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Superconvergence Results for the Iterated Discrete Legendre Galerkin Method for Hammerstein Integral Equations

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Abstract: In this paper we analyse the iterated discrete Legendre Galerkin method for Fredholm-Hammerstein integral equation with a smooth kernel. Using a sufficiently accurate numerical quadrature rule, we obtain super-convergence rates for the iterated discrete Legendre Galerkin solutions in both infinity and L^2-norm. Numerical examples are given to illustrate the theoretical results.

Keywords: Hammerstein integral equations, Spectral method, Iterated discrete Galerkin, Numerical quadrature, Super-convergence.

1. Introduction

Let X be a Banach space and consider the following Hammerstein integral equation:

$$x(t) - \int_{\Omega} k(t,s)\psi\big(s,x(s)\big)ds = f(t), \ \ t \in \Omega, \ x \in X, \qquad (1)$$

where Ω is a closed bounded region in R. Hammerstein integral equations arise as a reformulation of various physical phenomena in different branches of study such as mathematical physics, vehicular traffic, biology, economics etc.

There has been a notable interest in the numerical analysis of solutions of intergal equations (see [1-11]). The Galerkin, collocation, Petrov-Galerkin, degenerate kernel and Nystr$\ddot{o}$m methods are the most frequently used projection methods for solving the equations of type (1). We are mainly interested in iterated discrete Galerkin method in this paper.

The projection methods for solving equation (1) lead to algebraic nonlinear system, in which the coefficients are integrals, it appeared due to inner products and integral operator. Replacement of these integrals by numerical quadrature rule gives rise to the discrete projection methods. The effect of quadrature error on the convergence rates of the approximate solution is considered in these discrete projection methods (see [1], [12-14]). Discrete projection methods for Fredholm nonlinear integral equations with spline bases and their super-convergence results have been studied by many authors such as Atkinson and Potra [3], Atkinson and Flores [15] and many others. Atkinson and Bogomolny [16] have shown that sufficiently accurate numerical quadrature rules can preserve the rates of convergence of the spline based Galerkin method. However, to get better accuracy in spline based discrete projection methods, the number of partition points should be increased. Hence in such cases, one has to solve a large system of nonlinear equations, which is computationally very much expensive. To overcome the computational complexities encountered in the existing piecewise polynomial based projection methods, we apply polynomially-based projection methods to nonlinear Fredholm integral equations ([4, 5, 17, 18]). We choose the approximating subspaces X_n to be global polynomial subspaces of degree n which has dimension $n+1$. The advantage of using global polynomials is that the projection method will imply smaller nonlinear systems, something which is highly desirable in practical computations. In particular here, we choose to use Legendre polynomials, which can be generated recursively with ease and possess nice property of orthogonality.

In a recent paper [18], we obtain that the discrete Legendre Galerkin solution of the equation (1) converges with the order $O(n^{-r+1})$ in both infinity and L^2-norm, n being the highest degree of the Legendre polynomial employed in the approximation and r is the smoothness of the kernel k the nonlinear function ψ, the right hand side function f and the solution, with $n \geq r$. In this paper, we investigate the superconvergence property of the iterated discrete Legendre Galerkin method for Fredholm-Hammerstein integral equation (1). Our purpose in this paper is to obtain similar superconvergence results in polynomially-based discrete Galerkin method for Fredholm-Hammerstein integral equation (1) with a smooth kernel as in the case of piecewise polynomial based discrete Galerkin method. By choosing a numerical quadrature rule of certain degree of precesion, we show that the iterated discrete Legendre Galerkin solution converges with the order $O(n^{-2r})$ in both infinity and L^2 -norm.

The organization of this paper is as follows. In Section 2 and 3, we discuss the discrete and iterated discrete Legendre spectral Galerkin methods and obtain superconvergence results. In Section 4, numerical examples are given to illustrate the theoretical results. Throughout this paper, we assume that c is a generic constant.

2. Discrete Legendre Galerkin Method

In this section, we describe the discrete Galerkin method for solving Fredholm-Hammerstein integral equations using global polynomial basis functions.

Let $L^2[-1,1]$ be the space of real square-integrable functions on $[-1,1]$ and $X = C[-1,1] \subseteq L^2[-1,1]$. Consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi\big(s, x(s)\big)ds = f(t),\ -1 \le t \le 1, \quad (2)$$

where k, f and ψ are known real functions and x is the unknown function to be determined. For a fixed $t \in [-1,1]$, we denote $k_t(s) = k(t,s)$.

Throughout the paper, the following assumptions are made on f, $k(.,.)$ and ψ:

I. $f \in C[-1,1]$,

II. $k(t,s) \in C([-1,1] \times [-1,1])$ and $M_1 = sup_{t,s\in[-1,1]}|k(t,s)| < \infty$,

III. $\lim_{t\to t'} ||k_t(.) - k_{t'}(.)||_\infty = 0, \quad t, t' \in [-1,1]$,

IV. the nonlinear function $\psi(s,x)$ is continuous in $s \in [-1,1]$ and is Lipschitz continuous in x, i.e., for any $x_1, x_2 \in R$, $\exists c_1 > 0$ such that
$$|\psi(s,x_1) - \psi(s,x_2)| \le c_1|x_1 - x_2|,$$

V. the partial derivative $\psi^{(0,1)}(s,x)$ of ψ w.r. to the second variable exists and is Lipschitz continuous in x, i.e., for any $x_1, x_2 \in R$, $\exists\ c_2 > 0$ such that
$$\left|\psi^{(0,1)}(s,x_1) - \psi^{(0,1)}(s,x_2)\right| \le c_2|x_1 - x_2|.$$

