1. Introduction

Consider the following integral operator \( K \) defined on \( X = L^2[-1,1] \) or \( C[-1,1] \) by

\[
Ku(s) = \int_{-1}^{1} k(s,t)u(t)dt, \quad s \in [-1,1].
\]

We are interested to find \( u \in X \) such that

\[
Ku = \lambda u, \quad \|u\| = 1.
\]  

We cannot solve the above integral equations explicitly. So, many authors are interested to solve the above equations approximately to obtain the eigenvalues. Some of the commonly used methods are projection (Galerkin and collocation), degenerate kernel methods, and Nyström methods to obtain the approximate eigenvalues of the eigenvalue problem of a compact integral operator \( K \). In recent decades, spectral methods are being successfully applied in many fields.

To solve the various integral equations and the eigenvalue problem, numerically spectral projection methods have been used by various researchers (see, [1-6]). Legendre spectral approximation method for eigenvalue problem of a compact integral operator is developed in [7]. In this paper, we use discrete Legendre spectral projection methods to solve the eigenvalue problem and evaluate the error bounds for approximate eigenvalues with the exact eigenvalues.

The super-convergence results have been obtained for eigenvalues and iterated eigenvectors in discrete Legendre Galerkin methods.

We organize this paper as follows. In Section 2, we set up the abstract framework for the method and in Section 3, we discuss the discrete Legendre Galerkin and discrete Legendre collocation methods for the eigenvalue problem with smooth kernel. In Section 4, we discuss the convergence rates for eigenvalues in the discrete Legendre projection (Galerkin and collocation) methods in \( L^2 \)-norm. In Section 5, we present numerical examples.

We assume \( c \) is a generic constant throughout this paper.

2. Abstract Framework

Let \( L^2[-1,1] \) be the space of complex valued square integrable functions with the inner product

\[
\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt, \quad f, g \in L^2[-1,1],
\]

and \( \|f\|_{L^2} = \langle f, f \rangle^{1/2} \).

Let \( X = C[-1,1] \subset L^2[-1,1] \) be the space of complex valued continuous functions on \([-1,1]\). Let

\[
Ku(s) = \int_{-1}^{1} k(s,t)u(t)dt, \quad s \in [-1,1],
\]

where the kernel \( k(s,t) \in C([-1,1] \times [-1,1]), u \in X \) and \( \lambda \in \mathbb{C} - \{0\}. \) Then \( K \) is a compact linear integral operator on \( C[-1,1] \) and \( L^2[-1,1] \).

We are interested to \( u \in X \) and \( \lambda \in \mathbb{C} - \{0\} \) such that

\[
Ku = \lambda u.
\]  

Assume \( \lambda \neq 0 \) be the eigenvalue of \( K \) with algebraic multiplicity \( m \) and ascent \( l \). Let \( \Gamma \subset \rho(K) \) be a simple closed rectifiable curve such that \( \sigma(K) \cap \text{int}(\Gamma) = \{\lambda\}, 0 \notin \text{int}(\Gamma), \) where \( \text{int}(\Gamma) \) denotes the interior of \( \Gamma \).

Since the above equations cannot be solved exactly, we are interested to use projection methods to solve the eigenvalue problem (3). To do this, we let \( X_n = \{\phi_0, \phi_1, \ldots, \phi_n\} \) be the sequence of Legendre polynomial subspaces of \( X \) of degree \( \leq n \), where \( \{\phi_0, \phi_1, \ldots, \phi_n\} \) forms an orthonormal basis for \( X_n \). The \( \phi_i \)'s are given by

\[
\{\phi_i \}_{i=0}^{n} = \text{Legendre polynomial basis}.
\]

The \( \phi_i \) of degree \( i \) is given by

\[
\phi_i(s) = \frac{1}{2^i i!} \frac{d^i}{ds^i} (1-s^2)^n |_{s=0}.
\]

The \( \phi_i \) form an orthonormal basis of \( L^2[-1,1] \), where \( \phi_0 = 1 \) and \( \phi_i = \phi_{2i} \).

The projection of \( u \) onto \( X_n \) is defined by

\[
P_n u = \sum_{i=0}^{n} \langle u, \phi_i \rangle \phi_i.
\]

The error \( e \) of the projection is given by

\[
e = u - P_n u,
\]

and the projection error \( E \) is given by

\[
E = \|u - P_n u\|.
\]

The projection methods are used to approximate the eigenvalues and eigenvectors of \( K \). The projection methods are based on the Galerkin and collocation methods.

3. Discrete Legendre Galerkin Method

The discrete Legendre Galerkin method is based on the Galerkin method. The weak form of the eigenvalue problem is given by

\[
\langle Ku, \phi_i \rangle = \lambda \langle u, \phi_i \rangle, \quad i = 0, 1, \ldots, n.
\]

This equation is known as the discrete eigenvalue problem. The discrete eigenvalue problem is solved numerically using the Galerkin method.

4. Convergence Rates

Let \( u_n \) be the \( n \)th degree polynomial approximation of \( u \). The convergence rate \( C \) of \( u_n \) to \( u \) is given by

\[
C = \|u - u_n\|.
\]

The convergence rate \( C \) is obtained using numerical quadrature rules.

5. Numerical Examples

Numerical examples are presented to illustrate the theoretical results. The results are obtained using various numerical quadrature rules.

The numerical results are obtained using different quadrature rules. The quadrature rules are chosen to obtain the error bounds for the eigenvalues and eigenvectors of \( K \). The numerical results are presented in the form of tables and graphs.

