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Appling Homotopy Analysis Method to Solve Optimal Control Problems Governed by Volterra Integral Equations

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Abstract: This paper contributes a scheme for solving linear optimal control problem in systems run by Volterra integral equations. At first, the original optimal control problem of Volterra integral equations is transformed into an integral problem via the Pontryagins maximum principle, and then, we applied homotopy analysis method (HAM) to solve the integral problem. The proposed method is compared with other methods. Numerical results show that the HAM method is more powerful and faster.

Keywords: optimal control problems of volterra integral equations, homotopy analysis method, pontryagins maximum principle.

1. Introduction

The purpose of development of the theory of optimal control was originally dealing with controlled ordinary differential equations systems [1]. Ordinary differential equations cannot adequately describe many physical, biological, technological, and socio-economic problems. Volterra integral equations can describe a broad category of systems. A class of systems of controlled Volterra integral equations (VIEs) can model systems of controlled integro differential equations (IDEs) or controlled ordinary differential equations [2, 3].

To solve OCPs governed by VIEs different techniques have been suggested during the past two decades. Some studies, for example, can be mentioned such as the Schmidt's methods [4, 5], Belbas and Schmidt [6] and Belbas [7, 8]. In this paper, we are going to solve Volterra optimal control problems. To do so, we applied the HAM method to solve the problem. This method has been successfully applied to solve many types of problems [9, 10]. HAM is a powerful and easy tool to use and does not require small parameters in equations. In addition, it includes the *h* auxiliary parameter which provides a simple way for us for adjusting and controlling the convergence region solution series. This paper includes the following parts: The Volterra optimal control problem and some elementary related results are stated in section 2. Section 3 is devoted to description of the HAM method. In section 4, we discuss about the convergence analysis of the method. In Section 5, we demonstrate the accuracy of the method by considering two test examples. Section 6 consists of conclusion.

2. Statement of the problem

Consider a controlled Volterra integral equation of the form

$$x(t) = x(a) + \int_a^t f(t, s, x(s), u(s))ds. \qquad (1)$$

In such system, $x(t)$ equals the n-dimensional continuous state function, and $u(t)$ equals the m-dimensional piecewise continuous control function in a region $U, U \subseteq R^m$. It is assumed that $x(a)$ is fixed. In this paper, we consider the following optimal control problem:

$$\min \quad j = \int_a^b F(t, x(t), u(t))dt, \qquad (2)$$

under the Volterra integral equation given by (1). We say that a pair of functions (x, u) on $[a, b]$ is an admissible pair if x is a continuous function on $[a, b]$ and u is a piecewise continuous function on $[a, b]$ with values in a region U, $U \subseteq R^m$, and the relations (1) are satisfied. We look for an admissible pair (x^*, u^*) which solves the following problem:

$$\min \quad j = \int_a^b F(t, x(t), u(t))dt,$$

$$s.t \quad x(t) = x(a) + \int_a^t f(t, s, x(s), u(s))ds. \quad (3)$$

It is well known that the extremum principles give necessary conditions for an optimal pair (x^*, u^*) of the problem (3), so in the paper we adopt that the following conditions are satisfied:

i) There is the continuity of the function f for all s, t with $s \le t$, together with some Lipschitz conditions to guarantee the existence solution of equation (3), [11].

ii) The partial derivatives f_x and f_u exist and are continuous, and for all t, s with $t \le s$, $f(t, s, x, u) = 0$.

iii) $F(t, x, u)$ is a smooth function.

Following [12], using the above assumptions, the Pontryagin maximum principle can be stated as follows: Suppose that the optimal control $u^*(t), a \le t \le b$, and corresponding trajectory $x^*(t)$, solve the problem (3). Then, there exists a continuous real valued multiplier vector $\lambda^*(t)$, and a Hamiltonian $H(t, x, u, \lambda(.))$ defined by

$$H(t,x,u,\lambda(.)) = F(t,x,u) + \int_t^b f(s,t,x,u)\lambda(s)ds,$$

such that,

1. The vector $u^*(t)$ maximizes the real valued function $H(t,x^*(t),u,\lambda(.))$ for almost all and hence $\frac{\partial H}{\partial u}|_{u=u^*(t)} = 0$ in which $u^*(t)$ is an inner point of the control set U .