Let

$$Ky(t) = \int_{-1}^{1} k(t,s)y(s)ds,\ t \in [-1,1],\ y \in X.$$

Note that, using Holder's inequality, we have for any $y \in X$

$$\begin{aligned}||Ky||_\infty &= \sup_{t\in[-1,1]} |Ky(t)| \\ &= \sup_{t\in[-1,1]} \left|\int_{-1}^{1} k(t,s)y(s)ds\right| \\ &\le \sup_{t,s\in[-1,1]} |k(t,s)| \int_{-1}^{1} |y(s)|ds \\ &\le \sqrt{2}M_1||y||_{L^2}, \end{aligned} \quad (3)$$

and

$$||Ky||_{L^2} \le \sqrt{2}||Ky||_\infty \le 2M_1||y||_{L^2}. \quad (4)$$

This implies

$$||K||_{L^2} \le 2M_1. \quad (5)$$

We will use Kumar and Sloan's [8] technique for finding the approximate solution of the equation (2). To do this, we let

$$z(t) = \psi\big(t, x(t)\big),\ t \in [-1,1]. \quad (6)$$

Then the solution x of Hammerstein integral equation (2) is obtained by

$$x = f + Kz. \quad (7)$$

For our convenience, we define a nonlinear operator $\Psi: X \to X$ by

$$\Psi(x)(t) \coloneqq \psi\big(t, x(t)\big). \quad (8)$$

Then (6) takes the form

$$z = \Psi(Kz + f). \quad (9)$$

Let $T(u) \coloneqq \Psi(Ku + f), u \in X$, then the equation (9) can be written as $z = Tz$.

Theorem 1. Let $X = C[-1,1]$, $f \in X$ and $k(.,.) \in C([-1,1] \times [-1,1])$ with $M_1 = sup_{t,s\in[-1,1]}|k(t,s)| < \infty$. Let $\psi\big(s, y(s)\big) \in C([-1,1] \times R)$ satisfy the Lipschitz condition in the second variable, i.e.

$$|\psi(s,y_1) - \psi(s,y_2)| \le c_1|y_1 - y_2|,\ \ y_1, y_2 \in R,$$

with $2M_1c_1 < 1$. Then the operator equation $z = Tz$ has a unique solution $z_0 \in X$, i.e., we have $z_0 = Tz_0$.

Proof of the above theorem can be easily done using similar technique given in Theorem 2.4 of [19]. We denote x_0 be the solution of equation (7) corresponding to the solution z_0 of (9), i.e., $x_0 = Kz_0 + f$.

Now, we describe the discrete Legendre Galerkin method for the solution of Hammerstein integral equation (2). To do this, we let $X_n = span\{\varphi_0,\ \varphi_1,\ \varphi_2, \dots, \varphi_n\}$ be the sequence of Legendre polynomial subspaces of X of degree n, where $\{\varphi_0,\ \varphi_1,\ \varphi_2, \dots, \varphi_n\}$ forms an orthonormal basis for X_n. Here $\varphi_i{}'s$ are given by

$$\varphi_i(s) = \sqrt{\tfrac{2i+1}{2}}L_i(s),\ i = 0, 1, \dots, n, \quad (10)$$

where $L_i{}'s$ are the Legendre polynomials of degree $\le i$. These Legendre polynomials can be generated by the following three-term recurrence relation

$$L_0(s) = 1,\ L_1(s) = s, \qquad s \in [-1,1],$$

and for $i = 1, 2, \dots,\ n-1$ and $s \in [-1,1]$

$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s).$$

In practice, the integrals in Galerkin method for solving (7) and (9), appearing due to inner products and the integral operator K cannot be evaluated exactly. The replacement of these integrals by numerical quadrature gives rise to the discrete Galerkin method, which we describe below.

First we choose a numerical integration scheme

$$\int_{-1}^{1} f(t)dt = \sum_{p=1}^{M(n)} w_p f(t_p), \quad (11)$$

where

(i) the weights w_p are such that

$$w_p > 0, \ \ p = 1, 2, \dots, M(n), \quad (12)$$

(ii) the degree of precision d of the quadrature rule is at least $2n$, that is

$$\int_{-1}^{1} f(t)dt = \sum_{p=1}^{M(n)} w_p f(t_p), \quad (13)$$

for all polynomials of degree $\leq 2n \leq d$.

For the notational convenience, from now on we set $M(n) = M$. Using the above quadrature rule (11)-(13) (see Golberg [12], Sloan [20]), we define the discrete inner product

$$< f, g >_M = \sum_{p=1}^{M} w_p f(t_p) g(t_p), \ \ f, g \in C[-1,1]. \quad (14)$$

For the approximation of the integral operator K, using the quadrature rule (11)-(13), we consider the Nystr$\ddot{o}$m operator K_n defined by

$$(K_n z)(t) = \sum_{p=1}^{M} w_p k(t, t_p) z(t_p). \quad (15)$$

For the rest of the paper, we assume that $z_0 \in C^d[-1,1]$, $k(.,.) \in C^d([-1,1] \times [-1,1]), \psi \in C^d([-1,1] \times R)$ and $f \in C^d[-1,1]$, where $C^d[-1,1]$ denotes the space of d-times continuously differentiable functions on [-1,1] and d is the degree of precision of the numerical quadrature rule. For $z \in C^d[-1,1]$, let $||z||_{d,\infty} = max\{||z^{(j)}||_\infty : 0 \leq j \leq d\}$. We denote

$$||k||_{d,\infty} = max\{||D^{(l,p)} k||_\infty : 0 \leq l, p \leq d\},$$

where

$$(D^{(l,p)} k)(t, s) = \frac{\partial^{l+p}}{\partial t^l \partial s^p} k(t, s), \quad t, s \in [-1,1].$$

In the following theorem, we give the error bounds for the integral operator K and the Nystr$\ddot{o}$m operator K_n defined by (15).