6. Conclusion

The discrete Legendre projection methods are successfully applied to solve the eigenvalue problem of a compact integral operator. The discrete Legendre Galerkin and discrete Legendre collocation methods are used to approximate the eigenvalues and eigenvectors of \( K \). The convergence rates for eigenvalues and eigenvectors are obtained using numerical quadrature rules.

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\[ \phi_i(s) = \sqrt{\frac{2i+1}{2}} L_i(s), \ i = 0, 1, 2, ..., n, \]

where \( L_i \)'s are the Legendre polynomials of degree \( \leq i \).

The Legendre polynomials can be generated by the following recurrence relation

\[ L_n(s) = 1, \ L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s). \]

Since \( \phi_i \) and \( \phi_j \)'s are polynomials, note that

\[ \langle \phi_i, \phi_j \rangle = \int_{-1}^{1} \phi_i(t)\phi_j(t) \, dt = \delta_{i,j} \] (4)

for \( i, j = 0, 1, ..., n \). Now to solve the eigenvalue problem (3) by using projection methods, i.e., Galerkin and collocation methods, we need to evaluate the integrals, which will appear due to the inner products and the integral operator \( \mathbf{K} \). However, it is not possible to calculate the integrals exactly. So, we will replace the integrals with numerical quadrature rule and the method is named as discrete Legendre projection method.

To do this, we approximate the integration by the following numerical quadrature rule:

\[ \int_{-1}^{1} f(t) \, dt \approx \sum_{p=1}^{M(n)} w_p f(t_p), \] (5)

where \( M(n) \) is a constant depend upon \( n \) and

(i) \( w_p > 0 \), \( p = 1, 2, ..., M(n) \). (6)

(ii) the above rule has degree of precision \( d \) which is at least \( 2n \) that is

\[ \int_{-1}^{1} f(t) \, dt \approx \sum_{p=1}^{M(n)} w_p f(t_p), \]

for all polynomial of degree \( \leq 2n \).

From now onwards, we set \( M(n) = M \). Using (5), the discrete inner product is defined by

\[ \langle f, g \rangle_M = \sum_{p=1}^{M} w_p f(t_p)g(t_p), \ f, g \in C[-1,1] \] (8)

Using (5), the integral operator \( \mathbf{K} \) is approximated by the Nyström operator \( \mathbf{K}_n \) defined by

\[ (\mathbf{K}_n u)(s) = \sum_{p=1}^{M} w_p k(s, t_p) u(t_p). \] (9)

For the rest of the paper we set the following notations.

Let \( C'[-1,1] \) denote the space of \( r \) times continuously differentiable complex valued function on \([−1,1]\).

For \( u \in C'[-1,1] \), let

\[ \|u\|_{r,\infty} = \max \{ \|u^{(i)}\|_{\infty}, 1 \leq i \leq r \}, \]

where \( u^{(i)} \) denote the \( i \) th derivative of \( u \). Assume \( k(\cdot, \cdot) \in C^d([-1,1] \times [-1,1]) \), where \( d \) is the degree of precision of the numerical quadrature rule and \( d \geq 2n > n \geq r > 1 \). For fixed \( s \in [-1,1] \), we denote \( k_s(t) = k(s, t) \).

\[ 2 = \int_{-1}^{1} ds = \sum_{p=1}^{M} w_p, \] (10)

it follows that for \( j = 0, 1, 2, ..., d, \)

\[ \| (\mathbf{K}_n u)^{(j)} \|_{r,\infty} = \sup_{s \in [-1,1]} \| (\mathbf{K}_n u)^{(j)}(s) \|_{r,\infty} = \sup_{s \in [-1,1]} \sum_{p=1}^{M} w_p \frac{\partial^j}{\partial s^j} k(s, t_p) u(t_p) \]

\[ \leq \sum_{p=1}^{M} w_p \sup_{s \in [-1,1]} \left| \frac{\partial^j}{\partial s^j} k(s, t_p) \right| \|u(t_p)\|_{r,\infty} \]

\[ \leq 2\|k\|_{r,\infty} \|u\|_{r,\infty}, \]

where \( \|k\|_{r,\infty} = \max_{s \in [-1,1] \times [-1,1]} \left| \frac{\partial^{i+j}}{\partial s^i \partial t^j} k_s(t) \right| \). Then

\[ \| \mathbf{K}_n u \|_{r,\infty} = \max_{j} \| (\mathbf{K}_n u)^{(j)} \|_{r,\infty}, 0 \leq j \leq r \]

\[ \leq c \|k\|_{r,\infty} \|u\|_{r,\infty}, \] (11)

Also, for \( j = 0, 1, 2, ..., d, \) we have

\[ \| (\mathbf{K} u)^{(j)} \|_{\infty} = \max_{s \in [-1,1]} \| (\mathbf{K} u)^{(j)}(s) \|_{\infty} = \max_{s \in [-1,1]} \left| \int_{-1}^{1} \frac{\partial}{\partial s} k_s(t) u(t) \, dt \right| \]

\[ \leq 2\|k\|_{\infty} \|u\|_{\infty}. \] (12)

In the next theorem, the error bounds of Nyström operator (9) with the integral operator \( \mathbf{K} \) defined in equation (2) are being quoted.

**Theorem 2.1** [4]: Let \( k(\cdot, \cdot) \in C^d([-1,1] \times [-1,1]) \), then for any \( u \in C^d([-1,1]) \), we have
\[
\| (K_n - K) u \|_\infty \leq c n^{-d} \| K \|_{d,\infty} \| u \|_{d,\infty},
\]  

(13)

where \( c \) is a constant independent of \( n \).