2. $\lambda^*(t) = \frac{\partial H}{\partial x}|_{u=u^*(t), x=x^*(t)}$, for each *t*.

3. Basic idea of HAM

Given the following equation,

$N[u(t)] = 0,$ (4)

where *N* is a non-linear operator, *u*(*t*) is an unknown function and *t* represents independent variable. By generalizing the traditional homotopy method, the so-called zero-order deformation equation is constructed by Liao[13].

$$(1-q)L[\phi(t;q) - u_0(t)] = qhH(t)N[\phi(t;q)], \quad (5)$$

where $q \in [0,1]$ stands for an embedding parameter, *h* is for a non-zero auxiliary parameter, *H*(*t*) is a function of non-zero auxiliary, *L* is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$ and $\phi(t;q)$ function is unknown. It is worth mentioning that, choosing auxiliary objects such as *h* and *L* in HAM, is optional. At $q = 0$ and $q = 1$, we have $\phi(t;0) = u_0(t)$ and $\phi(t;1) = u(t),$ respectively. So, as the embedding parameter $q \in [0,1]$ goes up from 0 to 1, the solution $\phi(t;q)$ changes from the initial guess $u_0(t)$ to the solution $u(t).$

Expanding $\phi(t;q)$ in Taylor series with respect to q, we have

$$\phi(t;q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)q^m, \quad (6)$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0} . \quad (7)$$

The series equation (6) converges at $q = 1$, if the auxiliary linear operator, the initial guess, the auxiliary parameter *h*, and the auxiliary function are suitably adopted, then and we have

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t), \quad (8)$$

which based on Liao[9], is one of solutions of the original non-linear equation. As $h = -1$ and $H(t) = 1,$ Eq. (5) becomes

$$(1-q)L[\phi(t;q) - u_0(t)] + qN[\phi(t;q)] = 0, \quad (9)$$

which is frequently applied in the method of homotopy perturbation [14]. Based on (7), the zero-order deformation equations may be resulted in the governing equations (5). Define the vector

$$\vec{u}_n = \{u_0(t), u_1(t), \dots, u_n(t)\}. \quad (10)$$

Differentiating Eq. (5) *m* times with respect to the embedding parameter *q* and then setting *q* = 0 and finally dividing them by *m*! we have the so-called mth-order deformation equations

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)R_m(\vec{u}_{m-1}), \quad (11)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}|_{q=0}, \quad (12)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (13)$$

It must be highlight that $u_m(t)\, for\, (m \ge 1)$ is ruled by linear equation (5) under the linear boundary condition from the original problem that could be easily worked out by symbolic computation software like Mathematica and Maple.

4. Convergence Analyses

In this part, we show that, the HAM solution series (8) is convergent and it must be the solution of the suggested problem.

Theorem 1: When the series

$$u_0(t) + \sum_{m=1}^{\infty} u_m(t), \quad (14)$$

is convergent, if the high-order deformation equation (11) under the definitions (12) and (13) governs $u_m(t)$, it must be a solution of Eq.(4) [13].

Proof: Since, by hypothesis, the series is convergent, it holds

$$s(t) = \sum_{m=0}^{\infty} u_m(t). \quad (15)$$

So, the necessary condition for the convergence of the series is valid, that is,

$$\lim_{m\to\infty} u_m(t) = 0. \quad (16)$$

Using (11) and (16), we have

$$hH(t)\sum_{m=1}^{\infty} R_m(u_{m-1}(t)) = \sum_{m=1}^{\infty} L[u_m(t) - \chi_m u_{m-1}(t)] =$$

$$\lim_{n\to\infty}\sum_{m=1}^{n} L[u_m(t) - \chi_m u_{m-1}(t)] = \quad (17)$$

$$L[\lim_{n\to\infty}\sum_{m=1}^{n}(u_m(t) - \chi_m u_{m-1}(t))] = L[\lim_{n\to\infty}(u_n(t))] = 0.$$