Theorem 2. Let $k(.,.) \in C^d([-1,1] \times [-1,1])$, then for any $z \in C^d([-1,1])$, there hold

$$||(K_n - K)z||_\infty \leq c n^{-d} ||k||_{d,\infty} ||z||_{d,\infty},$$

where c is a constant independent of n and d is the degree of precision of the quadrature rule.

Proof: Let us consider the error function

$$E_d(f) = \int_{-1}^{1} f(s)ds - \sum_{j=1}^{M} w_j f(t_j).$$

Using the quadrature rule defined by (11)-(13), we see that $E_d(p) = 0, \ \forall p \in P_d$, where P_d is the space of all polynomials of degree $\leq d$.

For all $p \in P_d$, consider

$$|(K_n - K)z(t)| = \left| \int_{-1}^{1} k(t, s) z(s) ds - \sum_{j=1}^{M} w_j k(t, t_j) z(t_j) \right|$$

$$= |E_d(k(t,.)z(.)) - E_d(p)|$$

$$= |E_d[k_t(.)z(.) - p]|. \quad (16)$$

Since weight functions $w_j > 0$, choosing $p(x) = 1$, we have $\int_{-1}^{1} ds = \sum_{j=1}^{M} w_j = 2$. Using this we get

$$|E_d[k_t(.)z(.) - p]|$$

$$= \left| \int_{-1}^{1} [k(t, s) z(s) - p(s)] ds - \sum_{j=1}^{M} w_j [k(t, t_j) z(t_j) - p(t_j)] \right|$$

$$\leq ||k_t(.)z(.) - p||_\infty \left[\int_{-1}^{1} ds + \sum_{j=1}^{M} w_j \right]$$

$$\leq 4 \, ||k_t(.)z(.) - p||_\infty.$$

As $p \in P_d$ is arbitrary, from the above estimate and using Jackson's Theorem [21], we obtain

$$|E_d[k_t(.)z(.) - p]| \leq 4 \inf_{p \in P_d} ||k_t(.)z(.) - p||_\infty$$

$$\leq 4c ||(k_t(.))^d||_\infty ||z||_{d,\infty} d^{-d}. \quad (17)$$

Since from (13), we have $d \geq 2n$, it follows that

$$|E_d[k_t(.)z(.) - p]| \leq c ||k||_{d,\infty} ||z||_{d,\infty} (2n)^{-d}, \quad (18)$$

where c is a constant independent of n.

Now combining estimates (16) and (18), we get

$$||(K_n - K)z||_\infty \leq c n^{-d} ||k||_{d,\infty} ||z||_{d,\infty},$$

where c is a constant independent of n. This completes the proof. □

Note that from Theorem 2, we see that K_n converges pointwise in infinity norm. Hence, $||K_n||_\infty$ is pointwise bounded and since $X = C[-1,1]$ is a Banach space with $||.||_\infty$ norm, by Uniform Boundedness Principle we have K_n is uniformly bounded, i.e.

$$||K_n||_\infty \leq p < \infty, \tag{19}$$

where p is a constant independent of n.

Discrete orthogonal projection operator: Discrete orthogonal projection namely hyperinterpolation operator $Q_n: X \to X_n$ (see Sloan [20]) is defined by

$$Q_n u = \sum_{j=0}^{n} < u, \varphi_j >_M \varphi_j, \tag{20}$$

and Q_n satisfy

$$< Q_n u, \varphi >_M = < u, \varphi >_M, \quad \forall \varphi \in X_n. \tag{21}$$

We quote some crucial properties of Q_n from Sloan [20] which are needed for the convergence analysis of the approximate solutions.

Lemma 1. Let $Q_n: X \to X_n$ be the hyperinterpolation operator defined by (20). Then for any $u \in X$, the following result holds

$$< u - Q_n u, u - Q_n u >_M = \min_{\chi \in X_n} < u - \chi, u - \chi >_M.$$

Lemma 2. Let $Q_n: X \to X_n$ be the hyperinterpolation operator defined by (20). Then the following results hold

i. $||Q_n u||_{L^2} \leq p_1 ||u||_\infty$, where p_1 is a constant independent of n.
ii. There exists a constant $c > 0$ such that for any $n \in N$ and $u \in X$,
$$||Q_n u - u||_{L^2} \leq c \inf_{\varphi \in X_n} ||u - \varphi||_{L^2} \to 0, \quad as\ n \to \infty. \tag{22}$$
iii. In particular, if $u \in C^r[-1,1]$, then there holds,
$$||Q_n u - u||_{L^2} \leq c n^{-r} ||u||_{r,\infty}, \quad n \geq r, \tag{23}$$
where c is a constant independent u and n.

