Journal of Computer Science & Computational Mathematics, Volume 6, Issue 4, December 2016
DOI: 10.20967/jcscm.2016.04.001

Discrete Legendre Projection Methods for the Eigenvalue Problem of a Compact Integral Operator

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Abstract: In this paper, we consider the discrete Legendre projection methods to solve the eigenvalue problem. Using sufficiently accurate numerical quadrature rule, we obtain the error bounds for gap between the spectral subspaces, eigenvalues and iterated eigenvectors for the eigenvalue problem in L^2 norm. We also obtain the superconvergence results for eigenvalues and iterated eigenvectors in discrete Legendre Galerkin methods. Numerical examples are presented to illustrate the theoretical results.

Keywords: discrete projection methods, Eigenvalue problem, compact integral operator, Legendre polynomial bases**.**

1. Introduction

Consider the following integral operator K defined on $X = L^2[-1,1]$ *or* $C[-1,1]$ by

$$Ku(s) = \int_{-1}^{1} k(s,t)u(t)dt, \quad s \in [-1,1].$$

We are interested to find $u \in X$ such that

$$Ku = \lambda u, \quad \|u\| = 1. \tag{1}$$

We cannot solve the above integral equations explicitly. So, many authors are interested to solve the above equations approximately to obtain the eigenelements. Some of the commonly used methods are projection (Galerkin and collocation), degenerate kernel methods, and Nyström methods to obtain the approximate eigenelements of the eigenvalue problem of a compact integral operator K. In recent decades, spectral methods are being successfully applied in many fields.

To solve the various integral equations and the eigenvalue problem, numerically spectral projection methods have been used by various researchers (see, [1-6]). Legendre spectral approximation method for eigenvalue problem of a compact integral operator is developed in [7]. In this paper, we use discrete Legendre spectral projection methods to solve the eigenvalue problem and evaluate the error bounds for approximate eigenelements with the exact eigenelements.
The super-convergence results have been obtained for eigenvalues and iterated eigenvectors in discrete Legendre Galerkin method.hne

We organize this paper as follows. In Section 2, we set up the abstract framework for the method and in Section 3, we discuss the discrete Legendre Galerkin and discrete Legendre collocation methods for the eigenvalue problem with smooth kernel. In Section 4, we discuss the convergence rates for eigenelements in the discrete Legendre projection (Galerkin and collocation) methods in L^2-norm. In Section 5, we present numerical examples.

We assume c is a generic constant throughout this paper.

2. Abstract Framework

Let $L^2[-1,1]$ be the space of complex valued square integrable functions with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt, \quad f, g \in L^2[-1,1],$$

and $\|f\|_{L^2} = \langle f, f \rangle^{1/2}$.

Let $X = C[-1,1] \subset L^2[-1,1]$ be the space of complex valued continuous functions on $[-1,1]$. Let

$$Ku(s) = \int_{-1}^{1} k(s,t)u(t)\,dt, \quad s \in [-1,1], \tag{2}$$

where the kernel $k(s,t) \in C([-1,1] \times [-1,1])$, $u \in X$ and $\lambda \in \mathrm{C} - \{0\}$. Then K is a compact linear integral operator on $C[-1,1]$ and $L^2[-1,1]$.

We are interested to $u \in X$ and $\lambda \in \mathrm{C} - \{0\}$ such that

$$Ku = \lambda u. \tag{3}$$

Assume $\lambda \neq 0$ be the eigenvalue of K with algebraic multiplicity m and ascent l. Let $\Gamma \subset \rho(K)$ be a simple closed rectifiable curve such that $\sigma(K) \cap \mathrm{int}(\Gamma) = \{\lambda\}$, $0 \notin \mathrm{int}(\Gamma)$, where $\mathrm{int}(\Gamma)$ denotes the interior of Γ.

Since the above equations cannot be solved exactly, we are interested to use projection methods to solve the eigenvalue problem (3). To do this, we let $X_n = \{\phi_0, \phi_1, \ldots, \phi_n\}$ be the sequence of Legendre polynomial subspaces of X of degree $\leq n$, where $\{\phi_0, \phi_1, \ldots, \phi_n\}$ forms an orthonormal basis for X_n. The ϕ_i 's are given by

$$\phi_i(s) = \sqrt{\frac{2i+1}{2}} L_i(s), \ i = 0,1,2,...,n,$$

where L_i 's are the Legendre polynomials of degree $\le i$. The Legendre polynomials can be generated by the following recurrence relation

$$L_0(s) = 1, \ L_1(s) = s, \ s \in [-1,1],$$

and for $i = 1,2,...,n-1,$

$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s).$$

Since ϕ_i and ϕ_j 's are polynomials, note that

$$\langle \phi_i, \phi_j \rangle = \int_{-1}^{1} \phi_i(t)\overline{\phi_j(t)}\, dt = \int \phi_i(t)\phi_j(t)dt = \delta_{i,j} \quad (4)$$

for $i, j = 0,1,...,n.$ Now to solve the eigenvalue problem (3) by using projection methods, i.e., Galerkin and collocation methods, we need to evaluate the integrals, which will appear due the inner products and the integral operator K. However, it is not possible to calculate the integrals exactly. So, we will replace the integrals with numerical quadrature rule and the method is named as discrete Legendre projection method.

To do this, we approximate the integration by the following numerical quadrature rule:

$$\int_{-1}^{1} f(t)dt \cong \sum_{p=1}^{M(n)} w_p f(t_p), \quad (5)$$

(5)

where $M(n)$ is a constant depend upon n and

(i) w_p are the weights such that

$$w_p > 0, \ p = 1,2,...,M(n). \quad (6)$$

(ii) the above rule has degree of precision d which is at least $2n$ that is

$$\int_{-1}^{1} f(t)dt = \sum_{p=1}^{M(n)} w_p f(t_p), \quad (7)$$

for all polynomial of degree $\le 2n \le d$.

