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Divisor Graphs and Powers of Trees

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***Abstract*:** Suppose that T is a tree. AbuHijleh has shown that T^2 is a divisor graph iff T is a caterpillar with $diam(T) \le 5$. In this paper we characterize when T^3 and T^4 are divisor graphs.

***Keywords*:** tree, divisor graph, power of a graph, r-starlike tree.

1. Introduction

A graph G is called a divisor graph if there is a bijection $f : V(G) \to S$, for some finite nonempty set S of the positive integers such that $uv \in E(G)$ if and only if $\gcd(f(u), f(v)) = \min\{f(u), f(v)\}$ (This means $uv \in E(G)$ if and only if $f(u) | f(v)$ or $f(v) | f(u)$). The function f is called a divisor labeling of G.

Moreover, for a finite nonempty set S of the positive integers, the divisor graph $G(S)$ of S has S as its vertex set and two distinct vertices i and j are adjacent if $i | j$ or $j | i$. A graph G is a divisor graph if G is isomorphic to $G(S)$, for some S. While the divisor digraph $D(S)$ of S has vertex set S and (i, j) is an arc of $D(S)$ iff i divides j. In a digraph D, a transmitter is a vertex having indegree 0, a receiver is a vertex having outdegree 0, while a vertex v is a transitive vertex if it has both positive outdegree and positive indegree such that $(u,w) \in E(D)$ whenever (u,v) and $(v,w) \in E(D)$. An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of G.

The length $g(n)$ of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, ..., n\}$ was studied before [1-3]. The concept of a divisor graph involving finite nonempty sets of integers rather than positive integers was introduced by Singh, and Santhosh [4]. It was shown by them [4] that odd cycles of length greater than three are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs, as shown in another study [5]. Divisor graphs do not contain induced odd cycles of length greater than three, but they may contain triangles, for instance, complete graphs are divisor graphs [5].

The distance between any two vertices x and y, is the length of a shortest path between them. We denote this distance by $d_G(x, y)$. The diameter of a graph G is equal to $\sup\{d_G(x, y) : x, y \in V(G)\}$, denoted by d_G or $diam(G)$. The neighborhood of a vertex u is the set of all vertices that are adjacent to u, we denote the neighborhood of u by $N(u)$. The k^{th} power of a graph G is denoted by G^k, where the vertex set of G^k is $V(G)$ and two vertices x and y are adjacent if and only if $d_G(x, y) \le k$. A complete characterization of powers of paths, cycles, hypercubes, folded hypercubes, and caterpillars that are divisor graphs were studied before [6-10].

More results on divisor graphs can be found in [5, 11, 12]. For undefined notions and terminology, the reader is referred to Agnarsson, & Greenlaw [13].

2. Preliminaries

The following two theorems were used in characterizing divisor graphs, see [5].

Theorem 1. Let G be a graph. Then G is a divisor graph if and only if G has a divisor orientation.

Theorem 2. Every induced subgraph of a divisor graph is a divisor graph.

The following result was shown in [9], which determines when some powers of a graph is not a divisor graph.

Theorem 3. For any integer $k \ge 2$, if G is a graph of diameter $d \ge 2k + 2$, then G^k is not a divisor graph.

In [6, 7], a full characterization is a divisor graph was given when T^2. We state this characterization in the following theorem.

Theorem 4. Suppose T is a tree. Then T^2 is a divisor graph if and only if T is a caterpillar with $diam(T) \le 5$.

Next, we state the definition of the starlike tree.

Definition 1. A starlike tree T is represented by a subdivision of the edges of a star graph into paths (call each one of these paths a leg). Moreover, the starlike tree with central vertex u, where $\deg(u) = r$ and legs are of lengths $a_1, ..., a_r$, is called an r-starlike tree, see [14].

3. Some Powers of Trees that are not Divisor Graphs

Here we determine some powers of trees that are not divisor graphs. We start with the following definition.