Now the discrete Legendre Galerkin method for (9) is to find $z_n \in X_n$, such that

$$z_n = Q_n \psi(K_n z_n + f). \tag{24}$$

Letting $T_n(u) := Q_n \psi(K_n u + f)$, $u \in X$, the equation (24) can be written as $z_n = T_n z_n$. The corresponding discrete approximate solution x_n of x is defined by

$$x_n = K_n z_n + f. \tag{25}$$

We quote the following results from [18], which gives the convergence rates of the discrete Legendre Galerkin solutions in both infinity and L^2-norm.

Theorem 3. Let $z_0 \in C^r[-1,1]$, $n \geq r$, be an isolated solution of the equation (9). Let Q_n be the discrete orthogonal projection operator defined by (20). Assume that 1 is not an eigenvalue of the linear operator $T'(z_0)$, then for sufficiently n, the approximate solution z_n defined by the equation (24) is the unique solution in the sphere $B(z_0, \delta) = \{z: ||z - z_0||_\infty \leq \delta\}$ for some $\delta > 0$. Moreover, there exists a constant $0 < q < 1$, independent of n such that

$$\frac{\alpha_n}{1+q} \leq ||z_n - z_0||_\infty \leq \frac{\alpha_n}{1-q},$$

where $\alpha_n = ||(I - T_n'(z_0))^{-1}(T_n(z_0) - T(z_0))||_\infty$, and

$$||z_n - z_0||_\infty = O(n^{-r+1}).$$

Theorem 4. Let $z_0 \in C^r[-1,1]$ be an isolated solution of the equation (9) and x_0 be a isolated solution of the equation (7) such that $x_0 = K z_0 + f$. x_n be the discrete Legendre Galerkin approximation of x_0. Then the following holds

$$||x_0 - x_n||_\infty, \ ||x_0 - x_n||_{L^2} = O(n^{-r+1}).$$

3. Iterated Discrete Legendre Galerkin Method

In order to obtain more accurate approximation solution, we further consider the iterated discrete Legendre Galerkin approximation of z_0 in this section. To this end, we define the iterated discrete solution as

$$\tilde{z}_n = \psi(K_n z_n + f). \tag{26}$$

Applying Q_n on both sides of the equation (26), we obtain

$$Q_n \tilde{z}_n = Q_n \psi(K_n z_n + f). \tag{27}$$

From equations (24) and (27), it follows that $Q_n \tilde{z}_n = z_n$. Using this, we see that the iterated solution $\tilde{z}_n$ satisfies the following equation

$$\tilde{z}_n = \psi(K_n Q_n \tilde{z}_n + f). \tag{28}$$

Letting $\tilde{T}_n(u) := \psi(K_n Q_n u + f)$, $u \in X$, the equation (28) can be written as $\tilde{z}_n = \tilde{T}_n \tilde{z}_n$. Corresponding approximate solution $\tilde{x}_n$ of x is given by

$$\tilde{x}_n = K_n \tilde{z}_n + f. \tag{29}$$

Next we discuss the existence of the iterated approximate solutions and their error bounds. To do this, we first recall the following definition of υ-convergence and a lemma from [22].

Definition 1. Let X be Banach space and $BL(X)$ be space of bounded linear operators from X into X. Let $F_n, F \in BL(X)$. We say F_n is υ-convergent to F if

$||F_n|| \leq c < \infty$, $||(F_n - F)F|| \to 0$, $||(F_n - F)F_n|| \to 0$, as $n \to \infty$.

Lemma 3. (Ahues *et al.* [22]) Let X be a Banach space and F_n, F be bounded linear operators on X. If F_n is υ-convergent to F and $(I - F)^{-1}$ exists, then $(I - F_n)^{-1}$ exists and uniformly bounded on X, for sufficiently large n.

Lemma 4. Let $k(.,.) \in C^d([-1,1] \times [-1,1])$, $d \geq 2n > n \geq r$, then for any $u \in X$, the following hold

$$||(K_nQ_n - K)u||_\infty \to 0,\ as\ n \to \infty.$$

In particular if $u \in C^r[-1,1]$, then

$$||(K_nQ_n - K)u||_\infty = O(n^{-r}).$$

Proof: For any $u_n \in X_n$, it follows that

$$\left(\sum_{p=1}^{M} w_p[u_n(t_p)]^2\right)^{\frac{1}{2}} = \left(\int_{-1}^{1}[u_n(s)]^2\,ds\right)^{\frac{1}{2}} = ||u_n||_{L^2}. \quad (30)$$

We denote

$$e(k_t(.)) = |(K - K_n)u_n(t)|$$

$$= \left|\int_{-1}^{1} k(t,s)u_n(s)ds - \sum_{p=1}^{M} w_p k(t,t_p)u_n(t_p)\right|.$$

Using Cauchy-Schwartz inequality and estimate (30), we get

$$e(k_t(.)) = \left|\int_{-1}^{1} k(t,s)u_n(s)ds - \sum_{p=1}^{M} w_p k(t,t_p)u_n(t_p)\right|$$

$$= \left(\int_{-1}^{1}|k(t,s)|^2ds\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|u_n(s)|^2ds\right)^{\frac{1}{2}}$$

$$+\left(\sum_{p=1}^{M} w_p|k(t,t_p)|^2\right)^{\frac{1}{2}}\left(\sum_{p=1}^{M} w_p\,|u_n(t_p)|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sqrt{2} + \left(\sum_{p=1}^{M} w_p\right)^{\frac{1}{2}}\right)||k||_\infty ||u_n||_{L^2}$$

$$\leq 2\sqrt{2}\,||k||_\infty ||u_n||_{L^2}.$$

Since for any $y \in X_n$, $e(y) = 0$, we have

$$e(k_t(.)) = e(k_t(.) - y) \leq 2\sqrt{2}\inf_{y\in X_n}||k - y||_\infty ||u_n||_{L^2}.$$