From now onwards, we set $M(n) = M.$ Using (5), the discrete inner product is defined by

$$\langle f, g \rangle_M = \sum_{p=1}^{M} w_p f(t_p)\overline{g(t_p)}, \quad f, g \in C[-1,1] \quad (8)$$

.

Using (5), the integral operator K is approximated by the Nyström operator K_n defined by

$$(K_n u)(s) = \sum_{p=1}^{M} w_p k(s,t_p)u(t_p). \quad (9)$$

For the rest of the paper we set the following notations. Let $C^r[-1,1]$ denote the space of r times continuously differentiable complex valued function on $[-1,1].$

For $u \in C^r[-1,1],$ let

$$\|u\|_{r,\infty} = \max\{\|u^{(i)}\|_\infty, 1 \le i \le r\},$$

where $u^{(i)}$ denote the i th derivative of $u.$ Assume $k(.,.) \in C^d([-1,1]\times[-1,1]),$ where d is the degree of precision of the numerical quadrature rule and $d \ge 2n > n \ge r > 1.$ For fixed $s \in [-1,1],$ we denote $k_s(t) = k(s,t).$

$$2 = \int_{-1}^{1} ds = \sum_{p=1}^{M} w_p, \quad (10)$$

it follows that for $j = 0,1,2,...,d,$

$$\begin{aligned}\|(K_n u)^{(j)}\|_\infty &= \sup_{s\in[-1,1]} |(K_n u)^{(j)}(s)| \\ &= \sup_{s\in[-1,1]} \left|\sum_{p=1}^{M} w_p \frac{\partial^j}{\partial s^j} k(s,t_p)u(t_p)\right| \\ &\le \sum_{p=1}^{M} w_p \sup_{s\in[-1,1]} \left|\frac{\partial^j}{\partial s^j} k(s,t_p)\right| |u(t_p)| \\ &\le 2\|u\|_\infty \|k\|_{j,\infty},\end{aligned}$$

where $\|k\|_{j,\infty} = \max_{s\in[-1,1],\, 0\le i,l\le j} \left|\frac{\partial^{i+l}}{\partial s^i \partial t^l} k_s(t)\right|$. Then

$$\begin{aligned}\|K_n u\|_{r,\infty} &= \max_j\{\|(K_n u)^{(j)}\|_\infty, 0 \le j \le r\} \\ &\le c\|k\|_{r,\infty}\|u\|_\infty. \quad (11)\end{aligned}$$

Also, for $j = 0,1,2,...,d,$ we have

$$\begin{aligned}\|(Ku)^{(j)}\|_\infty &= \max_{s\in[-1,1]} |(Ku)^{(j)}(s)| \\ &= \max_{s\in[-1,1]} \left|\int_{-1}^{1} \frac{\partial^j}{\partial s^j} k_s(t)u(t)dt\right| \\ &\le 2\|k\|_{j,\infty}\|u\|_\infty. \quad (12)\end{aligned}$$

In the next theorem, the error bounds of Nyström operator (9) with the integral operator K defined in equation (2) are being quoted.

Theorem 2. 1 [4] : Let $k(\cdot,\cdot) \in C^d([-1,1]\times[-1,1]),$ then for any $u \in C^d[-1,1],$ we have

$$\|(K_n - K)u\|_\infty \le cn^{-d}\|k\|_{d,\infty}\|u\|_{d,\infty}, \tag{13}$$

where c is a constant independent of n.

Lemma 2.2: [8] Let S is a relatively compact subset of a Banach space X. Let T and T_n be the bounded linear operators from X into X. If $\|T_n - T\| \to 0,$ as $n \to \infty$ for each $x \in S$, then $\|T_n - T\| \to 0,$ uniformly for all $x \in S$.

3. Discrete Legendre projection methods:

In this section, we will discuss on the discrete Legendre projection (Galerkin and collocation) methods to solve the eigenvalue problem of a compact integral operator with smooth kernel. To discuss discrete Legendre Galerkin methods first, discrete orthogonal projection operators have been introduced in the following manner.

Discrete Legendre orthogonal projection operator:

To discuss on the discrete Legendre Galerkin methods, we need to introduce the discrete orthogonal projection operator. Discrete orthogonal projection namely hyper interpolation operator $Q_n^G : X \to X_n$ (Sloan [9]) is defined by

$$Q_n^G u = \sum_{j=0}^{n} \langle u, \phi_j \rangle_M \phi_j, \quad u \in X, \tag{14}$$

for $j = 0,1,2,\dots,n,$ and Q_n^G satisfy

$$\langle Q_n^G u, \phi \rangle_M = \langle u, \phi \rangle_M \quad \text{for all } \phi \in X_n.$$

Now we quote some properties of Q_n^G from [9, 4].

Lemma 3.1: Let $Q_n^G : X \to X_n$ be the hyperinterpolation operator defined as above. Then the following results hold.

(i) For any $u \in X$

$$\langle u - Q_n^G u,\, u - Q_n^G u \rangle_M = \min_{\chi \in X_n} \langle u - \chi, u - \chi \rangle_M. \tag{15}$$

(ii) For any $u \in X$,

$$\|Q_n^G u\|_{L^2} \le \sqrt{2}\|u\|_\infty, \tag{16}$$

and

$$\|Q_n^G u - u\|_{L^2} \le 2\sqrt{2} \inf_{u_n \in X_n} \|u - u_n\|_{L^2} \to 0 \text{ as } n \to \infty.$$

(iii) In particular, for $u \in C^r[-1,1],$ and $n \ge r,$

$$\|Q_n^G u - u\|_{L^2} \le cn^{-r}\|u^{(r)}\|_\infty, \tag{17}$$

where c is a constant independent of n.