Since $h \neq 0$ and $H(t) \neq 0,$ we must have

$$\sum_{m=1}^{\infty} R_m(\overrightarrow{u}_{m-1}(t)) = 0. \qquad (18)$$

$$\sum_{m=1}^{\infty} R_m(\overrightarrow{u}_{m-1}) = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;q)]}{\partial q^{m-1}} |_{q=0} = 0. \qquad (19)$$

Generally, the original equation (4) is not satisfy $\phi(t;q)$. Let $\varepsilon(t;q) = N[\phi(t;q)]$,
denote the residual error of Eq.(4). Simply,
$\varepsilon(t;q) = 0$,
matches with the exact solution of the original equation Eq. (4). As noted in the above, the Maclaurin series of the residual error $\varepsilon(t;q)$ on the embedding parameter q is

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \varepsilon(t;q)}{\partial q^m} |_{q=0} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m N[\phi(t;q)]}{\partial q^m} |_{q=0}.$$

When $q = 1$, the above expression gives, using (19)

$$\varepsilon(t;q) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \varepsilon(t;q)}{\partial q^m} |_{q=0} = 0.$$

As defined in $\varepsilon(t;q)$, we get the exact solution of the original equation (4) if $q = 1$. Then, the series

$$u_0(t) + \sum_{m=1}^{\infty} u_m(t),$$

must be the solution of the original equation (4). This ends the proof. □

5. Applications

For assessing the accuracy and the advantages of HAM to solve optimal control problems of Volterra integral equations, the following examples will be considered. In all of the test problems, the initial control has been set to zero. All the computations have been done with Mathematica software. We use the following notations in the Tables:
SD: Steepest Descent method, SDN1: Hybridization of Steepest Descent and two-step Newton method stated at Eqs. (12) based on Peyghami *et al.* [15], SDN2: Hybridization of Steepest Descent and two-step Newton method are stated at Eqs. (13) in that study [15]. Let us set:

$$\frac{\partial H^{(i)}}{\partial u} = \frac{\partial H}{\partial u}(t, x^{(i)}(t), u^{(i)}(t), \lambda^{(i)}(t)), \qquad (20)$$

where $u^{(i)}(t) = \sum_{j=0}^{i} u_j(t), x^{(i)}(t) = \sum_{j=0}^{i} x_j(t)$ and

$\lambda^{(i)}(t) = \sum_{j=0}^{i} \lambda_j(t),$ are the control function, state function,

and multiplier function in the i^{th}-iteration of HAM method respectively. For a given $\varepsilon > 0, (x^{(i)}(t), u^{(i)}(t))$ is an ε-solution of problem (3), if $\forall i$

$$\| \frac{\partial H^{(i)}}{\partial u} \|_2^2 < \varepsilon, \qquad (21)$$

where $\| . \|_2$ is L_2-norm defined by

On the other side, as stated in (12), we have

$$\| \frac{\partial H^{(i)}}{\partial u} \|_2^2 \overset{\Delta}{=} \int_a^b [\frac{\partial H^{(i)}}{\partial u}]^2 dt. \qquad (22)$$

Example 1: Consider the minimization of the functional

$$J = \int_0^1 (tx(t) - u(t) + e^{2u(t)}) dt,$$

subject to the integral equation

$$x(t) = \int_0^t (tx(s) + tu(s)) ds. \qquad (23)$$

The Hamiltonian function can be stated as:

$$H(t) = tx(t) - u(t) + e^{2u(t)} + \int_t^1 \lambda(s)(sx(t) + su(t)) ds.$$

Let us suppose that u^* minimize H. Using the maximum principle, we express the necessary conditions for optimality as:

$$\lambda^*(t) = \frac{\partial H}{\partial x} = t + \int_t^1 (s\lambda^*(s)) ds, \qquad (24)$$

$$2e^{2u^*(t)} - 1 + \int_t^1 (s\lambda^*(s)) ds = 0, \qquad (25)$$

$$x^*(t) = \int_0^t (tx^*(s) + tu^*(s)) ds. \qquad (26)$$

The analytical solution of the integral equation (24) may be obtained as:

$$\lambda^*(t) = t + e^{\frac{t^2}{2}} [e^{\frac{-1}{2}} - \frac{1}{2}\sqrt{2\pi} erf(\frac{\sqrt{2}}{2}) - te^{-\frac{t^2}{2}} + \frac{1}{2}\sqrt{2\pi} erf(\frac{\sqrt{2}}{2}t)],$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

