Ulam-Hyers Stability of Euler-Lagrange-Jensen-(a,b)-Sextic Functional Equations in Quasi-\(\beta\)-Normed Spaces

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1. Introduction

A fascinating and renowned talk delivered by Ulam [1] in 1940, enthused to study the investigation of stability of functional equations. The foremost answer to the question of Ulam was provided by Hyers [2]. Hyers’ theorem was generalized by Aoki [3] in 1950 for additive mappings. In 1978, Th.M. Rassias [4] tried to weaken the stipulation for the Cauchy difference and thrived in proving what is now known to be the Hyers-Ulam-Rassias stability for the additive Cauchy equation. During 1982-1989, J.M. Rassias [5-7] provided a further generalization of the result of Hyers and established a theorem using weaker conditions controlled by a product of different powers of norms. This type of stability involving a product of powers of norms is recognized as Ulam-Gavruta-Rassias stability by Bouikhalene and Elquorachi [8], Nakmahachalasint [9, 10], Park and Najati [11], Pietrzyk [12] and Sibaha et al. [13].

In 1994, a further generalization of the Th.M. Rassias’ theorem was obtained by Gavruta [14] who replaced the bounds \(\varepsilon(\|x\|^p+\|y\|^p)\) and \(\varepsilon\|x\|^p\|y\|^p\) by a general control function \(q(x, y)\). This type of stability is celebrated as generalized Hyers-Ulam-stability.

In 2008, Ravi et al. [15] investigated the stability of a new quadratic functional equation \(Q(lx + y) + Q(lx - y) = 2Q(x + y) + 2Q(x - y) + 2(l^2 - 2)Q(x - y) - 2Q(y)\) for any arbitrary but fixed real constant \(l\) with \(l \neq 0; l \neq \pm 1; l \neq \pm \sqrt{2}\) using mixed product-sum of powers. The above mentioned stability is acknowledged as J.M. Rassias stability involving mixed product-sum of powers of norms by Ravi et al. [16, 17].

The Hyers-Ulam-Rassias stability theory has lots of applications in various type of mathematical problems such as non-linear analysis, fixed point theory, and asymptotic derivative of some non-linear operators. Jung [18] proved the Hyers-Ulam stability for Jensen’s equation on a restricted domain and his result is applied for studying an interesting property of additive mapping. Zhou [19] applied the stability result of the functional equation \(g(x + y) + g(x - y) = 2g(x)\) to show a conjecture of Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the related Bernstein polynomials. These stability results are applied in stochastic analysis [20], financial and actuarial mathematics, psychology and sociology.

Several mathematicians have remarkably investigated Hyers-Ulam stability of various functional equations in modern spaces like intuitionistic fuzzy normed spaces, random normed spaces, probabilistic normed spaces, non-Archimedean intuitionistic fuzzy normed spaces, non-Archimedean spaces, paranormed spaces, and random normed spaces, that can be referred to in a number of studies [21-31]. There are many monographs and textbooks available in the literature of some other studies [32-36].

In this paper, we prove various stabilities associated with Hyers, Th.M. Rassias, J.M. Rassias and Gavruta of the following Euler-Lagrange-Jensen-(a,b)-sextic functional equation

\[
\begin{align*}
Q(ax + by) + Q(bx + ay) + (a - b)^6 \left[ f \left( \frac{ax - by}{a - b} \right) + f \left( \frac{bx - ay}{b - a} \right) \right] \\
= 64(ab)^2(a^2 + b^2) \left[ f \left( \frac{x+y}{2} \right) + f \left( \frac{x-y}{2} \right) \right] + 2(a^2 - b^2)(a^4 - b^4)[f(x) + f(y)]
\end{align*}
\]
where \( a \neq b \), such that \( \mu \in \mathbb{R} \); \( \mu = a + b \neq 0, \pm 1 \) and \( \lambda = 1 + (a - b)^6 - 2(a^2 + b^2) - 6(ab)^2(a^2 + b^2) \neq 0 \), in quasi-\( \beta \)-normed spaces using direct method. It is easy to verify that the function \( f(x) = cx^6 \) is a solution of the equation (1.1). Hence we say that it is a sextic functional equation.