Note that for any $u \in C^r[-1,1],$ and $n \ge r,$ using Jackson's theorem [10] and estimates (10) and (15), we get

$$\begin{aligned}\langle u - Q_n^G u,\, u - Q_n^G u \rangle_M &= \min_{\chi \in X_n} \langle u - \chi, u - \chi \rangle_M^{1/2} \\ &= \min_{\chi \in X_n} \{\sum_{i=1}^{M} w_i (u-\chi)^2(t_i)\}^{1/2} \\ &= \left(\sum_{i=1}^{M} w_i\right)^{1/2} \inf_{\chi \in X_n} \|u - \chi\|_\infty \\ &\le \sqrt{2} cn^{-r}\|u\|_{r,\infty},\end{aligned} \tag{18}$$

where c is a constant independent of n.

Discrete Legendre interpolatory projection operator:

Let $\{\tau_0, \tau_1, \dots, \tau_n\}$ be the zeros of the Legendre polynomial of degree n+1 and define the interpolatory projection $Q_n^C : X \to X_n$ by

$$Q_n^C u \in X_n,$$
$$Q_n^C u(\tau_i) = u(\tau_i), \; i = 0,1,\dots,n, \; u \in X. \tag{19}$$

We quote some properties of Q_n^C [11, 2].

Lemma 3.2 : Let $Q_n^C : X \to X_n$ be the projection operator defined by (19). Then the following conditions hold:

(i) $\|Q_n^C u\|_{L^2} \le c\|u\|_\infty,\, u \in C[-1,1]$, where c is a constant independent of n.

(ii) There exists a constant $c > 0$ such that for any $u \in X$,

$$\|u - Q_n^C u\|_{L^2} \le c \inf_{\phi \in X_n} \|u - \phi\|_\infty \to 0 \;\; as \;\; n \to \infty.$$

(iii) For any $u \in C^{(r)}[-1,1],$ there exists a constant c independent of n such that

$$\|Q_n^C u - u\|_{L^2} \le cn^{-r}\|u^{(r)}\|_\infty. \tag{20}$$

Remark: If $M = n+1$ and the quadrature points used in the discrete inner product (8) and the collocation nodes in (19) are the same, then Q_n^G reduces to Q_n^C, i.e., in such case Q_n^G and Q_n^C are the same.

For our convenience, from now onwards for this section, we set $Q_n = Q_n^G$ or Q_n^C, according as the projection operator is taken to be the discrete orthogonal projection or interpolatory projection operator. From Lemma-3.1 and Lemma-3.2, we observe that

$$\|Q_n u\|_{L^2} \le p_1\|u\|_\infty,$$
$$\|u - Q_n u\|_{L^2} \to 0 \;\; as \;\; n \to \infty,$$

for all $u \in C[-1,1],$ and for any $u \in C^r[-1,1],$

$$\|Q_n u - u\|_{L^2} \le cn^{-r}\|u\|_{r,\infty}.$$

The discrete Legendre projection method for the eigenvalue problem (3) is

$$Q_n K_n u_n = \lambda_n u_n, \qquad (21)$$

If Q_n replaced by Q_n^G, the above method leads to the discrete Legendre Galerkin method, whereas if Q_n is replaced by Q_n^C we get the discrete Legendre collocation method. The iterated eigenvector is defined by $\tilde{u}_n = \frac{1}{\lambda_n} K_n u_n$. It follows that $Q_n \tilde{u}_n = u_n$.

Theorem 3.3: $Q_n K_n$ is $\upsilon-$ convergent to K in L^2- norm.

Proof: To show $Q_n K_n$ is υ convergent to K in L^2- norm, we need to show that

(i) $\|Q_n K_n\|_{L^2} \le M$,

(ii) $\|(Q_n K_n - K)K\|_{L^2} \to 0$,

(iii) $\|(Q_n K_n - K)Q_n K_n\|_{L^2} \to 0$,

as $n \to \infty$ [12].

Consider

$$\|Q_n K_n\|_{L^2} \le p_1 \|K_n\|_\infty.$$

From Theorem - 2.1, we see that $\{K_n\}$ converges to K pointwise. Thus $\{K_n\}$ is pointwise bound. Since X is a Banach space, it follows that $\|K_n\|_\infty \le p_2$, where p_2 is a constant independent of n by using Uniform boundedness principle. This shows that $\|Q_n K_n\|_{L^2}$, is uniformly bounded.

By using the estimate (17), we obtain

$$\begin{aligned}\|(Q_n - I)Ku\|_{L^2} &\le cn^{-r}\|(Ku)^{(r)}\|_\infty \\ &\le cn^{-r}\|k\|_{r,\infty}\|u\|_\infty. \qquad (22)\end{aligned}$$

Next consider

$$\begin{aligned}&|(Q_n K_n - K)u(t)| \\ &= |(Q_n K_n - Q_n K + Q_n K - K)u(t)| \\ &\le |Q_n (K_n - K)u(t)| + |(Q_n - I)Ku(t)|.\end{aligned}$$

Now using the estimate (16), (22) and (13), we see

$$\begin{aligned}\|(Q_n K_n - K)u\|_{L^2} &\le \|Q_n (K - K_n)u\|_{L^2} + \|(Q_n - I)Ku\|_{L^2} \\ &\le \|(K - K_n)u\|_\infty + \|(Q_n - I)Ku\|_{L^2} \\ &\le cn^{-r}\|k\|_{r,\infty}\|u\|_{r,\infty} + cn^{-r}\|k\|_{r,\infty}\|u\|_\infty \\ &\to 0, \ as \ n \to \infty. \qquad (23)\end{aligned}$$

This gives that $Q_n K_n$ are point wise converges to K.

