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An Algorithmic Analysis of Degressively Proportional Apportionments

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Abstract: This paper focuses on the analysis of degressively proportional allocation of seats in the European Parliament. We propose an approach, which involves a linear function holding boundary conditions on the minimum and the maximum number of seats for countries with the smallest and the greatest populations. On this basis and applying a branch and bound algorithm, we search the set of all feasible solutions to find degressively proportional allocations of seats such that the sum of the squared differences or the sum of the absolute differences between these allocations and the values provided by the defined linear function is minimized.

Keywords: degressive proportionality, elections, allocation of seats, combinatorial optimization, numerical analysis.

1. Introduction

Large disparities in the population of the member states of the European Union as well as the desire to ensure fair representation of each country on the other, led to the Treaty of Lisbon specification of the degressive proportionality as the principle of the apportionment of seats in the European Parliament among the member states. The Treaty of Lisbon [1] introduces a term of *degressively proportional* composition of the European Parliament. The definition concerning degressive proportionality is discussed by Lamassoure [2]. The rules clarify the idea contained in the Treaty, i.e. the principle of fair distribution (allocation) is that a country with a larger number of people (population) cannot be given fewer seats than a less populated country. Furthermore, the principle of relative proportionality states that the bigger the country, the greater number of voters should its members of parliament represent. On this basis, the following can be defined: sequence $s_1, s_2,\dots, s_n$ is degressively proportional with respect to $p_1 \le p_2 \le\dots\le p_n$ if and only if $s_1 \le s_2 \le\dots\le s_n$ and $p_1/s_1 \le p_2/s_2 \le\dots\le p_n/s_n$. Nevertheless, it does not determine explicitly the feasible allocation of seats $s_1, s_2,\dots, s_n$ for the given $p_1 \le p_2 \le\dots\le p_n$, e.g., which holds degressive proportionality, even if the sum of seats is fixed. Since there are lots of such feasible allocations of seats, thus, it may be necessary to introduce a condition pointing to a unique (unambiguous) solution or at least reduced number of such solutions. Otherwise, the composition of the European Parliament must be settled through political negotiations. For more details on the approaches of different allocations of seats in the European Parliament see the related studies [3-8].

According to the above studies, the principle of degressively proportional allocation of seats in the European Parliament to the member states, established in the Treaty of Lisbon, does not allow for a clear indication of the solution. At least one additional condition is required. Thus, what is proposed is to find a degressively proportional allocation solution which, in terms of existing proposed allocation functions, is the nearest one to a given linear function that holds some boundary conditions following the Treaty of Lisbon.

2. Allocation of seats under degressive proportionality

We propose a degressively proportional allocation which should be the nearest one to an allocation determined by a linear function potentially with real values. The prerequisite for taking this position is the desire to minimize the discretionary selection criteria for the European Parliament composition. In the upcoming part of this section, we will define this problem formally.

There are n countries, where p_i and s_i denote the population and the number of seats allocated to country i for i=1,…, n. The allocation of seats (i.e., a solution) is expressed as a tuple $\delta = (s_1, s_2, \dots, s_n)$ of n elements.

The Treaty of Lisbon specifies the minimum number $m = 6$ and the maximum number $M = 96$ of seats that can be allocated to a single country. An additional assumption we adopted is that only those allocations are considered for which $s_1 = m$ and $s_n = M$ and the sum of all allocated seats is $\sum_{i=1}^{n} s_i = H$, where $H = 751$. These assumptions are reasonable in the light of the provisions of the quoted resolution [2], because of *inter alia* it states that "*The minimum and maximum numbers set by the Treaty must be fully utilized to ensure that the allocation of seats in the European Parliament reflects as closely as possible the range of populations of the Member States*". Since the compliance with the proportional allocation can be understood in different ways, therefore, we suggest considering the following functions of the reference allocation (see [9]):

$$A(t) = m + \frac{M-m}{p_n - p_1}(t - p_1), \qquad (1)$$

where t is assigned a value of p_i for a given country i.

Let Π denote the set of all feasible solutions. The objective is to find such a feasible solution $\delta \in \Pi$ that minimizes the criterion value which is the sum of the squared differences between δ and A:

$$f_1(\delta) = \sum_{i=1}^{n} (A(p_i) - s_i)^2 , \tag{2}$$

or the sum of the absolute differences between δ and A

$$f_2(\delta) = \sum_{i=1}^{n} | A(p_i) - s_i |, \tag{3}$$

and the optimal solution δ_i^* can be defined as follows:

$$\delta_i^* = \arg\min_{\delta \in \Pi} \{ f_i(\delta) \} \tag{4}$$

for $i \in \{1, 2\}$. In other words, it is looking for solutions which proportionally reflect the population structure within the range specified by the minimum and maximum number of seats allocated.