2. Preliminaries

Here, we will present some basic facts concerning quasi-\( \beta \)-normed spaces and some preliminary results. We fix a real number \( \beta \) with \( 0 < \beta \leq 1 \) and let \( \mathbb{K} \) denote either \( \mathbb{R} \) or \( \mathbb{C} \).

Let \( X \) be a linear space over \( \mathbb{K} \). A quasi-\( \beta \)-norm \( \| \cdot \| \) is a real-valued function on \( X \) satisfying the following:

- (i) \( \| x \| \geq 0 \) for all \( x \in X \) and \( \| x \| = 0 \) if and only if \( x = 0 \).
- (ii) \( \| \mu x \| = |\mu| \| x \| \) for all \( \mu \in \mathbb{K} \) and all \( x \in X \).
- (iii) There is a constant \( K \geq 1 \) such that
  \[ \| x + y \| \leq K (\| x \| + \| y \|), \]
  for all \( x, y \in X \).

The pair \( (X, \| \cdot \|) \) is called a quasi-\( \beta \)-normed space if \( \| \cdot \| \) is a quasi-\( \beta \)-norm on \( X \). The smallest possible \( K \) is called the modulus concavity of \( \| \cdot \| \). A quasi-\( \beta \)-Banach space is a complete quasi-\( \beta \)-normed space.

A quasi-\( \beta \)-norm \( \| \cdot \| \) is called a \((\beta,p)\)-norm (0 < \( \beta \leq 1 \)) if

\[ \| x + y \| \leq \| x \|^p + \| y \|^p \]

for all \( x, y \in X \). In this case, a quasi-\( \beta \)-Banach space is called a \((\beta,p)\)-Banach space.

Given a \( p \)-norm, the formula \( d(x,y) = \| x + y \|^p \) gives us a translation invariant metric on \( X \). By the Aoki-Rolewicz theorem [37] (see also [38]), each quasi-norm is equivalent to some \( p \)-norm, since it is much easier to work with \( p \)-norms than quasi-norms. Henceforth we restrict our attention mainly to \( p \)-norms.

3. Various stabilities of equation (1)

Throughout this section, we assume that \( \mathcal{A} \) is a linear space and \( \mathcal{B} \) is a \((\beta,p)\)-Banach space with \((\beta,p)\)-norm \( \| \cdot \|_B \). Let \( K \) be the modulus of concavity of \( \| \cdot \|_B \). For notational convenience, we define the difference operator for a given mapping \( f : \mathcal{A} \to \mathcal{B} \) as

\[ D_n f(x,y) = f(ax + by) + f(bx + ay) - f(ax - by) - f(bx - ay) \]

for all \( x, y \in \mathcal{X} \).

In this section, we prove various stabilities connected with Ulam-Hyers-Gavruta-Rassias stability of \((a,b)\)-sextic functional equations (1) in quasi-\( \beta \)-normed spaces using direct method.

**Theorem 1.** Let \( \Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty) \) be a mapping satisfying

\[ \sum_{j=0}^{\infty} \left( \frac{K}{\mu_j^{\beta p}} \right)^j \Phi(\mu^j x, \mu^j y) < \infty \]

for all \( x, y \in \mathcal{A} \). Let \( f : \mathcal{A} \to \mathcal{B} \) be a mapping with the condition \( f(0) = 0 \) such that

\[ \| D_n f(x,y) \|_B \leq \Phi(x,y) \]

for all \( x, y \in \mathcal{A} \). Then there exists a unique sextic mapping \( S : \mathcal{A} \to \mathcal{B} \) satisfying (1) and

\[ \| f(x) - S(x) \|_B \leq \frac{K}{\mu_j^{\beta p}} \sum_{j=0}^{\infty} \left( \frac{K}{\mu_j^{\beta p}} \right)^j \Phi(\mu^j x, \mu^j x) \]

for all \( x \in \mathcal{A} \).

\[ \begin{align*}
&\leq \frac{K}{\alpha \theta_0}(\|S'(\mu^n x) - f(\mu^n x)\|_B + \|f(\mu^n x) - S(\mu^n x)\|_B) \\
&\leq 2K \sum_{j=0}^{n} \frac{K}{(\mu)^{j+1}} \beta \phi(\mu^{n+j}, x^{n+j}),
\end{align*} \]
for all \( x \in \mathcal{A} \). By letting \( n \to \infty \), we immediately have the uniqueness of \( S \).