Let $B = \{u \in X, \|u\| \le 1\}$ be a closed unit ball in $C[-1,1]$. Since K is a compact operator, the set $S = \{Ku : u \in B\}$ is a relatively compact set in $C[-1,1]$. Then by Lemma- 2,2, we have

$$\begin{aligned}\|(Q_n K_n - K)K\|_{L^2} &= \sup\{\|(Q_n K_n - K)Ku\|_{L^2} : u \in B\} \\ &\le \sup\{\|(Q_n K_n - K)u\|_{L^2} : u \in S\} \\ &\to 0 \ as \ n \to \infty.\end{aligned}$$

Since Q_n is uniformly bounded over $K_n u,\ u \in B$ and K_n is compact, $S' = \{Q_n K_n u : u \in B\}$ is relatively compact set. Thus

$$\begin{aligned}&\|(Q_n K_n - K)Q_n K_n\|_{L^2} \\ &= \sup\{\|(Q_n K_n - K)Q_n K_n u\|_{L^2} : u \in B\} \\ &\le \sup\{\|(Q_n K_n - K)u\|_{L^2} : u \in S'\} \\ &\to 0 \ as \ n \to \infty.\end{aligned}$$

Hence $Q_n K_n$ is υ convergent to K in L^2 norm.

This completes the proof. □

Since $Q_n K_n$ is υ convergent to K in L^2- norm, for all small n s, the spectrum of $Q_n K_n$ inside Γ consists of m eigenvalues, say $\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,m}$ counted accordingly to their algebraic multiplicities in side Γ. Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + ... + \lambda_{n,m}}{m}$$

denote the arithmetic mean and we approximate λ by $\hat{\lambda}_n$. Let P^S, P_n^S and $P_n^{S,P}$ be the spectral projections of $K, K_n, and\ Q_n K_n$ respectively, associated with their corresponding spectral inside Γ. Let $R(P^S), R(P_n^S)$ and $R(P_n^{S,P})$ be the ranges of the spectral projections P^S, P_n^S and $P_n^{S,P}$, respectively.

If Q_n is replaced by Q_n^G, $P_n^{S,G}$ will be the spectral projection, whereas if Q_n is replaced by Q_n^C, $P_n^{S,C}$ will be the spectral projection. Let $R(P_n^{S,G})$ and $R(P_n^{S,C})$ be the ranges of the spectral projections $P_n^{S,G}$ and $P_n^{S,C}$, respectively.

To discuss the closeness of eigenvectors of the integral operator K and those of the approximate operators $Q_n K_n$ recall the concept of the gap between the spectral subspaces. For nonzero subspaces Y_1 and Y_2 of X, let

$$\delta_2(Y_1, Y_2) = \sup\{dist_p(y, Y_2) : \ y \in Y_1, \|y\|_{L^2} = 1\},$$

then

$$\hat{\delta}_2(Y_1, Y_2) = \max\{\delta_2(Y_1, Y_2), \delta_2(Y_2, Y_1)\}$$

denotes the gap between Y_1 and Y_2 in L^2- norm.

Theorem 3.4 [13]: Let $K, Q_n K_n \in BL(X)$ with $Q_n K_n$ is $\upsilon-$convergent to K in L^2-norm. Then for sufficiently large n there exists a constant c independent of n such that

$$\hat{\delta}_2(R(P_n^{S,P}),\ R(P^S)) \le c\left\|(K - Q_n K_n)K\,|_{R(P^S)}\right\|_{L^2}$$

In particular, for any $u_n \in R(P_n^{S,P})$, we have

$$\left\|u_n - P^S u_n\right\|_{L^2} \le c\left\|(K - Q_n K_n)K\,|_{R(P^S)}\right\|_{L^2}.$$

Theorem 3.5: For sufficiently large n, there exists a constant c independent of n such that

$$\hat{\delta}_2(K_n R(P_n^{S,P}),\ R(P^S)) \le c\left\|K_n(Q_n - I)K_n\right\|_{L^2} + c\left\|(K_n - K)K_n\right\|_{L^2}$$

In particular, for any $u_n \in R(P_n^{S,P})$, we have

$$\left\|K_n u_n - P^S K_n u_n\right\|_{L^2} \le c\left\|K_n(Q_n - I)K_n\right\|_{L^2} + c\left\|(K_n - K)K_n\right\|_{L^2}.$$

Proof: Let x_n be arbitrary element of $R(P_n^{S,P})$. We consider

$$K_n x_n - P^S K_n x_n = (I - P_n^S)K_n x_n + (P_n^S - P^S)K_n x_n. \qquad (24)$$

Using $P_n^{S,P} x_n = x_n$ the first term of the right hand side of the above equation, we obtain

$$\begin{aligned}\left\|(I - P_n^S)K_n x_n\right\|_{L^2} &= \left\|K_n(I - P_n^S)x_n\right\|_{L^2}\\ &= \left\|K_n(P_n^{S,P} - P_n^S)P_n^{S,P} x_n\right\|_{L^2}\\ &\le \left\|K_n(Q_n K_n - K_n)Q_n K_n x_n\right\|_{L^2}.\end{aligned}$$

Since $Q_n K_n$ is uniformly bounded in $L^2 -$ norm, it follows that

$$\begin{aligned}&\left\|(I - P_n^S)K_n x_n\right\|_{L^2}\\ &\le \left\|K_n(Q_n K_n - K_n)\right\|_{L^2}\left\|Q_n K_n x_n\right\|_{L^2}\\ &\le c\left\|K_n(Q_n - I)K_n\right\|_{L^2}\left\|x_n\right\|_{L^2}. \qquad (25)\end{aligned}$$

