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A Class of 3-Generated Groups

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Abstract: In this paper, the order of group

$G(n) = <a,b,c \mid a^n b^2 = 1, aca = c, a^{-1}(bc)^3 a = cb^{-1}>$,

is considered, where *n* is a positive odd integer. Further, it is proved that the order of this class is equal with $n.7^n$. In this way, the coset enumeration algorithm is used.

Keywords: **p**resentation of algebraic structure, deficiency zero theory**.**

1. Introduction

Detailed information on the deficiency of a presentation of a finitely presented group may be found [1-6]. In this paper, the Modified Todd-Coxeter enumeration algorithm is used as given in [7-8] to get a presentation for all subgroups of *G(n)*. Further an application of this algorithm may be found in [4-5, 9]. Our notation is standard and follows [6]; in our calculations to certain results of [4, 6, 7, 10] is referred. The main results of this paper are the following lemma A and theorem B.

2. The Deficiency Zero Groups

$$G(n) = <a,b,c \mid a^n b^2 = 1, aca = c, a^{-1}(bc)^3 a = cb^{-1}>$$

wherein *n* is a positive odd integer. The subgroup $H = <a, bc>$ of *G(n)* is of index 2 and by letting $a = A$ and $bc = B$ the following presentation for H is obtained.

$$H = <A, B \mid A^n B A^n = B, A^{B^{-1}} = A^{B^3} A = AB^{-1}A^{-1}B^9 A> = \\ A^n B A^n = B, BAB^{-1} = A^{-1}B^{-3}AB^3 A = AB^{-1}A^{-1}B^9 A>.$$

The subgroup $M = <B^8, A, BAB^{-1}, \ldots, B^7 A B^{-7}>$ of H has index 8 in H and letting $a_{i+1} = B^i A B^{-i}, (i = 0,1,2,\ldots,7), a_9 = B^8$ gives us preliminary presentation for *M* by 9 generated and 24 relations. A first manipulation of the relation gives us the following presentation:

$$M = <a_1, a_2, \ldots, a_9 \mid R_i, i = 1, 2, \ldots, 18>$$

Where

$R_1 : a_1^n a_2^n = 1$, $R_2 : a_1^{-n} a_9^n a_1^n = a_9$,

$R_3 : a_4 a_5 a_4^{-1} = a_1$, $R_4 : a_5 a_6 a_5^{-1} = a_2$,

$R_5 : a_6 a_7 a_6^{-1} = a_3$, $R_6 : a_7 a_8 a_7^{-1} = a_4$,

$R_7 : a_8 a_9 a_1 a_9^{-1} a_8^{-1} = a_5$,

$R_8 : a_1 a_2 a_1^{-1} = a_9^{-1} a_6 a_9$,

$R_9 : a_2 a_3 a_2^{-1} = a_9^{-1} a_7 a_9$,

$R_{10} : a_3 a_4 a_3^{-1} = a_9^{-1} a_8 a_9$,

$R_{11} : a_1 a_9^{-1} a_8 a_9^{-1} = a_2 a_1^{-1}$,

$R_{11+i} : a_{i+1} a_i^{-1} a_9 = a_{i+2} a_{i+1}^{-1}$, $i = 1, 2, \ldots, 7$.

Indeed, the relation

$a_2^n a_3^n = a_3^n a_4^n = a_5^n a_6^n = a_6^n a_7^n = a_7^n a_8 = 1$

is redundant. Then the following Lemma is proved.

Lemma A: M is an abelian group

Proof: In the first step, the generators $a_1, a_2, \ldots, a_8$ is eliminated. Producing the relations R_{11}, R_{12}, R_{13} and R_{14} consecutively. And considering R_3 yields:

$$a_8^{-1} = a_9 a_1^{-1} a_4^{-1} a_1 a_9^{-1} \tag{1}$$

In a similar way, $R_{12}, R_{13}, R_{14}, R_{15}$ together with R_4 yield:

$$a_5^{-1} = a_2 a_1^{-1} a_9^4 a_2^{-1} \tag{2}$$

The relation $R_{13}, R_{14}, R_{15}, R_{16}$ together with R_5 yield:

$$a_6^{-1} = a_3 a_2^{-1} a_9^4 a_3^{-1} \tag{3}$$

And finally, $R_{14}, R_{15}, R_{16}, R_{17}$ together with R_6 the following relation is obtained:

$$a_7^{-1} = a_4 a_3^{-1} a_9^4 a_4^{-1} \tag{4}$$

Since then, substituting for a_8 in R_{11} yields:

$$a_4^{-1} = a_2 a_1^{-1} a_9^3 a_1^{-1} \tag{5}$$

Also, the relation R_{12} yields:

$$a_3^{-1} = a_2^{-1} a_9^{-1} a_1 a_2^{-1} \tag{6}$$

The relations (1) to (6) prove that $M = <a_1, a_2, a_3>$. To prove the abelianity of *M*, the following method is applied: Substituting for a_4 and a_5 in R_3 yields:

$$a_9 = a_1^{-1} a_9^{-3} a_1 \tag{7}$$

Producing the relation R_{12} to R_{18} and substituting for a_8 the following relation is obtained:

$$(a_2a_1^{-1})a_9(a_2a_1^{-1}) = a_9. \quad (8)$$

Raising both sides of (7) to the power 3 and considering (8) the next relation is obtained:

$$a_9 = a_2a_9^{-3}a_2^{-1} \quad (9)$$

Now, substituting for a_5, a_6 and a_3 in a_4 and using the key relation (7) and (9) yields that $a_9 = a_2a_9^{-3}a_2^{-1}$ so the following relation is obtained:

$$[a_9, a_2^2] = 1 \quad (10)$$

On the other hand, since $a_5^n = a_1^n$ then raising both sides of (7) to the power *n*, and considering R_2 yields:

$$a_9^{n-1} = 1, [a_9, a_1^n] = [a_9, a_2^n] = 1 \quad (11)$$

The relation of (10) and (11) yields: $[a_9, a_2] = 1$ (12) for *n* is odd. In a similar way, the following relation is obtained: $[a_9, a_1] = 1$ (13)

Finally, it is shown that $[a_1, a_2] = 1$. Indeed, the relations R_3 and R_4 are equivalent to $a_9^4 = 1$ and $a_9^4 = a_1a_2a_1^{-1}a_2^{-1}$, respectively (by substituting for a_4 and $a_4 = a_5$ and considering (12) and (13)). So, $a_1a_2 = a_2a_1$. This completes the proof of the group *M*. Abelianization of *M* gives the following result:

$$\frac{M}{M'} = <a_1, a_2, a_3 \mid a_1^n a_2^n = 1, a_9^2 = 1, a_9 = a_1^2 a_2^{-2}, [a_1, a_2] = [a_1, a_9] = 1>$$

Or

$$M = < a_1, a_2 \mid a_1^n a_2^n = 1, a_1^4 = a_2^4, [a_1, a_2] = 1 >.$$

Case 1: *n*=4k+1

$$\Rightarrow a_1^{n-1} = a_2^{n-1}$$
$$\Rightarrow a_1a_2^{-1} = a_2^{n-1}$$
$$\Rightarrow a_1 = a_2^{-2n+1}$$
$$\Rightarrow M = < a_2 \mid a_2^{-2n^2+n} = 1, a_2^4 >$$
$$\Rightarrow M = < a_2 \mid a_2^{8n} = 1, a_2^{2n(n-1)} = 1 >$$

$$\Rightarrow M = < a_2^{8n} = 1 >, (for, n \equiv 1 \ (\mathrm{mod} 4))$$

Case 2: *n*=2k-1

$$\Rightarrow a_1^{n+1} = a_2^{n+1}$$
$$\Rightarrow a_1a_2^{-n} = a_2^{n+1}$$
$$\Rightarrow a_1 = a_2^{2n+1}$$
$$\Rightarrow M = < a_2 \mid a_2^{2n^2+n} = 1, a_2^n = 1, a_2^{8n+4} = a_2^4 = 1 >$$
$$\Rightarrow M = < a_2 \mid a_2^{8n} = 1, a_2^{2n(n+1)} = 1 > = < a_2 \mid a_2^{8n} = 1 >.$$

Case 3: n=4k+2

$$M = < a_1, a_2 \mid a_1^n a_2^n = 1, a_1^4 = a_2^4, [a_1, a_2] = 1 >$$

is also of order 8*n*. This may be proved easily by considering the subgroup $K = < a_1^2, a_2^2 >$ of *M* which is index 4 and $K \cong Z_{2n}$.

By the use of Lemma A, the following Theorem is obtained.

Theorem B: For every integer, if $n \neq 0 (\mathrm{mod}\ 4), G(n)$ is a finite group of order $2^7.n$.

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