**Theorem 2.** Let \( \phi: \mathcal{A} \times \mathcal{A} \to [0, \infty) \) be a mapping satisfying
\[ \sum_{j=0}^{\infty} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{y}{\mu^{j}} \right) < \infty \]  \hspace{1cm} (13)
for all \( x, y \in \mathcal{A} \). Let \( f: \mathcal{A} \to \mathcal{B} \) be a mapping with the condition \( f(0) = 0 \) satisfying (3) for all \( x, y \in \mathcal{A} \). Then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[ \|f(x) - S(x)\|_B \leq \frac{K}{\alpha \theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (14)
for all \( x \in \mathcal{A} \). The mapping \( S(x) \) is defined by
\[ S(x) = \lim_{n \to \infty} \mu^{n} f\left( \frac{x}{\mu^{n}} \right) \]  \hspace{1cm} (15)
for all \( x \in \mathcal{A} \).

**Proof.** Plugging \((x, y)\) into \( (\frac{x}{\mu^{j}}, x) \) in (3), we obtain
\[ \|f(x) - \mu^{n-j} f\left( \frac{x}{\mu^{j}} \right)\|_B \leq \frac{1}{\alpha \theta_0} \| f\left( \frac{x}{\mu^{j}}, x \right) \|_B \]  \hspace{1cm} (16)
for all \( x \in \mathcal{A} \). Now, substituting \( x \) as \( \frac{x}{\mu^{j}} \) multiplying by \( \mu^{6\beta} \) in (16) and summing the resulting inequality with (16), we have
\[ \|f(x) - \mu^{6n} f\left( \frac{x}{\mu^{n}} \right)\|_B \leq \frac{K}{2\theta_0} \sum_{j=0}^{n-1} (K\mu^{n-j}) \phi\left( \frac{x}{\mu^{n-j+1}}, \frac{x}{\mu^{n-j+1}} \right) \]  \hspace{1cm} (17)
for all \( x \in \mathcal{A} \) and using induction arguments on a positive integer \( n \), we conclude that
\[ \|f(x) - \mu^{6n} f\left( \frac{x}{\mu^{n}} \right)\|_B \leq \frac{K}{2\theta_0} \sum_{j=0}^{n-1} (K\mu^{n-j}) \phi\left( \frac{x}{\mu^{n-j+1}}, \frac{x}{\mu^{n-j+1}} \right) \]  \hspace{1cm} (18)
for all \( x \in \mathcal{A} \). The rest of the proof is obtained by similar arguments as in Theorem 1.

**Corollary 1.** Let \( \theta \geq 0 \) be fixed. If a mapping \( f: \mathcal{A} \to \mathcal{B} \) satisfies the inequality
\[ \|D_{n} f(x, y)\|_B \leq \theta \]  \hspace{1cm} (20)
for all \( x, y \in \mathcal{A} \), then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta}{\alpha \theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (21)
for all \( x \in \mathcal{A} \).

**Proof.** Considering \( \phi(x, y) = \theta \) , for all \( x, y \in \mathcal{A} \) in Theorem 1, we have
\[ \|f(x) - S(x)\|_B \leq \frac{K}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \theta \]  \hspace{1cm} (22)
for all \( x \in \mathcal{A} \).

**Corollary 2.** Let \( \theta_i \geq 0 \) be fixed and \( r \neq 6\beta \). If a mapping \( f: \mathcal{A} \to \mathcal{B} \) satisfies the inequality
\[ \|D_{n} f(x, y)\|_B \leq \theta_i (\|x\|_B + \|y\|_B) \]  \hspace{1cm} (23)
for all \( x, y \in \mathcal{A} \), then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta_i}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (24)
for all \( x \in \mathcal{A} \).