The considered problem can be solved by generating all possible degressively proportional allocation of seats, and then finding the best solution for each of the considered criterion values.

3. Algorithmic approach

Let us briefly describe the algorithm proposed by Lyko and Rudek [10] that allows us to search the solution space and to find all feasible allocations of seats in a reasonable time.

The set of all possible solutions (including non-feasible solutions) is repeatedly partitioned into smaller subsets representing partial solutions, i.e. search using a dedicated strategy. For each of them some prune procedures are applied. If any of them holds, then the examined subset of solutions is excluded from the further considerations. Otherwise, the subset of solutions is further partitioned. This process is continued until all possible solutions are checked or excluded, thereby the optimal solution is found. In this paper, we extended this algorithm to calculate optimization criteria for each of feasible solutions for the analyzed metric.

The results presented in this paper were provided by the algorithm which was coded in C++ and simulations were run on a PC utilised with an Intel® Core™i7-2600K 3.40 GHz CPU and 8GB of RAM. The running time of the algorithm does not exceed 30 seconds to search the solution space and to find all feasible solutions for each of the analysed cases.

4. Numerical analysis

In this section, we present the calculations for the population data of the European Union member states in 2013 and analyse demographic stability of each solution in perspective of the Parliament's single term of office. Since the data for the beginning and ending years of the 2014-2019 parliamentary terms is not available, we will examine the closest 5-year period allowed by the available data which is 2015-2020. The populations for years 2013 and 2015 are taken from the source of Eurostat [11] and due to the lack of data of projections for years 2020 in the above reference [11], they are obtained from Wikipedia [12].

The analysis takes into consideration all possible degressively proportional sequences $s_1 \le s_2 \le \ldots \le s_n$ with respect to $p_1 \le p_2 \le \ldots \le p_n$, for which $s_1 = 6$, $s_n = 96$ and $H = 751$, where p_i is equal to the population in the considered year of the i-th European Union member state and $n = 28$.

Let Π_y denote the cardinality of all feasible solutions obtained for year y. Table 1 shows the best allocation of seats that minimises the criterion values and the number of possible seats allocated to each country for year 2013. Moreover, for year 2013 we can determine the real feasible values of the minimum $s_{\min}$ and of the maximum $s_{\max}$ number of seats for each country, i.e. values from these ranges appeared in feasible allocations of seats, e.g. in feasible allocations Belgium can have {15, 16, ..., 22} seats.

The best allocations of seats that minimise the criterion values for years 2015 and 2020 are given in Table 2. Note that the order of countries (Sweden, Hungary, Portugal, Czech Rep., Greece, Belgium or Cyprus and Estonia) according to their populations in 2020 is different than in 2015 (see italic names in Table 2 and 3). Hence, they were renumbered during calculations, but presented in order defined for 2015.

Table 3 shows the best allocations of seats calculated for populations concerning years 2015 and 2020 that are feasible for both of these periods. On this basis, the stability of the proposed approach can be analysed. It revealed that the degressive proportionality is very sensitive to population changes; however, the applied algorithm allows us to find all feasible solutions and the allocations of seats that minimise the criterion values. It can be seen that the number of solutions that are feasible only for years 2015 and 2020 (i.e. 6730538 and 556329, respectively) are significantly reduced to 106 if they have to be feasible for both of them (see Tables 5-7), but the best allocation is the same (Table 3). Note that the order of some countries is different for 2015 and 2020, then to guarantee that solutions are feasible for both periods, we analyse only allocations of seats, where these counties have the same number of seats (in their groups).

Additionally, we analyse diversification of values in allocation of seats for European Parliament. Let Π^l_y denote the cardinality of all feasible solutions for year y, where the number of different values of seats is l. For instance (see Table 1) for the allocation of seats δ = (6, 7, 8, 8, 9, 9, 10, 12, 12, 13, 13, 13, 15, 16, 18, 18, 19, 19, 19, 19, 26, 29, 49, 59, 73, 77, 79, 96), we have l = 19 different values of seats, i.e., {6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 19, 26, 29, 49, 59, 73, 77, 79, 96}. The best and worst criterion values for solutions that belong to each set Π^l_y, as well as their cardinality are given in Tables 4-6 concerning years 2013, 2015 and 2020, respectively. Additionally in Tables 7 and 8, similar results are given for solutions that are feasible in the same time for years 2015 and 2020 for criterions f_1 and f_2, respectively. On this basis, it can be seen that the differences between criterion values in reference to the diversification of the number of seats as well as the range of the diversification is reduced (i.e. values 10, 11, 12, 13, 19 and 20 are eliminated). Thus, our approach also reveals the diversity of different allocations of seats.