**Lemma 2.2**: \([8]\) Let \( S \) be a relatively compact subset of a Banach space \( X \). Let \( T \) and \( T_n \) be the bounded linear operators from \( X \) into \( X \). If \( \| T_n - T \| \to 0 \), as \( n \to \infty \) for each \( x \in S \), then \( \| T_n - T \| \to 0 \), uniformly for all \( x \in S \).

3. Discrete Legendre projection methods:

In this section, we will discuss on the discrete Legendre projection (Galerkin and collocation) methods to solve the eigenvalue problem of a compact integral operator with smooth kernel. To discuss discrete Legendre Galerkin methods first, discrete orthogonal projection operators have been introduced in the following manner.

**Discrete Legendre orthogonal projection operator**:

To discuss on the discrete Legendre Galerkin methods, we need to introduce the discrete orthogonal projection operator. Discrete orthogonal projection namely hyper interpolation operator \( Q^G_n : X \to X_n \) (Sloan [9]) is defined by

\[
Q^G_n u = \sum_{j=0}^{n} \langle u, \phi_j \rangle_M \phi_j, \quad u \in X,
\]

(14)

for \( j = 0, 1, 2, ..., n \), and \( Q^G_n \) satisfy

\[
\langle Q^G_n u, \phi \rangle_M = \langle u, \phi \rangle_M \quad \text{for all } \phi \in X_n.
\]

Now we quote some properties of \( Q^G_n \) from [9, 4].

**Lemma 3.1**: Let \( Q^G_n : X \to X_n \) be the hyperinterpolation operator defined as above. Then the following results hold.

(i) For any \( u \in X \)

\[
\left\langle u - Q^G_n u, u - Q^G_n u \right\rangle_M = \min_{\chi \in X_n} \left\langle u - \chi, u - \chi \right\rangle_M.
\]

(15)

(ii) For any \( u \in X \),

\[
\| Q^G_n u \|_{L^2} \leq \sqrt{2} \| u \|_X,
\]

and

\[
\| Q^G_n u - u \|_{L^2} \leq 2 \sqrt{2} \inf_{u_n \in X_n} \| u - u_n \|_{L^2} \to 0 \quad \text{as } n \to \infty.
\]

(iii) In particular, for \( u \in C^1[-1,1] \), and \( n \geq r \),

\[
\| Q^G_n u - u \|_{L^2} \leq c n^{-r} \| u^{(r)} \|_\infty,
\]

(17)

where \( c \) is a constant independent of \( n \).

Note that for any \( u \in C^1[-1,1] \), and \( n \geq r \), using Jackson’s theorem [10] and estimates (10) and (15), we get

\[
\left\langle u - Q^G_n u, u - Q^G_n u \right\rangle_M = \min_{\chi \in X_n} \left\langle u - \chi, u - \chi \right\rangle_M
\]

\[
= \min_{\chi \in X_n} \left\{ \sum_{i=1}^{M} w_i (u - \chi)^2 (t_i) \right\}^{1/2}
\]

\[
= \left( \sum_{i=1}^{M} w_i \right)^{1/2} \inf_{\chi \in X_n} \| u - \chi \|_X
\]

\[
\leq \sqrt{2} c n^{-r} \| u \|_{r, \infty},
\]

(18)

where \( c \) is a constant independent of \( n \).

**Discrete Legendre interpolatory projection operator**:

Let \( \{ \tau_0, \tau_1, ..., \tau_n \} \) be the zeros of the Legendre polynomial of degree \( n+1 \) and define the interpolatory projection \( Q^I_n : X \to X_n \) by

\[
Q^I_n u = \sum_{j=0}^{n} \left( u(\tau_j) \phi_j \right) \phi_j, \quad u \in X.
\]

(19)

We quote some properties of \( Q^I_n \) [11, 2].

**Lemma 3.2**: Let \( Q^I_n : X \to X_n \) be the projection operator defined by (19). Then the following conditions hold:

(i) \( \| Q^I_n u \|_{L^2} \leq c \| u \|_X \), \( u \in C[-1,1] \), where \( c \) is a constant independent of \( n \).

(ii) There exists a constant \( c > 0 \) such that for any \( u \in X \),

\[
\| u - Q^I_n u \|_{L^2} \leq c \inf_{\phi \in X_n} \| u - \phi \|_X \to 0 \quad \text{as } n \to \infty.
\]

(iii) For any \( u \in C^{(r)}[-1,1] \), there exists a constant \( c \) independent of \( n \) such that

\[
\| Q^I_n u - u \|_{L^2} \leq c n^{-r} \| u^{(r)} \|_X.
\]

(20)

Remark: If \( M = n+1 \) and the quadrature points used in the discrete inner product (8) and the collocation nodes in (19) are the same, then \( Q^G_n \) reduces to \( Q^I_n \), i.e., in such case \( Q^G_n \) and \( Q^I_n \) are the same.

For our convenience, from now onwards for this section, we set \( Q_n = Q^G_n \) or \( Q_n^I \), according as the projection operator is taken to be the discrete orthogonal projection or interpolatory projection operator. From Lemma-3.1 and Lemma-3.2, we observe that

\[
\| Q_n u \|_{L^2} \leq p_1 \| u \|_X,
\]

\[
\| u - Q_n u \|_{L^2} \to 0 \quad \text{as } n \to \infty.
\]

for all \( u \in C[-1,1] \), and for any \( u \in C^1[-1,1] \),

\[
\| Q_n u - u \|_{L^2} \leq c n^{-r} \| u \|_{r, \infty}.
\]

The discrete Legendre projection method for the eigenvalue problem (3) is
If \( Q_n \) replaced by \( Q_n^G \), the above method leads to the discrete Legendre Galerkin method, whereas if \( Q_n \) is replaced by \( Q_n^C \) we get the discrete Legendre collocation method. The iterated eigenvector is defined by \( \tilde{u}_n = \frac{1}{\lambda_n} K_n u_n \). It follows that \( Q_n \tilde{u}_n = u_n \).