**Proof.** By replacing \( \phi(x, y) = \theta_i (\|x\|_B + \|y\|_B) \) for all \( x, y \in \mathcal{A} \) and considering \( y < 6\beta \) in Theorem 1, we can have
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta_i}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (25)
for all \( x \in \mathcal{A} \).

**Corollary 3.** Let \( \theta_2 \geq 0 \) be fixed and \( r \neq 6\beta \). If a mapping \( f: \mathcal{A} \to \mathcal{B} \) satisfies the inequality
\[ \|D_{n} f(x, y)\|_B \leq \theta_2 \]  \hspace{1cm} (26)
for all \( x, y \in \mathcal{A} \), then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta_2}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (27)
for all \( x \in \mathcal{A} \).

**Proof.** By replacing \( \phi(x, y) = \theta_2 (\|x\|_B + \|y\|_B) \) for all \( x, y \in \mathcal{A} \) and considering \( y < 6\beta \) in Theorem 1, we can have
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta_2}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (28)
for all \( x \in \mathcal{A} \).

**Corollary 4.** Let \( \theta_3 \geq 0 \) be fixed and \( \gamma \neq 6\beta \). If a mapping \( f: \mathcal{A} \to \mathcal{B} \) satisfies the inequality
\[ \|D_{n} f(x, y)\|_B \leq \theta_3 \]  \hspace{1cm} (29)
for all \( x, y \in \mathcal{A} \), then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[ \|f(x) - S(x)\|_B \leq \frac{K\theta_3}{2\theta_0} \sum_{j=0}^{n} (K\mu^{j}) \phi\left( \frac{x}{\mu^{j}}, \frac{x}{\mu^{j}} \right) \]  \hspace{1cm} (30)
for all \( x \in \mathcal{A} \).
for all \( x, y \in \mathcal{A} \), then there exists a unique sextic mapping \( S: \mathcal{A} \to \mathcal{B} \) satisfying (1) and
\[
\|S(x) - f(x)\|_\mathcal{B} \leq \sqrt{\theta_1 K_3 + \sqrt{\theta_2 K_4}} \|x\|_\mathcal{A}^\gamma \quad \text{for} \quad \gamma > 6 \beta, \\
\frac{2 \gamma (\mu^{6\beta} - K_6 \mu^\beta)}{3 \theta_2 K} \|x\|_\mathcal{A}^\gamma \quad \text{for} \quad \gamma < 6 \beta,
\]
for all \( x \in \mathcal{A} \).

Proof. By putting
\[
\phi(x, y) = \theta_3 \left(\|x\|_\mathcal{A}^{3\gamma} \|y\|_\mathcal{A}^{\gamma} + \|x\|_\mathcal{A}^{\gamma} + \|x\|_\mathcal{A}^{\gamma}\right),
\]
for all \( x, y \in \mathcal{A} \) and taking \( \gamma < 6 \beta \) in Theorem 1, we get
\[
\|S(x) - f(x)\|_\mathcal{B} \leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta}} \sum_{j=0}^{\infty} (K \mu^j - 6 \beta)^j \|x\|_\mathcal{A}^{\gamma} \\
\leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta}} (1 - K \mu^{6 \beta} - 6 \beta)^{-1} \|x\|_\mathcal{A}^{\gamma} \\
\leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta} - K_6 \mu^\beta} \|x\|_\mathcal{A}^{\gamma} \tag{21}
\]
for all \( x \in \mathcal{A} \) and considering \( \gamma > 6 \beta \) in Theorem 2, we have
\[
\|S(x) - f(x)\|_\mathcal{B} \leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta}} \sum_{j=0}^{\infty} (K \mu^j - 6 \beta)^{-1} \|x\|_\mathcal{A}^{\gamma} \\
\leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta}} (1 - K \mu^{6 \beta} - 6 \beta)^{-1} \|x\|_\mathcal{A}^{\gamma} \\
\leq \frac{3 \theta_2 K}{2 \gamma \mu^{6\beta} - K_6 \mu^\beta} \|x\|_\mathcal{A}^{\gamma} \tag{22}
\]
for all \( x \in \mathcal{A} \). From (21) and (22), we arrive at the required results. \( \square \)

References