Now the second term of the equation (24) gives

$$\left\|(P_n^S - P^S)K_n x_n\right\|_{L^2} \le c\left\|(K_n - K)K_n\right\|_{L^2}\left\|x_n\right\|_{L^2}. \qquad (26)$$

Since x_n is an arbitrary element of $P_n^{S,P}$ and combining the estimates (24), (26) and (25), the result gives

$$\hat{\delta}_2(K_n R(P_n^{S,P}),\ R(P^S)) \le c\left\|K_n(Q_n - I)K_n\right\|_{L^2} + c\left\|(K_n - K)K_n\right\|_{L^2}.$$

In particular, for any $u_n \in R(P_n^{S,P})$, we have

$$\begin{aligned}\left\|K_n u_n - P^S K_n u_n\right\|_{L^2} &\le \hat{\delta}_2(K_n R(P_n^{S,P}),\ R(P^S))\\ &\le c\left\|K_n(Q_n - I)K_n\right\|_{L^2}\\ &\quad + c\left\|(K_n - K)K_n\right\|_{L^2}.\end{aligned}$$

This completes the proof. □

Theorem 3.6: [14] The following holds

$$\begin{aligned}\left|\lambda - \hat{\lambda}_n\right| &\le c\left\|Q_n K_n(K - Q_n K_n)K\right\|_{L^2}\\ &\le c\left\|K_n(K - Q_n K_n)K\right\|_{\infty},\\ \left|\lambda - \lambda_{n.i}\right|^l &\le c\left\|Q_n K_n(K - Q_n K_n)K\right\|_{L^2}\\ &\le c\left\|K_n(K - Q_n K_n)K\right\|_{\infty},\end{aligned}$$

for $i = 1, 2, \ldots, m.$

4. Convergence Rates:

In this section, we will discuss on various convergence rates, which helps to evaluate the error bounds of approximate eigenelements with exact eigenelements in the eigenvalue problem by using discrete Legendre Galerkin and discrete Legendre collocation methods.

Lemma 4.1: Let K_n be the Nyström operator defined by (9). Assume that $k(.,.) \in C^d([-1,1]\times[-1,1])$ and $d \ge 2n > n \ge r > 1,$ then

$$\left\|(K_n - K)K_n\right\|_{L^2} = O(n^{-d}).$$

Proof: By using equation (13) and (11), we obtain

$$\begin{aligned}\left\|(K_n - K)K_n u\right\|_{\infty} &\le cn^{-d}\left\|k\right\|_{d,\infty}\left\|K_n u\right\|_{d,\infty}\\ &\le c\,n^{-d}\left\|k\right\|_{d,\infty}^2\left\|u\right\|_{\infty}.\end{aligned}$$

Next,

$$\begin{aligned}\left\|(K_n - K)K_n u\right\|_{L^2} &\le c\left\|(K_n - K)K_n u\right\|_{\infty}\\ &\le c\,n^{-d}\left\|k\right\|_{d,\infty}^2\left\|u\right\|_{\infty}.\end{aligned}$$

This completes the proof. □

At first, we will evaluate the convergence rates by using discrete Legendre Galerkin methods.

4.1 Discrete Legendre Galerkin methods:

Theorem 4.2: Let K be a compact linear integral operator with a kernel $k(.,.) \in C^r([-1,1]\times[-1,1])$ and Q_n^G be the discrete Legendre orthogonal projection operator defined by (14). Then the following holds.

$$\left\|(K - Q_n^G K_n)Ku\right\|_{L^2} = O(n^{-r}).$$

Proof: By using equations (12) and (17), we get

$$\left\|(I - Q_n^G)Ku\right\|_{L^2} \le cn^{-r}\left\|(Ku)^{(r)}\right\|_{\infty} \le cn^{-r}\left\|k\right\|_{r,\infty}\left\|u\right\|_{\infty}. \quad (27)$$

Thus,

$$\|(I-Q_n^G)Ku\|_{L^2}=O(n^{-r}).$$

Using the estimates (13), (16) and (27), we obtain

$$\begin{aligned}\|(K-Q_n^GK_n)u\|_{L^2} &= \|(K-Q_n^GK+Q_n^GK-Q_n^GK_n)u\|_{L^2}\\ &\le \|(I-Q_n^G)Ku\|_{L^2}+\|Q_n^G(K-K_n)u\|_{L^2}\\ &\le \|(I-Q_n^G)Ku\|_{L^2}+\sqrt{2}\|(K-K_n)u\|_\infty\\ &\le cn^{-r}\|k\|_{r,\infty}\|u\|_\infty+cn^{-d}\|k\|_{d,\infty}\|u\|_{d,\infty}.\end{aligned}$$

By replacing u by Ku in the above equation, we obtain

$$\begin{aligned}\|(K-Q_n^GK_n)Ku\|_{L^2} &\le cn^{-r}\|k\|_{r,\infty}\|Ku\|_\infty+cn^{-d}\|k\|_{d,\infty}\|Ku\|_{d,\infty}\\ &\le cn^{-r}\|k\|_{r,\infty}\|k\|_\infty\|u\|_\infty+cn^{-d}\|k\|_{d,\infty}^2\|u\|_\infty.\end{aligned}$$

Thus,

$$\|(K-Q_n^GK_n)Ku\|_{L^2}=O(n^{-\min\{r,d\}})=O(n^{-r}).$$

This completes the proof. □

Theorem 4.3: The following results hold.