**Theorem 3.3:** \( Q_n K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \)-norm.

**Proof:** To show \( Q_n K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \)-norm, we need to show that

(i) \( \|Q_n K_n\|_{L^2} \leq M \),
(ii) \( \|Q_n K_n - K\|_{L^2} \to 0 \),
(iii) \( \|Q_n (K_n - K)Q_n\|_{L^2} \to 0 \),

as \( n \to \infty \) [12].

Consider

\[
\|Q_n K_n\|_{L^2} \leq p_1 \|K\|_c. 
\]

From Theorem 2.1, we see that \( \{K_n\} \) converges to \( K \) pointwise. Thus \( \{K_n\} \) is pointwise bound. Since \( X \) is a Banach space, it follows that \( \|K_n\|_c \leq p_2 \), where \( p_2 \) is a constant independent of \( n \) by using Uniform boundedness principle. This shows that \( \|Q_n K_n\|_{L^2} \), is uniformly bounded.

By using the estimate (17), we obtain

\[
\|Q_n - I\|_{L^2} \leq cn^{-r} \|Ku\|_{r,\infty} \leq cn^{-r} \|K\|_{r,\infty} \|u\|_{r,\infty}. 
\]

Next consider

\[
\|Q_n K_n - K\|_{L^2} \leq \|Q_n (K_n - K)u\|_{L^2} + \|Q_n - I\|_{L^2} \|Ku\|_{L^2} 
\]

\[
\leq \|K - K_n\|_{L^2} \|u\|_{L^2} + \|Q_n - I\|_{L^2} \|Ku\|_{L^2} 
\]

\[
\leq cn^{-r} \|K\|_{r,\infty} \|u\|_{r,\infty} + cn^{-r} \|K\|_{r,\infty} \|u\|_{r,\infty} 
\]

\[
\to 0, \quad \text{as} \quad n \to \infty. 
\]

This gives that \( Q_n K_n \) are point wise converges to \( K \).

Let \( B = \{u \in X, |u| \leq 1\} \) be a closed unit ball in \( C[-1,1] \). Since \( K \) is a compact operator, the set \( S = \{Ku : u \in B\} \) is a relatively compact set in \( C[-1,1] \). Then by Lemma-2.2, we have

\[
\|Q_n K_n - K\|_{L^2} \leq \sup \{\|Q_n K_n - K\|_{L^2} : u \in B\} 
\]

\[
\leq \sup \{\|Q_n K_n - K\|_{L^2} : u \in S\} 
\]

\[
\to 0 \quad \text{as} \quad n \to \infty. 
\]

Since \( Q_n \) is uniformly bounded on \( K_n u, u \in B \) and \( K_n \) is compact, \( S' = \{Q_n K_n u : u \in B\} \) is relatively compact set. Thus

\[
\|Q_n K_n - K\|_{L^2} \leq \sup \{\|Q_n K_n - K\|_{L^2} : u \in B\} 
\]

\[
\leq \sup \{\|Q_n K_n - K\|_{L^2} : u \in S'\} 
\]

\[
\to 0 \quad \text{as} \quad n \to \infty. 
\]

Hence \( Q_n K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \)-norm. This completes the proof.

Since \( Q_n K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \)-norm, for all small \( n \), the spectrum of \( Q_n K_n \) inside \( \Gamma \) consists of \( m \) eigenvalues, say \( \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m} \) counted accordingly to their algebraic multiplicities in side \( \Gamma \). Let

\[
\lambda_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \ldots + \lambda_{n,m}}{m} 
\]

denote the arithmetic mean and we approximate \( \lambda \) by \( \lambda_n \). Let \( P_n, P^S_n \) and \( P_n^{S,P} \) be the spectral projections of \( K, K_n, \) and \( Q_n K_n \) respectively, associated with their corresponding spectral inside \( \Gamma \). Let \( R(P^S_n), R(P_n^S) \) and \( R(P_n^{S,P}) \) be the ranges of the spectral projections \( P^S_n, P_n^S \) and \( P_n^{S,P} \), respectively.

If \( Q_n \) is replaced by \( Q_n^G \), \( P_n^{S,G} \) will be the spectral projection, whereas if \( Q_n \) is replaced by \( Q_n^C \), \( P_n^{S,C} \) will be the spectral projection. Let \( R(P_n^{S,G}) \) and \( R(P_n^{S,C}) \) be the ranges of the spectral projections \( P_n^{S,G} \) and \( P_n^{S,C} \), respectively.

To the closeness of eigenvectors of the integral operator \( K \) and those of the approximate operators \( Q_n K_n \) recall the concept of the gap between the spectral subspaces. For nonzero subspaces \( Y_1 \) and \( Y_2 \) of \( X \), let

\[
\delta_2(Y_1, Y_2) = \sup \{dist_\rho (y, Y_2) : y \in Y_1, \|y\|_{L^2} = 1\} 
\]

\[
\delta_2(Y_1, Y_2) = \max \{\delta_2(Y_1, Y_2), \delta_2(Y_2, Y_1)\} 
\]

denotes the gap between \( Y_1 \) and \( Y_2 \) in \( L^2 \)-norm.