So, from (25), we have

$$u^*(t) = \frac{1}{2} \ln[\frac{1}{2} - \frac{1}{2} \int_t^1 s\lambda^*(s) ds]. \qquad (27)$$

Now we should solve the following problem:

$$x(t) = \int_0^t \left(tx(s) + t(\frac{1}{2} \ln[\frac{1}{2} - \frac{1}{2} \int_s^1 t\lambda(t) dt]) \right) ds,$$

$$\lambda(t) = t + \int_t^1 s\lambda(s) ds, \qquad (28)$$

where the optimal control law is given by

$$u^*(t) = \frac{1}{2} \ln[\frac{1}{2} - \frac{1}{2} \int_t^1 s\lambda(s) ds].$$

To solve the system (28) by means of homotopy analysis method, we choose the linear operators

$$L_i[\phi_i(t;q)] = \phi_i(t;q), \qquad i = 1,2. \qquad (29)$$

We now define non-linear operators as:

$$N_1[\phi_1,\phi_2]=\phi_1(t;q)-$$
$$\int_0^t\left(t\phi_1(s;q)+t\left(\frac{1}{2}\ln\left[\frac{1}{2}-\frac{1}{2}\int_s^1 t\phi_2(t;q)dt\right]\right)\right)ds, \qquad (30)$$
$$N_2[\phi_1,\phi_2]=\phi_2(t;q)-t-\int_t^1 s\phi_2(s;q)ds. \qquad (31)$$

Using above definition, we construct the zeroth-order deformation equations

$$(1-q)L_1[\phi_1(t;q)-x_0(t)]=qh_1H_1(t)N_1[\phi_1,\phi_2],$$
$$(1-q)L_2[\phi_2(t;q)-\lambda_0(t)]=qh_2H_2(t)N_2[\phi_1,\phi_2].(32)$$

Differentiating Eq.(32) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the following mth-order $(m \geq 1)$ deformation equation

$$L_1[x_m(t)-\chi_m x_{m-1}(t)]=h_1H_1(t)R_{1,m}(x_{m-1}), (33)$$
$$L_2[\lambda_m(t)-\chi_m \lambda_{m-1}(t)]=h_2H_2(t)R_{2,m}(\lambda_{m-1}), (34)$$

where

$$R_{1,m}(x_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N_1[\phi_1,\phi_2]}{\partial q^{m-1}}|_{q=0},$$

$$R_{2,m}(\lambda_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N_2[\phi_1,\phi_2]}{\partial q^{m-1}}|_{q=0}.$$

Now the solution of the m-order $(m \geq 1)$ deformation equations (33) and (34) becomes

$$x_m(t)=\chi_m x_{m-1}(t)+h_1H_1(t)R_{1,m}(x_{m-1}),$$
$$\lambda_m(t)=\chi_m \lambda_{m-1}(t)+h_2H_2(t)R_{2,m}(\lambda_{m-1}).$$

By starting with an initial approximations

$$x_0(t)=t, \quad and \quad \lambda_0(t)=-5t^3,$$

and by choosing $H_i = 1, (i = 1,2)$ we suppose

$$x(t)\approx\sum_{m=0}^{3}x_m(t), \qquad \lambda(t)\approx\sum_{m=0}^{3}\lambda_m(t),$$

where the optimal control law is given by

$$u(t)=\frac{1}{2}\ln\left[\frac{1}{2}-\frac{1}{2}\int_t^1 s\left(\sum_{m=0}^{3}\lambda_m(s)\right)ds\right].$$

For finding a suitable value of h, the h-curves of $x(t)$ and $u(t)$ obtained form 3th-order HAM approximation are drawn in Figures. 1 and 2 respectively.

Figures 3 and 4 show the state $x(t)$ and the control $u(t)$ for 3th-order HAM approximation. Also we compared the results of HAM for u(t) with the analytical solution obtained by (27).