$$\|K_n(I-Q_n^G)u\|_{L^2}=O(n^{-2r}),$$

$$\|K_n(I-Q_n^G)K_n\|_{L^2}=O(n^{-2r}).$$

Proof: Using the orthogonality of Q_n^G , we obtain

$$\begin{aligned}|K_n(I-Q_n^G)u(s)| &= \left|\sum_{p=1}^M w_pk_s(t_p)(I-Q_n^G)u(t_p)\right|\\ &= |\langle (I-Q_n^G)u,\bar{k}_s\rangle_M|\\ &= |\langle (I-Q_n^G)u,(I-Q_n^G)\bar{k}_s\rangle_M|\\ &= \left|\sum_{p=1}^M w_p(I-Q_n^G)u(t_p)\overline{(I-Q_n^G)\bar{k}_s(t_p)}\right|\end{aligned}$$

Set $\bar{k}_s=l_s$. Now by using Cauchy Schwartz inequality, we obtain

$$\begin{aligned}|K_n(I-Q_n^G)u(s)| &\le \Big(\sum_{p=1}^M w_p|(I-Q_n^G)l_s(t_p)|^2\Big)^{1/2}\\ &\qquad \Big(\sum_{p=1}^M w_p|(I-Q_n^G)u(t_p)|^2\Big)^{1/2}\\ &\le \langle (I-Q_n^G)l_s,(I-Q_n^G)l_s\rangle_M^{1/2}\\ &\qquad \langle (I-Q_n^G)u,(I-Q_n^G)u\rangle_M^{1/2}\\ &\le cn^{-2r}\|l_s^{(r)}\|_\infty\|u^{(r)}\|_\infty.\end{aligned}$$

Now by taking the supremum of the above equation, we obtain

$$\begin{aligned}\|K_n(I-Q_n^G)u\|_\infty &= \sup_{s\in[-1,1]}|K_n(I-Q_n^G)u(s)|\\ &\le cn^{-2r}\|k\|_{r,\infty}\|u^{(r)}\|_\infty. \qquad (28)\end{aligned}$$

Now using the above inequality, we obtain

$$\begin{aligned}\|K_n(I-Q_n^G)u\|_{L^2} &\le \sqrt{2}\|K_n(I-Q_n^G)u\|_\infty\\ &\le cn^{-2r}\|k\|_{r,\infty}\|u\|_{r,\infty}. \qquad (29)\end{aligned}$$

Now by replacing u by K_nu in the estimate (28) and using the estimate (11), we obtain

$$\begin{aligned}\|K_n(I-Q_n^G)K_nu\|_\infty &\le cn^{-2r}\|k\|_{r,\infty}\|(K_nu)^{(r)}\|_\infty\\ &\le cn^{-2r}\|k\|_{r,\infty}^2\|u\|_\infty.\end{aligned}$$

So, we obtain $\|K_n(I-Q_n^G)K_n\|_\infty=O(n^{-2r})$. Thus,

$$\begin{aligned}\|K_n(I-Q_n^G)K_n\|_{L^2} &\le \sqrt{2}\|K_n(I-Q_n^G)K_n\|_\infty\\ &\le cn^{-2r}\|k\|_{r,\infty}^2\|u\|_\infty.\end{aligned}$$

This completes the proof. □

Theorem 4.4: The following holds.

$$\|K_n(K-Q_n^GK_n)Ku\|_\infty=O(n^{-2r}).$$

Proof: We have

$$\begin{aligned}\|K_n(K-Q_n^GK_n)u\|_\infty &\le \|K_n(I-Q_n^G)Ku\|_\infty\\ &\quad +\|K_n(Q_n^GK-Q_n^GK_n)u\|_\infty. \qquad (30)\end{aligned}$$

Now replacing u by Ku in the estimate (29) and using the estimate (12), we get

$$\begin{aligned}\|K_n(I-Q_n^G)Ku\|_\infty &\le cn^{-2r}\|k\|_{r,\infty}\|(Ku)^{(r)}\|_\infty\\ &\le cn^{-2r}\|k\|_{r,\infty}^2\|u\|_\infty. \qquad (31)\end{aligned}$$

Now the second term of the right hand side of (30)

$$\begin{aligned}|K_n(Q_n^GK-Q_n^GK_n)u(s)| &= |w_pk(s,t_p)Q_n^G(K_n-K)u(t_p)|\\ &\le \|Q_n^G(K_n-K)u\|_{L^2}\Big(\sum_{p=1}^M w_p^2|k(s,t_p)|^2\Big)^{1/2}\\ &\le c\|(K_n-K)u\|_\infty\|k_s\|_\infty\\ &\le cn^{-d}\|k_s\|_\infty\|k\|_{d,\infty}\|u\|_{d,\infty}.\end{aligned}$$

Thus

$$\begin{aligned}\|K_n(Q_n^GK-Q_n^GK_n)u\|_\infty &\le cn^{-d}\|k\|_{r,\infty}\|k\|_{d,\infty}\|u\|_{d,\infty}. \qquad (32)\end{aligned}$$

Now by using the estimates (31) and (32) in (30), we obtain

$$\left\|K_n(K-Q_n^G K_n)u\right\|_\infty \le cn^{-2r}\|k\|_{r,\infty}^2\|u\|_\infty + cn^{-d}\|k\|_{r,\infty}\|k\|_{d,\infty}\|u\|_{d,\infty}.$$

Now replacing u by Ku in the above estimate, we obtain

$$\left\|K_n(K-Q_n^G K_n)Ku\right\|_\infty = O(n^{-2r}).$$

This completes the proof. □

Next, we evaluate the error bound for eigenvector, iterated eigenvector and eigenvalues in the discrete Legendre Galerkin method.