**Theorem 3.4** [13]: Let \( K, Q_n K_n \in BL(X) \) with \( Q_n K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \)-norm. Then for sufficiently
large \( n \) there exists a constant \( c \) independent of \( n \) such that
\[
\hat{\delta}_2(R(P_n^{S,P}), R(P^S)) \leq c \left\lVert (K - Q_n K_n)K \right\lVert_{R(P^S^*)}.
\]
In particular, for any \( u_n \in R(P_n^{S,P}) \), we have
\[
\left\lVert u_n - P^S K_n u_n \right\lVert^2 \leq c \left\lVert (K - Q_n K_n)K \right\lVert_{R(P^S^*)}^2.
\]

**Theorem 3.5**: For sufficiently large \( n \), there exists a constant \( c \) independent of \( n \) such that
\[
\hat{\delta}_2(K_n R(P_n^{S,P}), R(P^S)) \leq c \left\lVert K_n(Q_n - I)K_n \right\lVert^2 + c \left\lVert (K_n - K)K_n \right\lVert^2.
\]
In particular, for any \( u_n \in R(P_n^{S,P}) \), we have
\[
\left\lVert K_n u_n - P^S K_n u_n \right\lVert^2 \leq c \left\lVert K_n(Q_n - I)K_n \right\lVert^2 + c \left\lVert (K_n - K)K_n \right\lVert^2.
\]

**Proof**: Let \( x_n \) be arbitrary element of \( R(P_n^{S,P}) \). We consider
\[
K_n x_n - P^S K_n x_n = (I - P_n^S)K_n x_n + (P_n^S - P^S)K_n x_n.
\]
Using \( P_n^S x_n = x_n \) the first term of the right hand side of the above equation, we obtain
\[
\left\lVert (I - P_n^S)K_n x_n \right\lVert^2 = \left\lVert K_n (I - P_n^S) x_n \right\lVert^2
\]
\[
= \left\lVert K_n (P_n^S - P^S) P_n^S x_n \right\lVert^2
\]
\[
\leq \left\lVert K_n(Q_n K_n - K_n) Q_n K_n x_n \right\lVert^2.
\]
Since \( Q_n K_n \) is uniformly bounded in \( L^2 \) norm, it follows that
\[
\left\lVert (I - P_n^S)K_n x_n \right\lVert^2 \leq \left\lVert K_n(Q_n K_n - K_n) \right\lVert^2 \left\lVert K_n x_n \right\lVert^2.
\]
Now the second term of the equation (24) gives
\[
\left\lVert P_n^S - P^S \right\lVert^2 \left\lVert K_n x_n \right\lVert^2 \leq c \left\lVert K_n - K \right\lVert^2 \left\lVert K_n x_n \right\lVert^2.
\]
Since \( x_n \) is an arbitrary element of \( P_n^S \) and combining the estimates (24), (26) and (25), the result gives
\[
\hat{\delta}_2(K_n R(P_n^{S,P}), R(P^S)) \leq c \left\lVert K_n(Q_n - I)K_n \right\lVert^2 + c \left\lVert (K_n - K)K_n \right\lVert^2.
\]
In particular, for any \( u_n \in R(P_n^{S,P}) \), we have
\[
\left\lVert K_n u_n - P^S K_n u_n \right\lVert^2 \leq \hat{\delta}_2(K_n R(P_n^{S,P}), R(P^S)) \leq c \left\lVert K_n(Q_n - I)K_n \right\lVert^2 + c \left\lVert (K_n - K)K_n \right\lVert^2.
\]
This completes the proof.

**Theorem 3.6**: [14] The following holds
\[
\left\lVert \lambda - \hat{\lambda}_n \right\lVert \leq c \left\lVert Q_n K_n (K - Q_n K_n) K \right\lVert^2,
\]
\[
\left\lVert \lambda - \hat{\lambda}_n \right\lVert \leq c \left\lVert Q_n K_n (K - Q_n K_n) K \right\lVert^2,
\]
for \( i = 1, 2, \ldots, m \).

4. **Convergence Rates**:

In this section, we will discuss on various convergence rates, which helps to evaluate the error bounds of approximate eigenelements with exact eigenelements in the eigenvalue problem by using discrete Legendre Galerkin and discrete Legendre collocation methods.

**Lemma 4.1**: Let \( K_n \) be the Nyström operator defined by (9). Assume that \( k(\cdot, \cdot) \in C^d([-1,1] \times [-1,1]) \) and \( d \geq 2n > n \geq r > 1 \), then
\[
\left\lVert (K_n - K) K_n u \right\lVert = O(n^{-d}).
\]
**Proof**: By using equation (13) and (11), we obtain
\[
\left\lVert (K_n - K) K_n u \right\lVert \leq c n^{-d} \left\lVert K_n u \right\lVert \leq c n^{-d} \left\lVert K_n u \right\lVert.
\]
Next,
\[
\left\lVert (K_n - K) K_n u \right\lVert \leq c n^{-d} \left\lVert K_n u \right\lVert.
\]
This completes the proof.

At first, we will evaluate the convergence rates by using discrete Legendre Galerkin methods.

4.1 **Discrete Legendre Galerkin methods**:

**Theorem 4.2**: Let \( K \) be a compact linear integral operator with a kernel \( k(\cdot, \cdot) \in C^r([-1,1] \times [-1,1]) \) and \( Q_n^G \) be the discrete Legendre orthogonal projection operator defined by (14). Then the following holds.
\[
\left\lVert (K - Q_n^G K_n) K u \right\lVert = O(n^{-r}).
\]
**Proof**: By using equations (12) and (17), we get
\[
\left\lVert (I - Q_n^G) K u \right\lVert \leq c n^{-r} \left\lVert K u \right\lVert \leq c n^{-r} \left\lVert K u \right\lVert.
\]
Thus,
Using the estimates (13), (16) and (27), we obtain
\[ \| (I - Q_n^G) K u \|_{L_2} = O(n^{-r}). \]

By replacing \( u \) by \( K u \) in the above equation, we obtain
\[ \| (K - Q_n^G K_n) K u \|_{L_2} \leq cn^{-r} \| K u \|_{r, \infty} + cn^{-d} \| K u \|_{d, \infty}. \]

Thus, we have
\[ \| (K - Q_n^G K_n) K u \|_{L_2} = O(n^{-\min(r, d)}) = O(n^{-r}). \]

This completes the proof.