Definition 2. Let $T_{k,l}$ be the 3-starlike tree with central vertex u and the set of vertices of $T_{k,l}$ is $\{u\}\cup\{a_i, b_i : i = 1,\ldots, k-l+1\}\cup\{c_i : i = 1,\ldots, l\}$. Moreover, the legs of $T_{k,l}$ are $\{a_1,\ldots,a_{k-l+1}\}$, $\{b_1,\ldots,b_{k-l+1}\}$, and $\{c_1,\ldots,c_l\}$. Note that, k and l are positive integers with $1 \le l \le \left\lfloor \frac{k-1}{2} \right\rfloor$ and $k \ge 3$, see Figure 1.

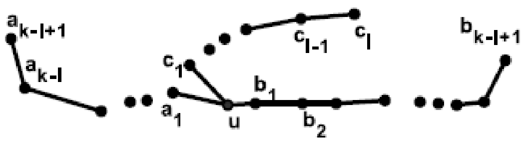

Figure 1: $T_{k,l}$

Theorem 5. Suppose that T is a tree containing an induced subgraph which is isomorphic to $T_{k,l}$ with $1 \le l \le \left\lfloor \frac{k-1}{2} \right\rfloor$ and $k \ge 3$, see Figure 1. Then T^k is not a divisor graph.

Proof: We show that $T_{k,l}^k$ is not a divisor graph. Suppose that D is a divisor orientation of $T_{k,l}^k$, where $(b_{k-l}, c_l) \in E(D)$. Since $b_{k-l+1}c_l \notin E(T_{k,l}^k)$, we have $(b_{k-l}, b_{k-l+1}) \in E(D)$. We get $(b_{k-l}, a_l) \in E(D)$ because $b_{k-l+1}a_l \notin E(T_{k,l}^k)$. But $a_{k-l+1}b_{k-l} \notin E(T_{k,l}^k)$, so we get $(a_{k-l+1}, a_l) \in E(D)$.

Moreover, we get $(a_{k-l}, c_l) \in E(D)$, because $a_{k-l}b_{k-l} \notin E(T_{k,l}^k)$. Since $a_{k-l+1}c_l \notin E(T_{k,l}^k)$, we must have $(a_{k-l}, a_{k-l+1}) \in E(D)$. We get $(a_{k-l}, b_l) \in E(D)$, because $a_{k-l+1}b_l \notin E(T_{k,l}^k)$. But $b_{k-l+1}a_{k-l} \notin E(T_{k,l}^k)$, so we get $(b_{k-l+1}, b_l) \in E(D)$. Now, since $b_{k-l+1}a_l \notin E(T_{k,l}^k)$, we get $(a_l, b_l) \in E(D)$. But $(a_{k-l+1}, a_l) \in E(D)$, so $(a_{k-l+1}, b_l) \in E(D)$, which is a contradiction. Hence, $T_{k,l}^k$ is not a divisor graph. Since $T_{k,l}$ is an induced subgraph of T then, by Theorem 2, T^k is not a divisor graph.

Also, if $diam(T) > 2k - 2(l-1)$, then T^k is not a divisor graph. □

Definition 3. Let T_s be the 3-starlike tree with central vertex u and $diam(T_s) \ge 6$, say d, where the set of vertices of T_s is $\{u\}\cup\{a_i, b_i, c_i : i = 1,\ldots,\frac{d}{2}\}$. Moreover, the legs of T_s are $\{a_1,\ldots,a_{\frac{d}{2}}\}$, $\{b_1,\ldots,b_{\frac{d}{2}}\}$, and $\{c_1,\ldots,c_{\frac{d}{2}}\}$, see Figure 2.

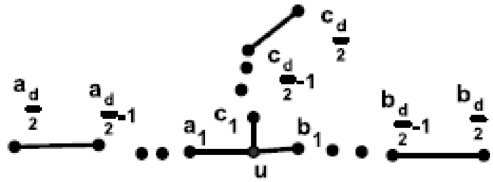

Figure 2: T_s

Theorem 6. For the 3-starlike tree T_s, T_s^{d-2} is not a divisor graph.