Theorem 4.5 : Let X be a Banach space and K, $Q_n^G K_n \in BL(X)$ with $Q_n^G K_n$ is $\upsilon-$convergent to K in L^2-norm. Then for sufficiently large n there exist a constant c independent of n such that

$$\hat{\delta}_2(R(P_n^{S,G}),\ R(P^S)) = O(n^{-r}).$$

In particular, for any $u_n^G \in R(P_n^{S,G})$, we have

$$\left\|u_n^G - P^S u_n^G\right\|_{L^2} = O(n^{-r}).$$

Proof: By using the Theorem -3.4 and Theorem-4.2, we obtain

$$\hat{\delta}_2(R(P_n^{S,G}),\ R(P^S)) \le c\left\|(K-Q_n^G K_n)K\,|_{R(P^S)}\right\|_{L^2} = O(n^{-r})$$

For any $u_n^G \in R(P_n^{S,G})$, we have

$$\left\|u_n^G - P^S u_n^G\right\|_{L^2} \le \left\|(K-Q_n^G K_n)K\,|_{R(P^S)}\right\|_{L^2} = O(n^{-r}).$$

This completes the proof. □

Theorem 4.6: For sufficiently large n, there exist a constant c independent of n such that

$$\delta_2(K_n R(P_n^{S,G}), R(P^S)) = O(n^{-2r}).$$

In particular, for any $u_n^G \in R(P_n^{S,G})$, we have

$$\left\|K_n u_n{}^G - P^S K_n u_n{}^G\right\|_{L^2} = O(n^{-2r}).$$

Proof: By using Theorem 4.3 and 3.5 with Lemma-4.1, we obtain

$$\delta_2(K_n R(P_n^{S,G}), R(P^S)) \le c\left\|K_n(Q_n^G - I)K_n\right\|_{L^2} + \left\|(K_n-K)K_n\right\|_{L^2} = O(n^{-2r}).$$

In particular, for any $u_n^G \in R(P_n^{S,G})$, we have

$$\left\|K_n u_n{}^G - P^S K_n u_n{}^G\right\|_{L^2} \le c\left\|K_n(Q_n^G - I)K_n\right\|_{L^2} + \left\|(K_n-K)K_n\right\|_{L^2} = O(n^{-2r}).$$

This completes the proof. □

Theorem 4.7: Suppose that λ be the eigenvalues of K with algebraic multiplicity m and ascent l. Let $\hat{\lambda}_n^G$ be the arithmetic mean of the eigenvalues $\lambda_{n,1}^G, \lambda_{n,2}^G, ...\lambda_{n,m}^G$. Then for sufficiently large n and for $i = 1, 2, ..., m,$ there exists a constant c independent of n such that

$$\left|\lambda - \hat{\lambda}_n^G\right| = O(n^{-2r}),$$

$$\left|\lambda - \hat{\lambda}_{n,i}^G\right|^l = O(n^{-2r}).$$

Proof: By using Theorem -3.6 and Theorem -4.4 for $i = 1,2,..., m,$ it follows that

$$\left|\lambda - \hat{\lambda}_n^G\right| \le \left\|K_n(K-Q_n^G K_n)K|_{R(P^S)}\right\|_\infty = O(n^{-2r}),$$

$$\left|\lambda - \hat{\lambda}_{n,i}^G\right|^l \le \left\|K_n(K-Q_n^G K_n)K|_{R(P^S)}\right\|_\infty = O(n^{-2r}).$$

This completes the proof. □

4.2 Discrete Legendre collocation methods:

In this subsection, we calculate the convergence rates for discrete Legendre collocation methods.

Theorem 4.8: Let K be a compact integral operator with the kernel $k(\cdot,\cdot) \in C^r([-1,1]\times[-1,1]), r \ge 1.$ Then the following holds,

$$\left\|(K-Q_n^C K_n)K\right\|_{L^2} = O(n^{-r}).$$

Proof: The proof is similar to the proof of Theorem-4.2.

Theorem 4.9: Let K be a ccompact linear integral operator with a kernel $k(.,.) \in C^d([-1,1]\times[-1,1]),$ and Q_n^C be the discrete interpolatory projection operator defined by equation (19). The following hold

$$\left\|K_n(I-Q_n^C)u\right\|_\infty = O(n^{-r}),$$

$$\left\|K_n(I-Q_n^C)u\right\|_{L^2} = O(n^{-r}).$$

Proof: Using the estimate (20) and Schwarz's inequality, we obtain

$$\left|K_n(I-Q_n^C)u(s)\right| = \left|\sum_{p=1}^{M} w_p k(s,t_p)(I-Q_n^C)u(t_p)\right| = \left|\left\langle k_s(\cdot),\ (I-Q_n^C)u\right\rangle_M\right| \le \left\|k_s(\cdot)\right\|_{L^2}\left\|(I-Q_n^C)u\right\|_{L^2} \le cn^{-r}\left\|u^{(r)}\right\|_\infty.$$

Thus,

$$\left\|K_n(I-Q_n^C)u\right\|_\infty = \sup_{s\in[-1,1]}\left|K_n(I-Q_n^C)u(s)\right| \le cn^{-r}\left\|u^{(r)}\right\|_\infty.$$

Now using the above inequality, we get

$$\left\|K_n(I-Q_n^C)u\right\|_{L^2} \le \sqrt{2}\left\|K_n(I-Q_n^C)u\right\|_\infty \le cn^{-r}\left\|u^{(r)}\right\|_\infty.$$