**Theorem 4.3:** The following results hold.
\[ \| K_n (I - Q_n^G) u \|_{L_2} = O(n^{-2r}), \]
\[ \| K_n (I - Q_n^G) K_n u \|_{L_2} = O(n^{-2r}). \]

**Proof:** Using the orthogonality of \( Q_n^G \), we obtain
\[ K_n (I - Q_n^G) u(s) = \sum_{p=1}^{M} w_p k_s(t_p) (I - Q_n^G) u(t_p). \]

Set \( \tilde{k}_s = I_s \). Now by using Cauchy Schwartz inequality, we obtain
\[ \| K_n (I - Q_n^G) u(s) \| \leq \sum_{p=1}^{M} w_p \| (I - Q_n^G) l_s(t_p) \|_{2}^{2} \frac{1}{2} \]
\[ \leq \sum_{p=1}^{M} w_p \| (I - Q_n^G) u(t_p) \|_{2}^{2} \frac{1}{2} \]
\[ \leq \langle (I - Q_n^G) l_s, (I - Q_n^G) l_s \rangle_{M} \]
\[ \langle (I - Q_n^G) u, (I - Q_n^G) u \rangle_{M} \]
\[ \leq cn^{-2r} \| l_s \|_{r, \infty} \| u \|_{r, \infty}. \]

Now by taking the supremum of the above equation, we obtain
\[ \| K_n (I - Q_n^G) u \|_{\infty} = \sup_{s \in [-1,1]} \| K_n (I - Q_n^G) u(s) \| \]
\[ \leq cn^{-2r} \| K u \|_{r, \infty} \| u \|_{r, \infty}. \]

By using the above inequality, we obtain
\[ \| K_n (I - Q_n^G) u \|_{L_2} \leq \sqrt{2} \| K_n (I - Q_n^G) u \|_{\infty} \]
\[ \leq cn^{-2r} \| K u \|_{r, \infty} \| u \|_{r, \infty}. \]

Now by replacing \( u \) by \( K_n u \) in the estimate (28) and using the estimate (11), we obtain
\[ \| K_n (I - Q_n^G) K_n u \|_{\infty} \leq cn^{-2r} \| K u \|_{r, \infty} \| (K_n u) \|_{r, \infty}. \]

Thus, we have
\[ \| K_n (I - Q_n^G) K_n u \|_{L_2} \leq \sqrt{2} \| K_n (I - Q_n^G) K_n u \|_{\infty} \]
\[ \leq cn^{-2r} \| K u \|_{2, \infty} \| u \|_{2, \infty}. \]

This completes the proof.

**Theorem 4.4:** The following holds.
\[ \| K_n (K - Q_n^G K_n) K u \|_{\infty} = O(n^{-2r}). \]

**Proof:** We have
\[ \| K_n (K - Q_n^G K_n) K u \|_{\infty} \leq \| K_n (I - Q_n^G) K u \|_{\infty} \]
\[ + \| K_n (Q_n^G K - Q_n^G K_n) K u \|_{\infty}. \]

Now replacing \( u \) by \( K u \) in the estimate (29) and using the estimate (12), we get
\[ \| K_n (I - Q_n^G) K u \|_{\infty} \leq cn^{-2r} \| K u \|_{r, \infty} \| (K u) \|_{r, \infty} \]
\[ \leq cn^{-2r} \| K u \|_{2, \infty} \| u \|_{2, \infty}. \]

Now the second term of the right hand side of (30)
\[ \| K_n (Q_n^G K - Q_n^G K_n) u(s) \| \]
\[ = \| w_p k(s, t_p) Q_n^G (K_n - K) u(t_p) \| \]
\[ \leq \| Q_n^G (K_n - K) u \|_{L_2} (\sum_{p=1}^{M} w_p k(s, t_p) )^{2} \frac{1}{2} \]
\[ \leq c \| (K_n - K) u \|_{\infty} \| k \|_{\infty} \]
\[ \leq c n^{-d} \| K u \|_{r, \infty} \| u \|_{r, \infty}. \]

Thus
\[ \| K_n (Q_n^G K - Q_n^G K_n) u \|_{\infty} \]
\[ \leq cn^{-d} \| K u \|_{r, \infty} \| u \|_{r, \infty}. \]

Now by using the estimates (31) and (32) in (30), we
obtain
\[ \left\| K_n (K - Q_n^G K_n) u \right\|_\infty \leq cn^{-2r} \left\| K \right\|_{\ell, \infty}^2 \left\| u \right\|_\infty + cn^{-d} \left\| K \right\|_{r, \infty} \left\| \frac{d}{d \tau} u \right\|_\infty. \]

Now replacing \( u \) by \( K u \) in the above estimate, we obtain
\[ \left\| K_n (K - Q_n^G K_n) K u \right\|_\infty = O(n^{-2r}). \]
This completes the proof. \( \square \)

Next, we evaluate the error bound for eigenvector, iterated eigenvector and eigenvalues in the discrete Legendre Galerkin method.