Using this and Jackson's Theorem [21], we obtain

$$||(K_n - K)u_n||_\infty \leq 2\sqrt{2}cn^{-r}||k||_{r,\infty}||u_n||_{L^2}. \quad (31)$$

Since $Q_nu \in X_n,\ u \in X$, from (31) we obtain

$$||(K - K_n)Q_nu||_\infty \leq 2\sqrt{2}cn^{-r}||k||_{r,\infty}||Q_nu||_{L^2}$$

$$\leq 2\sqrt{2}cn^{-r}||k||_{r,\infty}p_1||u||_\infty. \quad (32)$$

Since $d \geq 2r$, using estimates (3), (23) and (32), we have for any $u \in C^r[-1,1]$,

$$||(K_nQ_n - K)u||_\infty$$

$$\leq ||(K - K_n)Q_nu||_\infty + ||K(Q_n - I)u||_\infty$$

$$\leq 2\sqrt{2}cn^{-r}||k||_{r,\infty}p_1||u||_\infty + \sqrt{2}\,M_1||(Q_n - I)u||_{L^2} \quad (33)$$

$$= O(n^{-r}). \quad (34)$$

Note that form estimates (22) and (33), it follows that for any $u \in X$,

$$||(K_nQ_n - K)u||_\infty \to 0,\ as\ n \to \infty. \quad (35)$$

This completes the proof. □

Theorem 5. Let $z_0 \in C^d[-1,1]$, $d \geq 2n > n \geq r$, be an isolated solution of the equation (9). Assume that 1 is not an eigenvalue of $T'(z_0)$. Then for sufficiently large n, the operator $I - \tilde{T}_n'(z_0)$ is invertible on X and there exist a constant $L > 0$ independent of n such that $||\left(I - \tilde{T}_n'(z_0)\right)^{-1}||_\infty \leq L.$

Proof: Consider

$$||\tilde{T}_n'(z_0)||_\infty = ||\psi'(K_nQ_nz_0 + f)K_nQ_n||_\infty. \quad (36)$$

We have

$$||\psi'(K_nQ_nz_0 + f)||_\infty \leq ||\psi'(K_nQ_nz_0 + f) - \psi'(K_nz_0 + f)||_\infty$$

$$+||\psi'(K_nz_0 + f)||_\infty. \quad (37)$$

Form Lemma 1 and Jackson's Theorem [21], we have for any $z \in C^r[-1,1]$,

$$< (Q_n - I)z, (Q_n - I)z >_M^{\frac{1}{2}} = \min_{\chi\in X_n} < z - \chi, z - \chi >_M^{\frac{1}{2}}$$

$$= \min_{\chi\in X_n}\left\{\sum_{p=1}^{M} w_p(z - \chi)^2(t_p)\right\}^{\frac{1}{2}}$$

$$\leq \left(\sum_{p=1}^{M} w_p\right)^{\frac{1}{2}}\inf_{\chi\in X_n}||z - \chi||_\infty$$

$$\leq \sqrt{2}cn^{-r}||z||_{r,\infty}. \quad (38)$$

Using Lipschitz continuity of $\psi^{(0,1)}(.,.)$ and (38), we obtain

$$||\psi'(K_nQ_nz_0 + f) - \psi'(K_nz_0 + f)||_\infty$$

$$\leq c_2||K_n(Q_n - I)z_0||_\infty$$

$$= c_2\sup_{t\in[-1,1]}|K_n(Q_n - I)z_0(t)|$$

$$= c_2\sup_{t\in[-1,1]}\left|\sum_{p=1}^{M} w_p k(t,t_p)(Q_n - I)z_0(t_p)\right|$$

$$\leq c_2\sup_{t\in[-1,1]}\left(\sum_{p=1}^{M} w_p\,|k(t,t_p)|^2\right)^{\frac{1}{2}}\left(\sum_{p=1}^{M} w_p\,[(Q_n - I)z_0(t_p)]^2\right)^{\frac{1}{2}}$$

$$\leq c_2\left(\sum_{p=1}^{M} w_p\right)^{\frac{1}{2}} ||k||_\infty < (Q_n - I)z_0, (Q_n - I)z_0 >_M^{\frac{1}{2}}$$

$$\leq 2c_2cn^{-r}||k||_\infty||z_0||_{r,\infty} \to 0,\ as\ n \to \infty. \quad (39)$$

Using Theorem 2, Lipschitz continuity of $\psi^{(0,1)}(.,.)$ and boundedness of $||\psi'(Kz_0 + f)||_\infty$, we have

$$||\psi'(K_n z_0 + f)||_\infty$$
$$\leq ||\psi'(K_n z_0 + f) - \psi'(Kz_0 + f)||_\infty + ||\psi'(Kz_0 + f)||_\infty$$
$$\leq c_2||(K_n - K)z_0||_\infty + ||\psi'(Kz_0 + f)||_\infty$$
$$\leq c_2cn^{-d}||k||_{d,\infty}||z_0||_{d,\infty} + ||\psi'(Kz_0 + f)||_\infty < \infty . \quad (40)$$

From estimates (37), (39) and (40), it follows that

$$||\psi'(K_nQ_nz_0 + f)||_\infty \leq B_1 < \infty, \quad (41)$$

where B_1 is a constant independent of n.

