma\right)\right)$, for all $x, y \in \mathcal{A}$ and taking $\gamma < 6\beta$ in Theorem 1, we get

$$\begin{aligned} \|f(x) - S(x)\|_{\mathcal{B}} &\leq \frac{3\theta_3 K}{2^\beta \mu^{6\beta}} \sum_{j=0}^{\infty} \left(K\mu^{\gamma-6\beta}\right)^j \|x\|_{\mathcal{A}}^\gamma \\ &\leq \frac{3\theta_3 K}{2^\beta \mu^{6\beta}} \left(1 - K\mu^{\gamma-6\beta}\right)^{-1} \|x\|_{\mathcal{A}}^\gamma \\ &\leq \frac{3\theta_2 K}{2^\beta(\mu^{6\beta} - K\mu^r)} \|x\|_{\mathcal{A}}^\gamma \end{aligned} \tag{21}$$

for all $x \in \mathcal{A}$ and considering $\gamma > 6\beta$ in Theorem 2, we have

$$\begin{aligned} \|f(x) - S(x)\|_{\mathcal{B}} &\leq \frac{3\theta_3 K}{2^\beta} \sum_{j=0}^{\infty} \left(K\mu^{6\beta-\gamma}\right)^j \|x\|_{\mathcal{A}}^\gamma \\ &\leq \frac{3\theta_3 K}{2^\beta \mu^\gamma} \left(1 - K\mu^{6\beta-\gamma}\right)^{-1} \|x\|_{\mathcal{A}}^\gamma \\ &\leq \frac{3\theta_3 K}{2^\beta(\mu^r - K\mu^{6\beta})} \|x\|_{\mathcal{A}}^\gamma \end{aligned} \tag{22}$$

for all $x \in \mathcal{A}$. From (21) and (22), we arrive at the required results. □

References

[1] S.M. Ulam, “Problems in Modern Mathematics”, *Rend. Chap. VI*, Wiley, New York, 1960.

[2] D.H. Hyers, “On the stability of the linear functional Equation”, *Proc. Nat. Acad. Sci., U.S.A.*, vol. 27, pp. 222-224, 1941.

[3] T. Aoki, “On the stability of the linear transformation in Banach spaces”, *J. Math.Soc., Japan*, vol. 2, no. 1-2, pp. 64-66, 1950.

[4] Th.M. Rassias, “On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, vol. 72, pp. 297-300, 1978.

[5] J.M. Rassias, “On approximately of approximately linear mappings by linear mappings”, *J. Funct. Anal. USA*, vol. 46, pp. 126-130, 1982.

[6] J.M. Rassias, “On approximately of approximately linear mappings by linear mappings”, *Bull. Sci. Math.*, vol. 108, no. 4, pp. 445-446, 1984.

[7] J.M. Rassias, “Solution of problem of Ulam”, *J.Approx. Theory. USA*, vol. 57, no. 3, pp. 268-273, 1989.

[8] B. Bouikhalene and E. Elquorachi, “Ulam-Gavruta-Rassias stability of the Pexider functional equation”, *Int. J. Appl. Math. Stat.*, vol. 7, no. F07, pp. 7-39, 2007.

[9] P. Nakmahachalasint, “On the generalized Ulam-Gavruta-Rassias stability of a mixed type linear and Euler-Lagrange-Rassias functional equation”, *Int. J. Math. Math. Sci.*, vol. 2007, 2007.

[10] P. Nakmahachalasint, “Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities of an additive functional equation in several variables”, *Int. J. Math. Math.Sci.*, vol. 2007, 2007.

[11] C. Park and A. Najati, “Homomorphisms and Derivations in C*-Algebras”, *Abst. Appl. Anal.*, vol. 2007, Art. ID 80630, pp. 1-12, 2007.

[12] A. Pietrzyk, “Stability of the Euler-Lagrange-Rassias functional equation”, *Demonstratio Mathematica*, vol. 39, no. 3, pp. 523-530, 2006.

[13] A. Sibaha, B. Bouikhalene and E. Elquorachi, “Ulam-Gavruta-Rassias stability for a linear functional equation”, *Int. J. Appl. Math. Stat.*, vol. 7, pp. 157-168, 2007.

[14] P. Gavruta, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings”, *J. Math. Anal. Appl.*, vol. 184, pp. 431-436, 1994.

[15] K. Ravi, M. Arunkumar and J.M. Rassias, “Ulam stability for the orthogonally genera Euler-Lagrange type functional equation”, *Int. J. Math. Stat.*, vol. 3, pp. 36-46, 2008.