**Theorem 4.5:** Let \( X \) be a Banach space and \( K \), \( Q_n^G K_n \in \mathcal{B}(X) \) with \( Q_n^G K_n \) is \( \nu \)-convergent to \( K \) in \( L^2 \) norm. Then for sufficiently large \( n \) there exist a constant \( \epsilon \) independent of \( n \) such that
\[ \tilde{\delta}_2 (R(P_n^{S,G}), R(P^S)) = O(n^{-r}). \]

In particular, for any \( u_n^G \in R(P_n^{S,G}) \), we have
\[ \left\| u_n^G - P^S u_n^G \right\|_{L^2} = O(n^{-r}). \]
**Proof:** By using the Theorem 3.4 and Theorem 4.2, we obtain
\[ \tilde{\delta}_2 (R(P_n^{S,G}), R(P^S)) \leq \epsilon \left( \left\| (K - Q_n^G K_n) K \right\|_{R(P^S)}^2 \right)_{L^2} = O(n^{-r}). \]
For any \( u_n^G \in R(P_n^{S,G}) \), we have
\[ \left\| u_n^G - P^S u_n^G \right\|_{L^2} \leq \left\| (K - Q_n^G K_n) K \right\|_{R(P^S)} \leq O(n^{-r}). \]
This completes the proof. \( \square \)

**Theorem 4.6:** For sufficiently large \( n \), there exist a constant \( \epsilon \) independent of \( n \) such that
\[ \tilde{\delta}_2 (K_n R(P_n^{S,G}), R(P^S)) = O(n^{-2r}). \]
In particular, for any \( u_n^G \in R(P_n^{S,G}) \), we have
\[ \left\| K_n u_n^G - P^S K_n u_n^G \right\|_{L^2} = O(n^{-2r}). \]
**Proof:** By using Theorem 4.3 and 3.5 with Lemma 4.1, we obtain
\[ \tilde{\delta}_2 (K_n R(P_n^{S,G}), R(P^S)) \leq \epsilon \left( \left\| K_n (Q_n^G - I) K_n \right\|_{L^2} + \left\| (K_n - K) K_n \right\|_{L^2} \right) = O(n^{-2r}). \]
In particular, for any \( u_n^G \in R(P_n^{S,G}) \), we have
\[ \left\| K_n u_n^G - P^S K_n u_n^G \right\|_{L^2} \leq \epsilon \left( \left\| K_n (Q_n^G - I) K_n \right\|_{L^2} + \left\| (K_n - K) K_n \right\|_{L^2} \right) = O(n^{-2r}). \]
This completes the proof. \( \square \)

**Theorem 4.7:** Suppose that \( \lambda \) be the eigenvalues of \( K \) with algebraic multiplicity \( m \) and ascent \( I \). Let \( \tilde{\lambda}_n^G \) be the arithmetic mean of the eigenvalues \( \tilde{\lambda}_{n,1}^G, \tilde{\lambda}_{n,2}^G, \ldots, \tilde{\lambda}_{n,m}^G \). Then for sufficiently large \( n \) and for \( i = 1, 2, \ldots, m \), there exists a constant \( c \) independent of \( n \) such that
\[ |\lambda - \tilde{\lambda}_n^G| = O(n^{-2r}), \]
\[ |\lambda - \tilde{\lambda}_{n,i}^G| = O(n^{-2r}). \]
**Proof:** By using Theorem 3.6 and Theorem 4.4 for \( i = 1, 2, \ldots, m \), it follows that
\[ |\lambda - \tilde{\lambda}_n^G| \leq \left\| K_n (K - Q_n^G K_n) K \right\|_{R(P^S)} = O(n^{-2r}), \]
\[ |\lambda - \tilde{\lambda}_{n,i}^G| \leq \left\| K_n (K - Q_n^G K_n) K \right\|_{R(P^S)} = O(n^{-2r}). \]
This completes the proof. \( \square \)

**4.2 Discrete Legendre collocation methods:**

In this subsection, we calculate the convergence rates for discrete Legendre collocation methods.

**Theorem 4.8:** Let \( K \) be a compact integral operator with the kernel \( k(\cdot, \cdot) \in C^r([\lambda, \lambda] \times [\lambda, \lambda]) \), \( r \geq 1 \). Then the following holds,
\[ \left\| (K - Q_n^C K_n) K \right\|_{L^2} = O(n^{-r}). \]
**Proof:** The proof is similar to the proof of Theorem 4.2.

**Theorem 4.9:** Let \( K \) be a compact linear integral operator with a kernel \( k(\cdot, \cdot) \in C^d([-1, 1] \times [-1, 1]) \), and \( Q_n^C \) be the discrete interpolatory projection operator defined by equation (19). The following hold
\[ \left\| K_n (I - Q_n^C K_n) \right\|_{L^2} = O(n^{-r}), \]
\[ \left\| K_n (I - Q_n^C u_n^C) \right\|_{L^2} = O(n^{-r}). \]
**Proof:** Using the estimate (20) and Schwarz’s inequality, we obtain
\[ \left\| K_n (I - Q_n^C u(s)) \right\| = \sum_{p=1}^M w_p k(s, t_p) (I - Q_n^C u(t_p)) = \left\| (k(s, \cdot))_M (I - Q_n^C u) \right\|_{L^2} \leq \left\| (k(s, \cdot))_M \right\|_M \left\| (I - Q_n^C u) \right\|_{L^2} \leq cn^{-d} \left\| (I - Q_n^C u) \right\|_{L^2}. \]
Thus,
\[ \left\| K_n(I - Q_n^C)u \right\|_\infty = \sup_{s \in [-1,1]} \left| K_n(I - Q_n^C)u(s) \right| \leq cn^{-r} \left\| u'(r) \right\|_\infty. \]