Since $Q_nz \in X_n$, from estimate (30), we have

$$\left(\sum_{p=1}^{M} w_p[Q_nz(t_p)]^2\right)^{\frac{1}{2}} = ||Q_nz||_{L^2}. \quad (42)$$

Hence using Cauchy-Schwartz inequality and the estimate (42), we have

$$||K_nQ_nz||_\infty = \sup_{t\in[-1,1]} |K_nQ_nz(t)|$$
$$= \sup_{t\in[-1,1]} \left|\sum_{p=1}^{M} w_pk(t,t_p)Q_nz(t_p)\right|$$
$$\leq M_1\left(\sum_{p=1}^{M} w_p\right)^{\frac{1}{2}}\left(\sum_{p=1}^{M} w_p[Q_nz(t_p)]^2\right)^{\frac{1}{2}}$$
$$= M_1\sqrt{2}||Q_nz||_{L^2} \leq \sqrt{2}M_1p_1||z||_\infty < \infty. \quad (43)$$

This implies

$$||K_nQ_n||_\infty \leq \sqrt{2}M_1p_1. \quad (44)$$

Combining estimates (36), (41) and (44), we have

$$||\tilde{T}_n'(z_0)||_\infty \leq B_2 < \infty, \quad (45)$$

where B_2 is independent of n. Hence it follows that $||\tilde{T}_n'(z_0)||_\infty$ is uniformly bounded.

Using estimates (3), (41) and (45), we have

$$||[\tilde{T}_n'(z_0) - T'(z_0)]\tilde{T}_n'(z_0)||_\infty$$
$$= ||[\psi'(K_nQ_nz_0 + f)K_nQ_n - \psi'(Kz_0 + f)K]\tilde{T}_n'(z_0)||_\infty$$
$$\leq ||\psi'(K_nQ_nz_0 + f)(K_nQ_n - K)\tilde{T}_n'(z_0)||_\infty$$
$$+||[\psi'(K_nQ_nz_0 + f) - \psi'(Kz_0 + f)]K\tilde{T}_n'(z_0)||_\infty$$
$$\leq ||\psi'(K_nQ_nz_0 + f)||_\infty||(K_nQ_n - K)\tilde{T}_n'(z_0)||_\infty$$
$$+||\psi'(K_nQ_nz_0 + f) - \psi'(Kz_0 + f)||_\infty||K\tilde{T}_n'(z_0)||_\infty$$
$$\leq B_1||(K_nQ_n - K)\tilde{T}_n'(z_0)||_\infty$$
$$+2M_1B_2||\psi'(K_nQ_nz_0 + f) - \psi'(Kz_0 + f)||_\infty. \quad (46)$$

Using Lipschitz continuity of $\psi^{(0,1)}(.,.)$ and Lemma 4, we obtain

$$||\psi'(K_nQ_nz_0 + f) - \psi'(Kz_0 + f)||_\infty \leq c_2||(K_nQ_n - K)z_0)||_\infty$$
$$\to 0,\ as\ n \to \infty. \quad (47)$$

Next, Let $\bar{B} := \{x \in X: ||x||_\infty \leq 1\}$ be the closed unit ball in X. We have $\tilde{T}_n'(z_0) = \psi'(K_nQ_nz_0 + f)K_nQ_n$. Since $\{K_nQ_n\}$ is a sequence of compact operators and $\psi'(K_nQ_nz_0 + f)$ is uniformly bounded, $\tilde{T}_n'(z_0)$ are compact operators. Thus $S = \{\tilde{T}_n'(z_0)x : x \in \bar{B},\ n \in N\}$ is relatively compact set. Using estimate (35), we can conclude

$$||(K_nQ_n - K)\tilde{T}_n'(z_0)||_\infty = sup\{||(K_nQ_n - K)\tilde{T}_n'(z_0)x||_\infty : x \in \bar{B}\}$$
$$= sup\{||(K_nQ_n - K)y||_\infty : y \in S\},$$
$$\to 0,\ as\ n \to \infty . \quad (48)$$

Combining the estimates (46), (47) and (48), we have

$$||[\tilde{T}_n'(z_0) - T'(z_0)]\tilde{T}_n'(z_0)||_\infty \to 0, \quad as\ n \to \infty.$$

Following the similar steps and using the fact that $T'(z_0)$ is compact, it can be proved that

$$||[\tilde{T}_n'(z_0) - T'(z_0)]T'(z_0)||_\infty \to 0, \quad as\ n \to \infty.$$

Hence $\tilde{T}_n'(z_0)$ is υ-convergent to $T'(z_0)$ in $||.||_\infty$ norm. By direct application of Lemma 3, it follows that for sufficiently large n, $I - \tilde{T}_n'(z_0)$ is invertible, i.e., there exist a constant $L > 0$, independent of n such that $||\left(I - \tilde{T}_n'(z_0)\right)^{-1}||_\infty \leq L$. This completes the proof. □

Theorem 6. Let $z_0 \in C^d[-1,1]$, $d \geq 2n > n \geq r$, be an isolated solution of the equation (9). Let $Q_n: X \to X_n$ be the discrete orthogonal projection operator defined by (20). Assume that 1 is not an eigenvalue of $T'(z_0)$, then for sufficiently large n, the iterated solution $\tilde{z}_n$ defined by (28) is the unique solution in the sphere $B(z_0, \delta) = \{z: ||z - z_0||_\infty \leq \delta\}$. Moreover, there exists a constant $0 < q < 1$, independent of n such that

$$\frac{\beta_n}{1+q} \leq ||\tilde{z}_n - z_0||_\infty \leq \frac{\beta_n}{1-q} ,$$

where $\beta_n = ||(I - \tilde{T}_n'(z_0))^{-1}(\tilde{T}_n(z_0) - T(z_0))||_\infty$.