Proof: According to Figure 2 of T_s, it is easy to check that the induced subgraph on the set $\{a_{\frac{d}{2}}, a_{\frac{d}{2}-1}, b_{\frac{d}{2}}, b_{\frac{d}{2}-1}, c_{\frac{d}{2}}, c_{\frac{d}{2}-1}\}$ in T_s^{d-2} is isomorphic to G_1 (see Figure 3). But, G_1 is not a divisor graph, see [5]. So, T_s^{d-2} is not a divisor graph. □

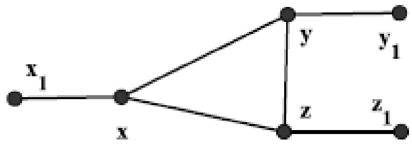

Figure 3: G_1

4. Characterizing when T^k is a Divisor Graph for $k = 3$ and 4

Let T be a tree, we decide when T^k, for $k = 3$ and 4, is a divisor graph. For $k = 3$, we have two general subcases to consider.

Theorem 7. Suppose T is a tree with $diam(T) \le 5$. Then T^3 is a divisor graph.

Proof: Firstly, assume that $diam(T) = 5$. To show that T^3 is a divisor graph, we will assume that the vertices of T are named as shown in Figure 4.

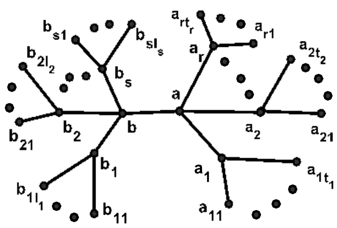

Figure 4: A tree T with $diam(T) = 5$

We give a divisor labeling f of T^3, as follows:

$f(a_1)=p_1,$ $\quad f(a_2)=p_1p_2,$

$\ldots,$ $\quad f(a_r)=\prod_{i=1}^{i=r}p_i,$

$f(a_{11})=p_1\prod_{i=1}^{i=r}p_i,$ $\quad f(a_{12})=p_1^2\prod_{i=1}^{i=r}p_i,$

$\ldots,$ $\quad f(a_{1t_1})=p_1^{t_1}\prod_{i=1}^{i=r}p_i,$

$f(a_{21})=p_2\prod_{i=1}^{i=r}p_i,$ $\quad \ldots,$

$f(a_{2t_2})=p_2^{t_2}\prod_{i=1}^{i=r}p_i,$ $\quad \ldots,$

$f(a_{r1})=p_r\prod_{i=1}^{i=r}p_i,$ $\quad \ldots,$

$f(a_{rt_r})=p_r^{t_r}\prod_{i=1}^{i=r}p_i,$ $\quad f(b_{11})=q_1,$

$\ldots,$ $\quad f(b_{1l_1})=q_1^{l_1},$

$f(b_{21})=q_2,$ $\quad \ldots,$

$f(b_{2l_2})=q_2^{l_2},$ $\quad \ldots,$

$f(b_{s1})=q_s,$ $\quad \ldots,$

$f(b_{sl_s})=q_s^{l_s},$ $\quad f(b_1)=\prod_{i=1}^{i=r}p_i\prod_{i=1}^{i=s}q_i^{l_i},$

$f(b_2)=q_1\prod_{i=1}^{i=r}p_i\prod_{i=1}^{i=s}q_i^{l_i},$ $\quad \ldots,$

$f(b_s)=q_1^{s-1}\prod_{i=1}^{i=r}p_i\prod_{i=1}^{i=s}q_i^{l_i},$

$f(b)=q_1^{s-1}\prod_{i=1}^{i=r}p_i^{1+t_i}\prod_{i=1}^{i=s}q_i^{l_i},$

$f(a)=q_1^{s}\prod_{i=1}^{i=r}p_i^{1+t_i}\prod_{i=1}^{i=s}q_i^{l_i},$

where, $\{p_i\}_{i=1}^{i=r}$ and $\{q_i\}_{i=1}^{i=s}$ are distinct primes. Hence, T^3 is a divisor graph. By the work above and Theorem 2, we get T^3 is a divisor graph when $diam(T)<5$. □

Theorem 8. Suppose T is a tree with $diam(T)=6$ or 7. Then T^3 is a divisor graph if and only if the center(s) of T has (have) degree two.

Proof: At first, assume that the center (a center) of T has degree more than two. Then T has an induced subgraph which is isomorphic to $T_{k,l}$ with $k=3$ and $l=1$. Hence, by Theorem 2 and Theorem 5, T^3 is not a divisor graph.