Now using the above inequality, we get
\[ \left\| K_n(I - Q_n^C)u \right\|_2 \leq \sqrt{2} \left\| K_n(I - Q_n^C)u \right\|_\infty \leq cn^{-r} \left\| u'(r) \right\|_\infty. \]

This completes the proof. \( \square \)

**Theorem 4.10:** The following holds
\[ \left\| K_n(K - Q_n^C K)u \right\|_\infty = O(n^{-r}). \]

**Proof:** We have
\[
\left\| K_n(K - Q_n^C K)u \right\|_\infty \leq \left\| K_n(I - Q_n^C)u \right\|_\infty + \left\| K_n(Q_n^C K - Q_n^C K)u \right\|_\infty. \tag{33}
\]

Now by replacing \( u \) by \( Ku \) in the above equation, we obtain
\[
\left\| K_n(I - Q_n^C)Ku \right\|_\infty \leq cn^{-r} \left\| (Ku)'(r) \right\|_\infty \leq cn^{-r} \left\| K \right\|_{r,\infty} \left\| u \right\|_{d,\infty}. \tag{34}
\]

Now by using Cauchy Schwarz inequality and the Theorem 2.1, we get
\[
\left\| K_n(Q_n^C K - Q_n^C K)u \right\|_\infty \leq cn^{-d} \left\| k \right\|_{r,\infty} \left\| u \right\|_{d,\infty}.
\]

The proof completes by combining the estimates (35), (34) with (33). \( \square \)

Now, we will evaluate the error bounds for eigenelements using discrete Legendre collocation methods.

**Theorem 4.11:** Then for sufficiently large \( n \), there exist a constant \( c \) independent of \( n \) such that
\[ \left| \hat{\lambda} - \lambda_n \right| = O(n^{-r}). \]

In particular, for any \( u_n^C \in R(P_n^{S,C}) \), we have
\[ \left\| u_n^C - P^S u_n^C \right\|_2 = O(n^{-r}). \]

**Proof:** The proof follows directly from the Theorem-3.4 and Theorem-4.8.

**Theorem 4.12:** Suppose that \( \lambda \) be the eigenvalues of \( K \) with algebraic multiplicity \( m \) and ascent \( l \). Let \( \hat{\lambda}_n \) be the arithmetic mean of the eigenvalues \( \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m} \). Then for sufficiently large \( n \) and for \( t = 1, 2, \ldots, m \), there exists a constant \( c \) independent of \( n \) such that
\[
\left| \lambda - \hat{\lambda}_n \right| = O(n^{-r}), \\
\left| \lambda - \lambda_{n,t} \right| = O(n^{-r}).
\]

**Proof:** The proof of this theorem is easy by using the Theorem-4.10 and Theorem-3.6. \( \square \)

5. Numerical Results:

In this section, we present the numerical results. Choose the approximating subspaces \( X_n \) to be the Legendre polynomial subspaces of degree less than equal to \( n \).

In Tables 1 and 2, we present the errors of approximated eigenvalues with exact eigenvalues in discrete Legendre Galerkin and discrete Legendre collocation methods in \( L^2 \)-norm. We denote \( u_n^G, \tilde{u}_n^G, u_n^C \) and \( \tilde{u}_n^C \) are the eigenvector and iterated eigenvector in discrete Legendre Galerkin and discrete Legendre collocation methods, respectively.

Let \( \lambda, \tilde{\lambda}_n \) be the exact eigenvalue and arithmetic mean of approximate eigenvalues, respectively. For different values of \( n \), we compute \( \lambda, \tilde{\lambda}_n, u_n^G, \tilde{u}_n^G, u_n^C \) and \( \tilde{u}_n^C \). The computed errors in \( L^2 \) norm are presented in the following Tables.

**Example 1:** We consider the eigenvalue problem
\[
\int_{-1}^{1} k(s,t) u(t) dt = \lambda u(s), \quad s \in [-1, 1],
\]
where the kernel \( k(s,t) = \frac{1}{\sqrt{1 + |s-t|}} \).

| \( n \) | \( \left| \lambda - \tilde{\lambda}_n \right| \) | \( \left\| u_n^G - P^S u_n^G \right\|_2 \) | \( \left\| \tilde{u}_n^G - P^S \tilde{u}_n^G \right\|_2 \) |
|---|---|---|---|
| 2 | 8.087394e-06 | 1.567136e-03 | 2.097737e-05 |
| 3 | 5.820741e-08 | 4.197827e-05 | 8.101305e-07 |
| 4 | 4.232207e-09 | 7.533855e-06 | 1.994845e-09 |
| 5 | 1.875486e-10 | 2.532415e-07 | 3.453123e-10 |
| 6 | 1.078914e-12 | 3.547901e-09 | 2.526611e-11 |

| \( n \) | \( \left| \lambda - \hat{\lambda}_n \right| \) | \( \left\| u_n^C - P^S u_n^C \right\|_2 \) | \( \left\| \tilde{u}_n^C - P^S \tilde{u}_n^C \right\|_2 \) |
|---|---|---|---|
| 2 | 7.456780e-05 | 4.777710e-04 | 1.726072e-05 |
| 3 | 5.050314e-08 | 2.952569e-05 | 7.824956e-06 |
| 4 | 1.583437e-06 | 1.930987e-05 | 2.819752e-07 |
| 5 | 1.495489e-07 | 1.253249e-05 | 1.653990e-07 |
| 6 | 1.495489e-07 | 8.166315e-07 | 6.348285e-09 |
| 7 | 2.410390e-09 | 5.459677e-07 | 4.73585e-09 |
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