This completes the proof. □

Theorem 4.10: The following holds

$$\left\|K_n(K-Q_n^CK_n)Ku\right\|_\infty = O(n^{-r}).$$

Proof: We have

$$\left\|K_n(K-Q_n^CK_n)u\right\|_\infty \le \left\|K_n(I-Q_n^C)Ku\right\|_\infty + \left\|K_n(Q_n^CK-Q_n^CK_n)u\right\|_\infty. \tag{33}$$

Now by replacing u by Ku in the above equation, we obtain

$$\left\|K_n(I-Q_n^C)Ku\right\|_\infty \le cn^{-r}\left\|(Ku)^{(r)}\right\|_\infty \le cn^{-r}\|k\|_{r,\infty}\|u\|_\infty. \tag{34}$$

Now by using Cauchy Schwartz inequality and the Theorem 2.1, we get

$$\begin{aligned}\left|K_n(Q_n^CK_n-Q_n^CK)u(s)\right| &= \left|\sum_{p=1}^{M} w_p k(s,t_p)Q_n^C(K_n-K)u(t_p)\right| \\ &\le \left\|Q_n^C(K_n-K)u\right\|_{L^2}\left(\sum_{p=1}^{M} w_p^2\left|k(s,t_p)\right|^2\right)^{\frac{1}{2}} \\ &\le c\left\|(K_n-K)u\right\|_\infty\left\|k_s\right\|_\infty \\ &\le cn^{-d}\left\|k_s\right\|_\infty\|k\|_{d,\infty}\|u\|_{d,\infty}\end{aligned}$$

Thus,

$$\left\|K_n(Q_n^CK_n-Q_n^CK)u\right\|_\infty \le cn^{-d}\|k\|_{r,\infty}\|k\|_{d,\infty}\|u\|_{d,\infty}. \tag{35}$$

The proof completes by combining the estimates (35), (34) with (33). □

Now, we will evaluate the error bounds for eigenelements using discrete Legendre collocation methods.

Theorem 4.11: Then for sufficiently large n, there exist a constant c independent of n such that

$$\widehat{\delta}_2(R(P_n^{S,C}), R(P^S)) = O(n^{-r}).$$

In particular, for any $u_n^C \in R(P_n^{S,C})$, we have

$$\left\|u_n^C - P^S u_n^C\right\|_{L^2} = O(n^{-r}).$$

Proof: The proof follows directly from the Theorem- 3.4 and Theorem-4.8.

Theorem 4.12: Suppose that λ be the eigenvalues of K with algebraic multiplicity m and ascent l. Let $\widehat{\lambda}_n^C$ be the arithmetic mean of the eigenvalues $\lambda_{n,1}^C, \lambda_{n,2}^C, ...\lambda_{n,m}^C$. Then for sufficiently large n and for $i = 1, 2, ..., m,$ there exists a constant c independent of n such that

$$\left|\lambda - \widehat{\lambda}_n^C\right| = O(n^{-r}),$$

$$\left|\lambda - \widehat{\lambda}_{n,i}^C\right|^l = O(n^{-r}).$$

Proof: The proof of this theorem is easy by using the Theorem-4.10 and Theorem -3.6. □

5. Numerical Results:

In this section, we present the numerical results. Choose the approximating subspaces X_n to be the Legendre polynomial subspaces of degree less than equal to n.

In Tables 1 and 2, we present the errors of approximated eigenelements with exact eigenelements in discrete Legendre Galerkin and discrete Legendre collocation methods in L^2-norm. We denote $u_n^G, \tilde{u}_n^G, u_n^C$ and $\tilde{u}_n^C$ are the eigenvector and iterated eigenvector in discrete Legendre Galerkin and discrete Legendre collocation methods, respectively.

Let $\lambda, \widehat{\lambda}_n$ be the exact eigenvalue and arithmetic mean of approximate eigenvalues, respectively. For different values of n, we compute $\lambda, \widehat{\lambda}_n, u_n^G, \tilde{u}_n^G, u_n^C$ and $\tilde{u}_n^C$. The computed errors in L^2 norm are presented in the following Tables.

Example 1: We consider the eigenvalue problem

$$\int_{-1}^{1} k(s,t)u(t)dt = \lambda u(s), \quad s \in [-1, 1],$$

where the kernel $k(s,t) = \dfrac{1}{\sqrt{1+|s-t|}}$.

Table-1: Legendre Galerkin Method

n	$\lvert\lambda - \hat{\lambda}_n\rvert$	$\left\|u_n^G - P^S u_n^G\right\|_{L^2}$	$\left\|\tilde{u}_n^G - P^S \tilde{u}_n^G\right\|_{L^2}$
2	8.087394e-06	1.567136e-03	2.097377e-05
3	5.820741e-08	4.197827e-05	8.101305e-07
4	4.232207e-09	7.533855e-06	1.994845e-09
5	1.875486e-10	2.532415e-08	3.453123e-10
6	1.078914e-12	3.547901e-09	2.526611e-11

Table 2: Legendre collocation Method

n	$\lvert\lambda - \hat{\lambda}_n\rvert$	$\left\|u_n^C - P^S u_n^C\right\|_{L^2}$	$\left\|\tilde{u}_n^C - P^S \tilde{u}_n^C\right\|_{L^2}$
2	7.456780e-05	4.777710e-04	1.726072e-05
3	5.050314e-06	2.9525691e-04	7.824956e-06
4	1.583437e-06	1.930987e-05	2.819752e-07
5	1.494589e-07	1.252349e-05	1.653909e-07
6	1.494589e-07	8.166315e-07	6.348285e-09
7	2.410390e-09	5.459677e-07	4.73585e-09

Acknowledgement

This work is supported under Fast track scheme for young scientists, Department of Science and Technology, Govt. of India (SR/FTP/MS-010/2012).

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