Conversely, assume that the center(s) of T has (have) degree two. To show that T^3 is a divisor graph, we give a divisor labeling of T^3. This divisor labeling is similar to that in Theorem 7.

If $diam(T)=6$, then we name the vertices of T as in Figure 4 and add a vertex u between a and b. The divisor labeling f of T^3 is:

$f(a)=p_1,$

$f(a_1)=p_1^2,$ $\quad f(a_2)=p_1^2p_2,$

$\ldots.,$ $\quad f(a_r)=p_1\prod_{i=1}^{i=r}p_i,$

$f(a_{11})=p_1^2\prod_{i=1}^{i=r}p_i,$ $\quad f(a_{12})=p_1^3\prod_{i=1}^{i=r}p_i,$

$\ldots..,$ $\quad f(a_{1t_1})=p_1^{1+t_1}\prod_{i=1}^{i=r}p_i,$

$f(a_{21})=p_1p_2\prod_{i=1}^{i=r}p_i,$ $\quad \ldots,$

$f(a_{2t_2})=p_1p_2^{t_2}\prod_{i=1}^{i=r}p_i,$ $\quad \ldots,$

$f(a_{r1})=p_1p_r\prod_{i=1}^{i=r}p_i,$ $\quad \ldots.,$

$f(a_{rt_r})=p_1p_r^{t_r}\prod_{i=1}^{i=r}p_i,$ $\quad f(b_{11})=q_1,$

$\ldots.,$ $\quad f(b_{1l_1})=q_1^{l_1},$

$f(b_{21})=q_2,$ $\quad \ldots.,$

$f(b_{2l_2})=q_2^{l_2},$ $\quad \ldots.,$

$f(b_{s1})=q_s,$ $\quad \ldots.,$

$f(b_{sl_s})=q_s^{l_s},$ $\quad f(b_1)=p_1\prod_{i=1}^{i=s}q_i^{l_i},$

$$f(b_2) = p_1 q_1 \prod_{i=1}^{i=s} q_i^{l_i},$$

….,

$$f(b_s) = p_1 q_1^{s-1} \prod_{i=1}^{i=s} q_i^{l_i},$$

$$f(b) = p_1 q_1^{s-1} \prod_{i=1}^{i=s} q_i^{l_i} \prod_{i=1}^{i=r} p_i,$$

$$f(u) = p_1 q_1^{s-1} \prod_{i=1}^{i=s} q_i^{l_i} \prod_{i=1}^{i=r} p_i^{1+t_i},$$

where, $\{p_i\}_{i=1}^{i=r}$ and $\{q_i\}_{i=1}^{i=s}$ are distinct primes. Hence, T^3 is a divisor graph.

For $diam(T) = 7,$ we name the vertices of T as shown in Figure 4 and add the vertices u_1 and u_2 between a and b. A divisor labeling f of T^3 is similar to the previous case and we omit it. □

In the case of $diam(T) \geq 8,$ one can use Theorem 3 to get T^3 that is not a divisor graph.

For $k = 4,$ we want to see when T^4 is a divisor graph. First suppose that T is a tree with $diam(T) = 7$ and T does not contain an induced subgraph that is isomorphic to T_s with $diam(T_s) = 6$. In this case the general form of T is given in Figure 5 and we name this general form by T_7.

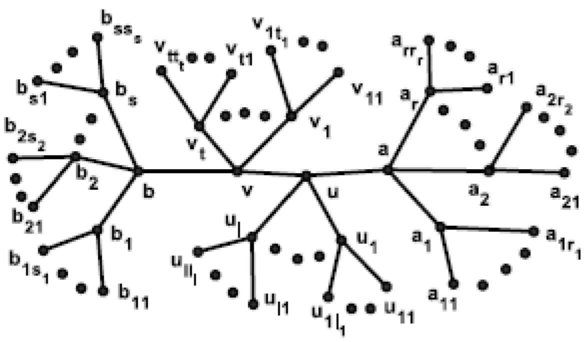

Figure 5: T_7

Lemma 1. Let T_7 be the graph given in Figure 5, then T_7^4 is a divisor graph.