Furthermore, the analysis of the best criterion values for diversification values l (Tables 4-6 and 8) shows that there are more than one allocation of seats with the same best criterion value f_2 for different values of l. Therefore, the sum of the absolute differences of f_2 does not support the unambiguous choice of the best allocation of seats for the European Parliament. The sum of the squared differences provides one solution, which is unambiguously the best according to criterion f_1 and populations. However, it is not a rule, and there can be more solutions with the same criterion value. On the other hand, for some values of parameters (e.g. $H = 738$ and population from 2013), a feasible solution does not exist.

5. Conclusions

The principle of degressively proportional allocation of seats in the European Parliament, established in the Treaty of Lisbon, does not allow a clear indication of the solution. At least one additional condition is required. The solution proposed in this paper involves a linear function holding boundary conditions on the minimum and the maximum number of seats for counties with the smallest and the greatest populations. On this basis and applying a branch and bound algorithm, we search the set of all feasible solutions to find degressively proportional allocations of seats such that the sum of the squared differences (or the sum of the absolute differences) between these allocations and the values provided by the defined linear function is minimised. Therefore, it was also possible to carry out the analysis with a view of selecting the optimal - in the considered sense - allocations that meet the condition of degressive proportionality during a full European Parliament term of office. The example analysis for the year 2013 indicated in this paper can be of course modified for the case of a finite number of moments of time for which the demographic data are either known or forecast as it was done for 2015 and 2020.

On the basis of the proposed algorithmic approach, the problem of allocation of seats in the European Parliament can take a new dimension, as the political negotiations on specific solutions can be moved to a discussion on setting the additional condition. Undoubtedly, the algorithmic approach to analyse the distribution of seats can contribute to the objectification of the problem. Elaborating and establishing one additional position, or one additional principle, allows a multiple generation of allocations that comply with this new condition and at the same time, let avoid tedious political negotiations each time a new condition is established.

Table 1. The population of countries in 2013 and the best allocation of seats for the analysed criterion

Country	Population	s_{min}	s_{max}	Seats	
				f_1	f_2
Malta	421 364	6	6	6	6
Luxembourg	537 039	6	7	7	7
Cyprus	865 878	6	11	8	7
Estonia	1 320 174	6	15	8	7
Latvia	2 023 825	6	15	9	8
Slovenia	2 058 821	6	15	9	8
Lithuania	2 971 905	7	16	10	9
Croatia	4 262 140	10	17	12	11
Ireland	4 591 087	10	17	12	11
Slovakia	5 410 836	10	18	13	12
Finland	5 426 674	10	18	13	12
Denmark	5 602 628	10	18	13	12
Bulgaria	7 284 552	13	19	15	14
Austria	8 451 860	14	20	16	16
Sweden	9 555 893	15	21	18	18
Hungary	9 908 798	15	21	18	18
Portugal	10 487 289	15	22	19	18
Czech Rep.	10 516 125	15	22	19	18
Greece	11 062 508	15	22	19	18
Belgium	11 161 642	15	22	19	18
Netherlands	16 779 575	21	31	26	25
Romania	20 020 074	25	36	29	28
Poland	38 533 299	47	63	49	51
Spain	46 727 890	56	73	59	61
Italy	59 685 227	71	84	73	77
UK	63 896 071	76	86	77	82
France	65 578 819	77	88	79	83
Germany	82 020 578	96	96	96	96
Criterion				**54.3**	**36.6**

Table 2. The population of countries in years 2015 and 2020 with the best allocation of seats