Proof: From Theorem 5, we have $\left(I - \tilde{T}_n'(z_0)\right)^{-1}$ exists and it is uniformly bounded in infinity norm, i.e., $\exists L > 0$ such that $||\left(I - \tilde{T}_n'(z_0)\right)^{-1}||_\infty \leq L$.

Using Lipschitz continuity of $\psi^{(0,1)}(.,.)$ and estimate (43), we have for any $z \in B(z_0, \delta)$,

$$||[\tilde{T}_n'(z) - \tilde{T}_n'(z_0)]v||_\infty$$
$$= ||[\psi'(K_n Q_n z + f) - \psi'(K_n Q_n z_0 + f)]K_n Q_n v||_\infty$$
$$\leq ||\psi'(K_n Q_n z + f) - \psi'(K_n Q_n z_0 + f)||_\infty ||K_n Q_n v||_\infty$$
$$\leq c_2 ||K_n Q_n (z_0 - z)||_\infty ||K_n Q_n v||_\infty$$
$$\leq 2c_2 {M_1}^2 p_1^2 ||z - z_0||_\infty ||v||_\infty \leq 2c_2 {M_1}^2 p_1^2 \delta ||v||_\infty.$$

This implies

$$\sup_{||z-z_0||_\infty \leq \delta} ||\left(I - \tilde{T}_n'(z_0)\right)^{-1} (\tilde{T}_n'(z) - \tilde{T}_n'(z_0))||_\infty \leq 2Lc_2 {M_1}^2 p_1^2 \delta$$
$$\leq q, (say),$$

where we choose δ in such a way that $q \in (0,1)$. This proves the estimate (4.4) of Theorem 2 in [23].

Now using the Lipschtiz continuity of $\psi(., x(.))$ and Lemma 4, we have

$$||\tilde{T}_n(z_0) - T(z_0)||_\infty = ||\psi(K_n Q_n z_0 + f) - \psi(K z_0 + f)||_\infty$$
$$\leq c_1 ||(K_n Q_n - K) z_0||_\infty \to 0, \ as \ n \to \infty.$$

Hence

$$\beta_n = ||\left(I - \tilde{T}_n'(z_0)\right)^{-1} (\tilde{T}_n(z_0) - T(z_0))||_\infty$$
$$\leq Lc_1 ||(K_n Q_n - K) z_0||_\infty \to 0, \qquad as \ n \to \infty.$$

Choose n large enough such that $\beta_n \leq \delta(1-q)$. We choose δ in such a way that $q \in (0,1)$. This proves the estimate (4.5) of Theorem 2 in [23]. Hence applying Theorem 2 of [23], we obtain

$$\frac{\beta_n}{1+q} \leq ||\tilde{z}_n - z_0||_\infty \leq \frac{\beta_n}{1-q},$$

where $\beta_n = ||(I - \tilde{T}_n'(z_0))^{-1}(\tilde{T}_n(z_0) - T(z_0))||_\infty$. This completes the proof. □

Theorem 7. Let $z_0 \in C^d[-1,1]$, $d \geq 2n > n \geq r$, be an isolated solution of the equation (9) and x_0 be a isolated solution of the equation (7) such that $x_0 = K z_0 + f$. $\tilde{x}_n$ be the iterated discrete Legendre Galerkin approximation of x_0. Then the following hold

$$||x_0 - \tilde{x}_n||_{L^2}, ||x_0 - \tilde{x}_n||_\infty = O(n^{-2r}).$$

Proof: From Theorem 6, we have

$$||\tilde{z}_n - z_0||_\infty \leq \frac{\beta_n}{1-q}$$
$$\leq c||(I - \tilde{T}_n'(z_0))^{-1}(\tilde{T}_n(z_0) - T(z_0))||_\infty. \quad (49)$$

Now using Lipschtiz continuity of $\psi(.,.)$, we have

$$||\tilde{T}_n(z_0) - T(z_0)||_\infty = ||\psi(K_n Q_n z_0 + f) - \psi(K z_0 + f)||_\infty$$
$$\leq c_1 ||(K_n Q_n - K) z_0||_\infty$$
$$\leq c_1 [||(K_n Q_n - K_n) z_0||_\infty + ||(K_n - K) z_0||_\infty]. \quad (50)$$

We have

$$||(K_n Q_n - K_n) z_0||_\infty = \sup_{t\in[-1,1]} |(K_n Q_n - K_n) z_0(t)|$$
$$= \sup_{t\in[-1,1]} \left| \sum_{p=1}^{M} w_p k(t, t_p)(Q_n - I) z_0(t_p) \right|$$
$$= \sup_{t\in[-1,1]} |< k_t(.), (Q_n - I) z_0 >_M|. \quad (51)$$