Proof: We give a divisor labeling f of T_7^4 as follows:

$$f(a) = p_1,$$

$$f(u_2) = p_1 p_2^2,$$

$$f(u_l) = p_1 p_2^l,$$

….,

$$f(a_{11}) = p_1^{r+2} p_2^l,$$

$$f(a_{1r_1}) = p_1^{1+r+r_1} p_2^l,$$

….,

…..,

….,

$$f(v_{11}) = p_1 q_2 p_2^l,$$

$$f(v_{1t_1}) = p_1 q_2^{t_1} p_2^l,$$

$$f(v_{tt_t}) = p_1 q_2^{\sum_{i=1}^{i=t} t_i} p_2^l,$$

….,

$$f(b_{21}) = q_1^{1+s_1},$$

$$f(b_{2s_2}) = q_1^{s_1+s_2},$$

$$f(b_{ss_s}) = q_1^{\sum_{i=1}^{i=s} s_i},$$

$$f(b_1) = p_1 p_2^l q_2^{\sum_{i=1}^{i=t} t_i} q_1^{\sum_{i=1}^{i=s} s_i},$$

$$f(b_2) = p_1 p_2^l q_2^{\sum_{i=1}^{i=t} t_i} q_1^{1+\sum_{i=1}^{i=s} s_i},$$

$$f(b_s) = p_1 p_2^l q_2^{\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

….,

….,

$$f(u_1) = p_1 p_2,$$

…,

$$f(a_1) = p_1^2 p_2^l,$$

$$f(a_r) = p_1^{r+1} p_2^l,$$

…,

$$f(a_{21}) = p_1^{2+r+r_1} p_2^l,$$

$$f(a_{2r_2}) = p_1^{1+r+r_1+r_2} p_2^l,$$

$$f(a_{r1}) = p_1^{2+r+\sum_{i=1}^{i=r-1} r_i} p_2^l,$$

$$f(a_{rr_r}) = p_1^{1+r+\sum_{i=1}^{i=r} r_i} p_2^l,$$

….,

….,

$$f(b_{11}) = q_1,$$

$$f(b_{1s_1}) = q_1^{s_1},$$

…,

….,

….,

$$f(u_{11}) = p_1^{r+1} p_2^{l+1},$$

$$f(u_{1l_1}) = p_1^{r+1} p_2^{l+l_1},$$

$$f(u_{ll_l}) = p_1^{r+1} p_2^{l+\sum_{i=1}^{i=l} l_i},$$

$$f(v_1) = p_1^{r+1} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(v_2) = p_1^{r+1} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{1+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i}, \qquad \ldots,$$

$$f(v_t) = p_1^{r+1} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{t-1+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(b) = p_1^{r+1} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(v) = p_1^{r+1+\sum_{i=1}^{i=r} r_i} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(u) = p_1^{r+1+\sum_{i=1}^{i=r} r_i} p_2^{l+\sum_{i=1}^{i=l} l_i} q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s+\sum_{i=1}^{i=s} s_i},$$

where, $\{p_i\}_{i=1}^{i=2}$ and $\{q_i\}_{i=1}^{i=2}$ are distinct primes. Hence, T_7^4 is a divisor graph. □

We get a similar result when $diam(T)=6$. The proof of this result is similar to that of the previous lemma. We state that, in the following lemma.

Lemma 2. Let T be a tree with $diam(T)=6$ and T does not contain an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$. Then T^4 is a divisor graph.

In the case, $diam(T) \le 5$, then T^4 is an induced subgraph of T_7^4 and hence T^4 is a divisor graph. We state this in the following lemma.

Lemma 3. Suppose T is a tree with $diam(T) \le 5$. Then T^4 is a divisor graph.

In Theorem 6, we have seen that T_s^{d-2} with $diam(T_s) = d$ is not a divisor graph. Hence, we get the following result.

Lemma 4. Suppose T is a tree with $diam(T)=6$ or 7 and T contains an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$. Then T^4 is not a divisor graph.

We summarize all of these results in the following theorem.

Theorem 9. Suppose T is a tree with $diam(T) \le 7$. Then T^4 is a divisor graph if and only if T does not contain an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$.

Next, we will discuss the case where $diam(T)=8$ or 9.