Country	Population 2015	Seats		Population 2020	Seats	
		f_1	f_2		f_1	f_2
Malta	429344	6	6	420000	6	6
Luxembourg	562958	7	7	629000	7	7
Cyprus	847008	8	7	1267000	7	7
Estonia	1313271	8	7	1229000	7	7
Slovenia	1986096	9	8	1882000	8	8
Latvia	2062874	9	8	1952000	8	8
Lithuania	2921262	10	9	2732000	9	9
Croatia	4225316	11	11	4428000	11	11
Ireland	4625885	12	11	5177000	12	12
Finland	5421349	13	12	5441000	12	12
Slovakia	5471753	13	12	5572000	12	12
Denmark	5659715	13	12	5643000	12	12
Bulgaria	7202198	15	15	6967000	14	14
Austria	8584926	16	17	8860000	17	17
Sweden	9747355	18	19	10203000	18	18
Hungary	9849000	18	19	9772000	18	18
Czech Rep.	10374822	18	19	10843000	18	18
Belgium	10538275	18	19	10703000	18	18
Greece	10812467	18	19	10743000	18	18
Portugal	11258434	18	19	11721000	19	19
Netherlands	16900726	26	25	17281000	25	25
Romania	19861408	29	28	21303000	30	30
Poland	38005614	49	49	38283000	49	49
Spain	46439864	59	59	50016000	62	62
Italy	60795612	75	77	62403000	76	76
UK	64767115	79	80	65762000	80	80
France	66352469	80	81	67849000	82	82
Germany	81174000	96	96	80160000	96	96
Criterion		**39.2**	**30.0**		**8.0**	**10.5**

Table 3. The population of countries in years 2015 and 2020 with the best allocations of seats for each year among feasible for both years

Country	Population 2015	Seats		Population 2020	Seats	
		f_1	f_2		f_1	f_2
Malta	429344	6	6	420000	6	6
Luxembourg	562958	7	7	629000	7	7
Cyprus	847008	8	7	1267000	8	7
Estonia	1313271	8	7	1229000	8	7
Latvia	1986096	9	8	1882000	9	8
Slovenia	2062874	9	8	1952000	9	8
Lithuania	2921262	10	9	2732000	10	9
Croatia	4225316	11	11	4428000	11	11
Ireland	4625885	12	12	5177000	12	12
Slovakia	5421349	12	12	5441000	12	12
Finland	5471753	12	12	5572000	12	12
Denmark	5659715	12	12	5643000	12	12
Bulgaria	7202198	14	14	6967000	14	14
Austria	8584926	16	16	8860000	16	16
Sweden	9747355	17	17	10203000	17	17
Hungary	9849000	17	17	9772000	17	17
Portugal	10374822	17	17	10843000	17	17
Czech Rep.	10538275	17	17	10703000	17	17
Greece	10812467	17	17	10743000	17	17
Belgium	11258434	17	17	11721000	17	17
Netherlands	16900726	25	25	17281000	25	25
Romania	19861408	29	29	21303000	29	29
Poland	38005614	51	52	38283000	51	52
Spain	46439864	62	63	50016000	62	63
Italy	60795612	77	78	62403000	77	78
UK	64767115	81	82	65762000	81	82
France	66352469	82	83	67849000	82	83
Germany	81174000	96	96	80160000	96	96
Criterion		**79.6**	**34.0**		**22.4**	**19.1**

Table 4. The cardinality of Π^{l}_{2013} and the related best and worst criterion values among feasible solutions for 2013

l	Cardinality	f_1		f_2	
		Best	Worst	Best	Worst
10	3	176.7	210.8	52.4	57.6
11	1 503	87.5	344.9	39.1	74.6
12	54 708	72.2	440.2	36.9	75.2
13	603 445	65.9	473.1	**36.6**	74.5
14	2 907 368	61.9	479.9	**36.6**	72.8
15	7 015 461	59.5	485.2	**36.6**	72.7
16	8 801 880	57.8	441.6	**36.6**	70.4
17	5 722 274	56.2	386.1	**36.6**	68.7
18	1 861 648	55.2	322.8	**36.6**	59.0
19	275 588	**54.3**	183.6	**36.6**	57.3
20	14 447	57.6	167.5	**36.6**	54.9
21	125	94.0	143.1	42.7	54.7
Total	**27 258 450**				

Table 5. The cardinality of Π^{l}_{2015} the related best and worst criterion values among feasible solutions for 2015

l	Cardinality	f_1		f_2	
		Best	Worst	Best	Worst
10	5	131.5	171.1	44.7	47.6
11	919	57.9	227.8	32.7	53.0
12	25 574	52.3	299.9	31.4	69.5
13	244 828	48.1	358.6	30.4	69.2
14	1 020 202	45.1	365.4	30.1	69.2
15	2 083 607	42.7	399.9	**30.0**	58.2
16	2 121 444	41.2	396.7	**30.0**	57.4
17	1 012 275	40.2	355.3	**30.0**	57.1
18	206 534	39.3	253.2	**30.0**	50.9
19	14 975	**39.2**	187.4	30.1	49.8
20	175	57.3	97.8	32.3	43.6
Total	**6 730 538**				