Considering $\varepsilon = 10^{-30}$, the value of $\|\frac{\partial H^{(i)}}{\partial u}\|_2^2$ in each iteration have been shown in Table1. Table 2 provides the errors between the exact solution u^* and approximate solution u, which is obtained from 3th-order HAM for h = -0.3035728986424, with the following formula:

$$\| u-u^* \| = \int_a^b | u(t)-u^*(t) | dt. \qquad (35)$$

The obtained values for J^* from 3th-order HAM are given in Table 3.

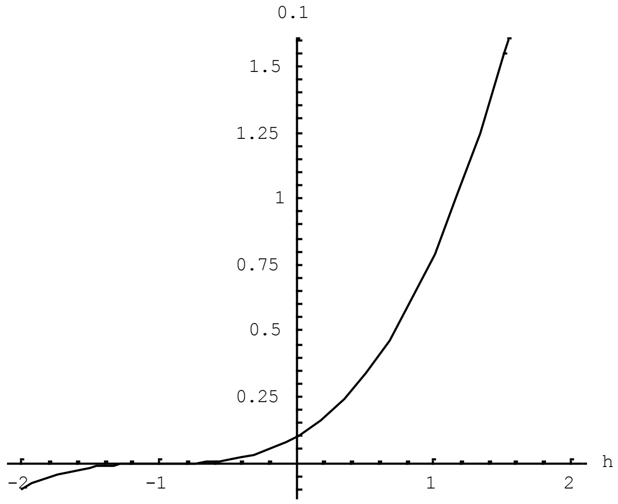

Figure 1. h - curve of x(t)

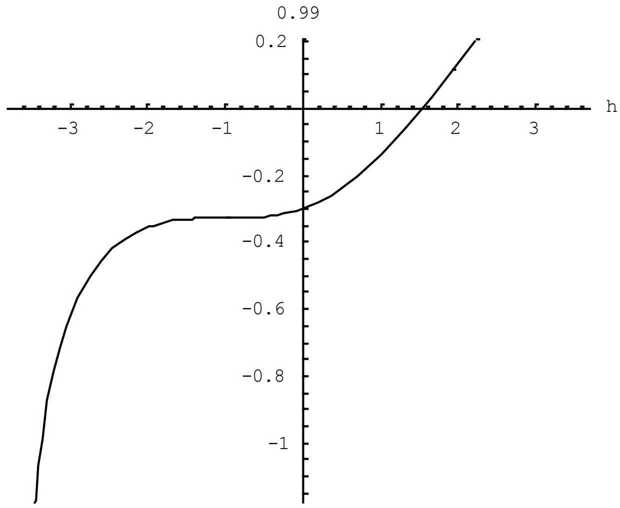

Figure 2. h - curve of u(t)

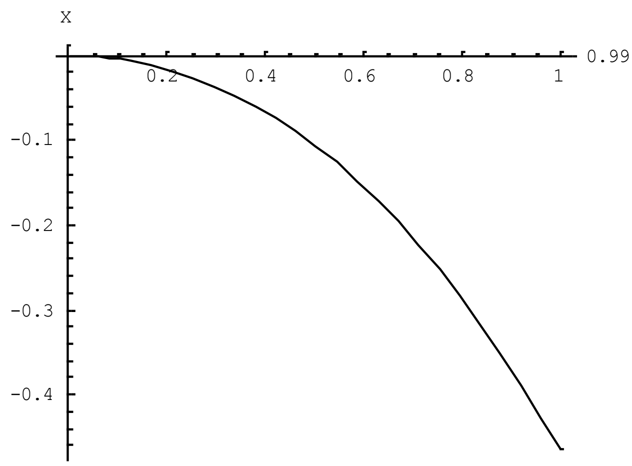

Figure 3. Approximation solution for x(t) with h=-1

Table 1: $\| \frac{\partial H^{(i)}}{\partial u} \|_2^2$ *for* h =-0.3035728986424compared with the methods in [15]