Theorem 10. Suppose T is a tree with $diam(T)=8$ or 9. Then T^4 is a divisor graph if and only if the center(s) of T has (have) degree two and T does not contain an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$.

Proof: At first, assume that the center (a center) of T has degree more than two. Then T has an induced subgraph which is isomorphic to $T_{k,l}$ with $k=4$ and $l=1$, then by Theorem 5 and Theorem 2, T^4 is not a divisor graph. Also, when T has an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$. Then, by Theorem 6, T^4 is not a divisor graph.

Conversely, assume that the center(s) of T has (have) degree two and T does not have an induced subgraph that is isomorphic to T_s with $diam(T_s)=6$. To show that T^4 is a divisor graph, we give a divisor labeling of T^4. This divisor labeling is similar to the one in Lemma 1. We give this divisor labeling for $diam(T)=9$. In this case we name the vertices of T as shown in Figure 5 and add the vertices c_1 and c_2 between v and u ($c_1 \in N(v)$ and $c_2 \in N(u)$). The divisor labeling f of T^4 is:

$$f(c_2) = p_1, \qquad f(u) = p_1^2,$$

$$f(a) = p_1^3, \qquad f(u_1) = p_1^3 p_2,$$

$$f(u_2) = p_1^4 p_2, \qquad \ldots,$$

$$f(u_l) = p_1^{l+2} p_2, \qquad f(a_1) = p_1^{l+3} p_2,$$

$$\ldots, \qquad f(a_r) = p_1^{l+r+2} p_2,$$

$$f(u_{11}) = p_1^{l+r+3} p_2, \qquad \ldots,$$

$$f(u_{1l_1}) = p_1^{l+r+l_1+2} p_2, \qquad \ldots,$$

$$f(u_{ll_l}) = p_1^{l+r+2+\sum_{i=1}^{i=l} l_i} p_2,$$

$$f(a_{11}) = p_1^{l+r+2} p_2^2, \qquad \ldots,$$

$$f(a_{1r_1}) = p_1^{l+r+2} p_2^{1+r_1}, \qquad \ldots,$$

$$f(a_{rr_r}) = p_1^{l+r+2} p_2^{1+\sum_{i=1}^{i=r} r_i},$$

$$f(v_{11}) = p_1 q_2, \qquad \ldots,$$

$$f(v_{1t_1}) = p_1 q_2^{t_1}, \qquad \ldots,$$

$$f(v_{tt_t}) = p_1 q_2^{\sum_{i=1}^{i=t} t_i}, \qquad f(b_{11}) = q_1,$$

$$\ldots., \qquad f(b_{1s_1}) = q_1^{s_1},$$

$$\ldots., \qquad f(b_{ss_s}) = q_1^{\sum_{i=1}^{i=s} s_i},$$

$$f(b_1) = p_1 q_2^{\sum_{i=1}^{i=t} t_i} q_1^{\sum_{i=1}^{i=s} s_i},$$

$$f(b_2) = p_1 q_2^{\sum_{i=1}^{i=t} t_i} q_1^{1+\sum_{i=1}^{i=s} s_i}, \qquad \ldots,$$

$$f(b_s) = p_1 q_2^{\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(v_1) = p_1^2 q_2^{\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(v_2) = p_1^2 q_2^{1+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i}, \qquad \ldots,$$

$$f(v_t) = p_1^2 q_2^{t-1+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i}, \qquad \ldots,$$

$$f(b) = p_1^2 q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(v) = p_1^{l+2} p_2 q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

$$f(c_1) = p_2 p_1^{l+r+2+\sum_{i=1}^{i=l} l_i} q_2^{t+\sum_{i=1}^{i=t} t_i} q_1^{s-1+\sum_{i=1}^{i=s} s_i},$$

where $\{p_i\}_{i=1}^{i=2}$ and $\{q_i\}_{i=1}^{i=2}$ are distinct primes. Hence, T^4 is a divisor graph.

In the case that $diam(T) = 8$, we name the vertices of T as shown in Figure 5 and add the vertex c_1 between u and v. A divisor labeling f of T^4 is similar to the previous case and we omit it. □

In the case of $diam(T) \geq 10$, one can use Theorem 3 to get T^4 that is not a divisor graph.

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