Table 6. The cardinality of Π^{l}_{2020} and the related best and worst criterion values among feasible solutions for 2020

l	Cardinality	f_1		f_2	
		Best	Worst	Best	Worst
11	11	66.8	158.1	30.3	55.4
12	554	24.9	265.9	19.8	69.3
13	9 667	20.6	270.5	17.8	70.9
14	61 551	12.4	325.8	13.7	70.9
15	167 019	11.4	327.4	13.2	66.9
16	199 356	10.0	307.7	11.7	61.6
17	100 113	9.2	255.4	11.0	55.6
18	17 376	**8.0**	150.9	**10.5**	50.0
19	682	14.1	79.2	16.1	36.7
Total	**556 329**				

Table 7. The cardinality of $\Pi^{l}_{2015+2020}$ and the related best and worst criterion values among feasible solutions for years 2015 and 2020 – criterion f_1

l	Cardinality	Best 2015	Worst 2015	Best 2020	Worst 2020
14	8	92.8	122.9	31.9	43.4
15	46	84.6	125.0	26.3	44.0
16	38	80.7	124.4	22.5	39.5
17	13	83.3	119.2	23.6	31.3
18	1	**79.6**	79.6	22.4	22.4
Total	**106**				

Table 8. The cardinality of $\Pi^{l}_{2015+2020}$ and the related best and worst criterion values among feasible solutions for years 2015 and 2020 – criterion f_2

l	Cardinality	Best 2015	Worst 2015	Best 2020	Worst 2020
14	8	37.2	40.3	22.5	26.2
15	46	**34.0**	40.3	19.3	26.2
16	38	**34.0**	40.3	19.3	26.0
17	13	**34.0**	34.2	**19.1**	19.7
18	1	**34.0**	34.0	19.3	19.3
Total	**106**				

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References

[1] The Treaty of Lisbon, http://europa.eu/lisbon treaty/full text/index en.htm [accessed: 2010.05.18].

[2] A. Lamassoure, A. Severin, European Parliament Resolution on "Proposal to amend the Treaty provisions concerning the composition of the European Parliament", adopted on 11 October 2007 (INI/2007/2169).

[3] G. R. Grimmett, J.-F. Laslier, F. Pukelsheim, V. Ramírez González, R. Rose, W. Słomczyński, M. Zachariasen, K. Życzkowski, "The allocation between the EU Member States of the seats in the European

Parliament”, *European Parliament Studies*, PE 432.760, 2011.

[4] K. Cegielka, P. Dniestrzanski, J. Lyko, A. Misztal, “Division of Seats in the European Parliament”, *Journal for Perspectives of Economic Political and Social Integration*, vol. 16, no. 1-2, pp. 39-55, 2010.

[5] P. Dniestrzanski, “Proposal for measure of degressive proportionality”, *Procedia - Social and Behavioral Sciences*, vol. 110, pp. 140-147, 2014.

[6] P. Serafini, “Allocation of the EU Parliament seats via integer linear programming and revised quotas”, *Mathematical Social Science*, vol. 63, no. 2 pp. 107-113, 2012.

[7] V. R. Gonzalez, J. M. Aroza, A. M. Garcia, “Spline methods for degressive proportionality in the composition of the European Parliament”, *Mathematical Social Science*, vol. 63, no. 2, pp. 114-120, 2012.

[8] J. Florek, “A numerical method to determine a degressive proportional distribution of seats in the European Parliament”, *Mathematical Social Sciences*, vol. 63, no. 2, pp. 121-129, 2012.

[9] W. Slomczynski, K. Zyczkowski, “Mathematical aspects of degressive proportionality”, *Mathematical Social Science*, vol. 63, no. 2, pp. 94-101, 2012.

[10] J. Lyko, R. Rudek, “A fast exact algorithm for the allocation of seats for the EU Parliament”, Expert *Systems with Applications*, vol. 40, no. 13, pp. 5284-5291, 2013.

[11] Population of EU countries, http://appsso.eurostat.ec.europa.eu/nui/show.do [accessed: 2016.01.28]

[12] List of countries by past and future population, http://en.wikipedia.org/wiki/List_of_countries_by_past_and_future_population [accessed: 2016.01.28]