Using the estimate (38), orthogonality property of Q_n and Cauchy-Schwartz inequality, we obtain

$$|< k_t(.), (Q_n - I) z_0 >_M|$$
$$= |< (Q_n - I) k_t(.), (Q_n - I) z_0 >_M|$$
$$= \left| \sum_{p=1}^{M} w_p (Q_n - I) k(t, t_p)(Q_n - I) z_0(t_p) \right|$$
$$\leq \left(\sum_{p=1}^{M} w_p \left[(Q_n - I) k(t, t_p)\right]^2 \right)^{\frac{1}{2}} \left(\sum_{p=1}^{M} w_p \left[(Q_n - I) z_0(t_p)\right]^2 \right)^{\frac{1}{2}}$$
$$= < (Q_n - I) k_t(.), (Q_n - I) k_t(.) >_M^{\frac{1}{2}}$$
$$\times < (Q_n - I) z_0, (Q_n - I) z_0 >_M^{\frac{1}{2}}$$
$$\leq 2cn^{-2r} ||k||_{r,\infty} ||z_0||_{r,\infty}. \quad (52)$$

Since $d \geq 2r$, using Theorem 2 and estimates (49), (50), (51) and (52), we have

$$||\tilde{z}_n - z_0||_\infty = O\left(n^{-\min\{2r,d\}}\right) = O(n^{-2r}). \quad (53)$$

From estimates (7), (19) and (29), we get

$$||\tilde{x}_n - x_0||_\infty = ||K_n \tilde{z}_n - K z_0||_\infty$$
$$\leq ||K_n(\tilde{z}_n - z_0)||_\infty + ||(K_n - K) z_0||_\infty$$
$$\leq p||(\tilde{z}_n - z_0)||_\infty + ||(K_n - K) z_0||_\infty.$$

Hence using Theorem 2 and estimate (53) and the fact that $d \geq 2r$, we obtain

$$||x_0 - \tilde{x}_n||_\infty = O\left(n^{-\min\{2r,d\}}\right) = O(n^{-2r}).$$

Also as $||x_0 - \tilde{x}_n||_{L^2} \leq \sqrt{2} ||x_0 - \tilde{x}_n||_\infty$, we get

$$||x_0 - \tilde{x}_n||_{L^2} = O(n^{-2r}).$$

This completes the proof. □

4. Conclusions

From Theorem 4, we see that the discrete Legendre Galerkin solution of Hammerstein integral equation converges with the optimal order, $O(n^{-r+1})$ in both L^2 and infinity norm. Result of Theorem 7 imply that, if the numerical quadrature is of certain degree of precision, the iterated discrete Legendre Galerkin solution converges with the order $O(n^{-2r})$ in both L^2 and infinity norm. This shows that the iterated discrete Legendre Galerkin approximation exhibits superconvergence.

5. Numerical Example

In this section, we present the numerical results. For that we take the Legendre polynomials as the basis functions for the subspace X_n. We present the errors of the approximate solutions under the discrete Legendre Galerkin and iterated discrete Legendre Galerkin methods in both infinity and L^2-norm.

For computations we use Gauss quadrature rule and the Newton-Raphson method to solve the nonlinear systems. The numerical algorithms are compiled by using Matlab.

In the following tables, n represents the highest degree of the Legendre polynomials employed in the computation.

Example 1. We consider the following integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi\big(s, x(s)\big)ds = f(t), \qquad -1 \le t \le 1,$$

with the kernel function $k(t,s) = \frac{3\pi\sqrt{2}}{16} cos\left(\frac{\pi|s-t|}{4}\right)$, $\psi\big(s, x(s)\big) = x(s)^2$ and the function $f(t) = \frac{-1}{4} cos\left(\frac{\pi t}{4}\right)$. The exact solution is given by $(t) = cos\left(\frac{\pi t}{4}\right)$.

Table 1

n	$\|\|x_0 - x_n\|\|_{L^2}$	$\|\|x_0 - x_n\|\|_\infty$	$\|\|x_0 - \tilde{x}_n\|\|_{L^2}$	$\|\|x_0 - \tilde{x}_n\|\|_\infty$
2	0.1661e-02	0.3408e-02	0.2714e-04	0.2122e-04
4	0.2044e-02	0.2217e-02	0.4427e-04	0.3544e-04
5	0.8676e-05	0.2102e-04	0.3172e-08	0.2479e-08
7	0.2410e-07	0.6456e-07	0.9931e-13	0.7871e-13
8	0.4172e-10	0.1132e-09	0.2168e-14	0.3109e-14

Example 2 We consider the following integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi\big(s, x(s)\big)ds = f(t), \quad -1 \le t \le 1,$$

with the kernel function $k(t,s) = \frac{1}{10} cos(\pi t)\, sin(\pi s)$, $\psi\big(s, x(s)\big) = x(s)^3$ and the function $f(t) = sin(\pi t)$. The exact solution is given by $x(t) = sin(\pi t) + \frac{1}{3}\big(20 - \sqrt{391}\big)cos(\pi t)$.

From Tables 1 and 2, we see that the iterated discrete Legendre Galerkin approximation converges much faster than the discrete Legendre Galerkin solution.

Table 2

n	$\|\|x_0 - x_n\|\|_{L^2}$	$\|\|x_0 - x_n\|\|_\infty$	$\|\|x_0 - \tilde{x}_n\|\|_{L^2}$	$\|\|x_0 - \tilde{x}_n\|\|_\infty$
3	0.1730e-01	0.5256e-01	0.1010e-02	0.1429e-02
4	0.4242e-03	0.2004e-02	0.6337e-05	0.7530e-05
6	0.5221e-05	0.2911e-04	0.9636e-08	0.1363e-07
8	0.3993e-07	0.2513e-06	0.5337e-11	0.7369e-11
10	0.4114e-09	0.1871e-08	0.9266e-14	0.2728e-13

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