[16] K. Ravi, R. Kodandan and P. Narasimman, “Ulam stability of a quadratic functional equation”, *Int. J. Pure and Appl. Math.*, vol. 51, no. 1, pp. 87-101, 2009.

[17] K. Ravi, P. Narasimman and R. Kishore Kumar, “Generalized Hyers-Ulam- Rassias stability and J.M. Rassias stability of a quadratic functional equation”, *Int. J. Math.Sci. & Engg. Appl.*, vol. 3, no. 2, pp. 79-94, 2009.

[18] S.M. Jung, “Hyers-Ulam-Rassias stability of Jensen's equation and its application”, *Proc. Amer. Math. Soc.*, vol. 126, no. 11, pp. 3137-3143, 1998.

[19] D.X. Zhou, “On a conjecture of Z. Ditzian”, *J. Approx. Theory*, vol. 69, pp. 167-172, 1992.

[20] P. Malliavin, “Stochastic Analysis”, *Springer*, Berlin, 1997.

[21] A. Alotaibi, M. Mursaleen, H. Dutta and S.A. Mohiuddine, “On the Ulam stability of Cauchy functional equation in IFN-spaces”, *Appl. Math. Inf. Sci.*, vol. 8, no. 3, pp.1135- 1143, 2014.

[22] L. Cadariu and V. Radu, “Fixed points and stability for functional equations in Probabilistic metric and random normed spaces”, *Fixed Point Theory and Applications*, 2009, Art. ID 589143, vol. 2009, 2009.

[23] M.B. Ghaemi, Hamid Majani and Majid Eshaghi Gordji, “Approximately quintic and sextic mappings on the probabilistic normed spaces”, *Bull. Korean Math. Soc.*, vol. 49, no. 2, pp. 339-352, 2012.

[24] J.R. Lee, D.Y. Shin and Ch. Park, “Hyers-Ulam stability of functional equations in matrix normed spaces”, *J. Inequl. Appl.*, vol. 2013, 2013.

[25] S.A. Mohiuddine, “Stability of Jensen functional equation in intuitionistic fuzzy normed space”, *Chaos, Solitons & Fract.*, vol. 42, pp. 2989-2996, 2009.

[26] S.A. Mohiuddine, A. Alotaibi and M. Obaid, “Stability of various functional equations in non-Archimedean intuitionistic fuzzy normed spaces”, *Discrete Dyn. Nat. Soc.*, vol. 2012, Article ID 234727, 2012.

[27] M.S. Moslehian and Th.M. Rassias, “Stability of functional equations in non-Archimedean spaces”, *Appl. Anal.Discrete Math.*, vol. 1, pp. 325-334, 2007.

[28] C. Park and D.Y. Shin, “Functional equations in paranormed spaces”, *Adv.Differences Equations*, vol. 23, 2012.
[29] C. Park, J.R. Lee and D.Y. Shin, “Generalized Ulam-Hyers stability of random homomorphisms in random normed algebras associated with the Cauchy functional equation”, *Appl. Math. Lett.*, vol. 25, pp. 200-205, 2012.
[30] M.J. Rassias, “Generalized Hyers-Ulam product-sum stability of a Cauchy type additive functional equation”, *Euro. J. Pure Appl.Math.*, vol. 4, no. 1, pp. 50-58, 2011.
[31] K. Ravi, M. Arunkumar, J.M. Rassias, “On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation”, *Int.J. Math. Stat.*, vol. 7, pp. 143-156, 2007.
[32] S. Czerwik, “Functional Equations and Inequalities in Several Variables”, *World Scientific Publishing Company*, New Jersey, London, Singapore and Hong Kong, 2002.
[33] D.H. Hyers, G. Isac and Th.M. Rassias, “Stability of Functional Equations in Several Variables”, *Birkhauser*, Basel, 1998.
[34] S.M. Jung, “Hyers-Ulam-Rassias stability of Functional Equations in Mathematical Analysis”, *Hardonic press*, Palm Harbor, 2001.
[35] Pl. Kannappan, “Quadratic functional equation and inner product spaces”, *Results Math.*, vol. 27, no. 3-4, pp. 368-372, 1995.
[36] P.K. Sahoo and Pl. Kannappan, “Introduction to Functional Equations”, *CRC Press*, Taylor & Francis Group, 2011.
[37] S. Rolewicz, “Metric Linear Spaces”, *Reidel*, Dordrecht, 1984.
[38] A. Bodaghi, “Stability of a quartic functional equation”, *The Scientific World Journal*, Art. ID 752146, vol. 2014, 2014.