Itr	SDN	SDN1	SDN2	HAM
1	0.185994612×10^{1}	0.185994612×10^{1}	0.185994612×10^{1}	0.180937×10^{-31}
2	0.292584772×10^{0}	0.292584772×10^{0}	0.292584772×10^{0}	0.846466×10^{-32}
3	0.292584772×10^{0}	$0.978499400\times 10^{-1}$	$0.978499400\times 10^{-1}$	0.164841×10^{-32}
4	$0.838472588\times 10^{-2}$	$0.838472588\times 10^{-2}$	$0.838472588\times 10^{-2}$	
5	$0.838472588\times 10^{-2}$	$0.461994962\times 10^{-5}$	$0.796714880\times 10^{-3}$	
6	$0.461993516\times 10^{-5}$	$0.2118656096\times 10^{-14}$	$0.601632398\times 10^{-10}$	
7	$0.270214306\times 10^{-11}$			

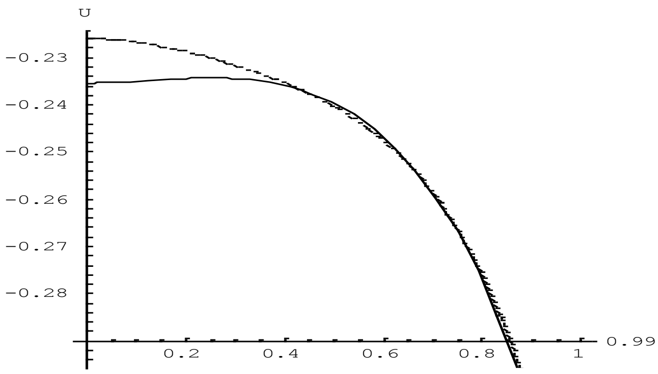

Figure 4. Approximation solution for u (t) with h =-0.3035728986424, solid line: analytical solution

Table 2: The errors of the actual and numerical control in Example1.

Method	Errors
SDN	0.47973×10^{-8}
SDN1	0.10391×10^{-10}
SDN2	0.23091×10^{-5}
HAM	0.152163×10^{-14}

Table 3: Numerical results for cost functional J^* in example1.

Itr	J*
1	1.00427
2	0.825434
3	0.74411

Example 2

$$\min J = \int_0^1 (tx(t) - u^4(t))dt,$$

$$s.t \quad x(t) = \int_0^t (tx(s) + tu(s))ds.$$

The Hamiltonian function can be stated as:

$$H = tx(t) - u^4(t) + \int_t^1 \lambda(s)(sx(t) + su(t))ds.$$

Let us suppose that u^* minimize H. Using the maximum principle, we express the necessary conditions for optimality as:

$$\lambda^*(t) = \frac{\partial H}{\partial x} = t + \int_t^1 (s\lambda^*(s))ds, \qquad (36)$$

$$-4(u^*(t))^3 + \int_t^1 (s\lambda^*(s))ds = 0, \qquad (37)$$

$$x^*(t) = \int_0^t (tx^*(s) + tu^*(s))ds. \qquad (38)$$

we should solve the following problem

$$x(t) = \int_0^t \left(tx(s) + t(\frac{1}{4}\int_s^1 t\lambda(t)dt)^{\frac{1}{3}} \right) ds,$$

$$\lambda(t) = t + \int_t^1 s\lambda(s)ds, \qquad (39)$$

where the optimal control law is given by

$$u^*(t) = \left(\frac{1}{4}\int_t^1 s\lambda(s)ds \right)^{\frac{1}{3}}.$$

For a proper value of h, the h-curves of $u(t)$ generated by 4th-order HAM approximation, is extracted in Figure 5. The value of $\| \frac{\partial H^{(i)}}{\partial u} \|_2^2$ produced at each iterations together with the corresponding iteration number of the algorithms are also listed in table 4.

Table 4. $\| \frac{\partial H^{(i)}}{\partial u} \|_2^2$ for h = -1 compared with the methods in [15]

Itr	SDN	SDN1	SDN2	HAM
1	0.3454434×10^{1}	0.345443×10^{1}	0.3454434×10^{1}	0
2	0.2857226×10^{1}	0.285722×10^{1}	0.2857226×10^{1}	0
3	0.1736043×10^{0}	0.41128×10^{-3}	0.4342500×10^{-1}	0
4	0.411285×10^{-3}	0.12175×10^{-18}	0.9061416×10^{-7}	
5	0.261778×10^{-8}	0.139000×10^{-19}	0.1199313×10^{-15}	
6	0.113420×10^{-18}			

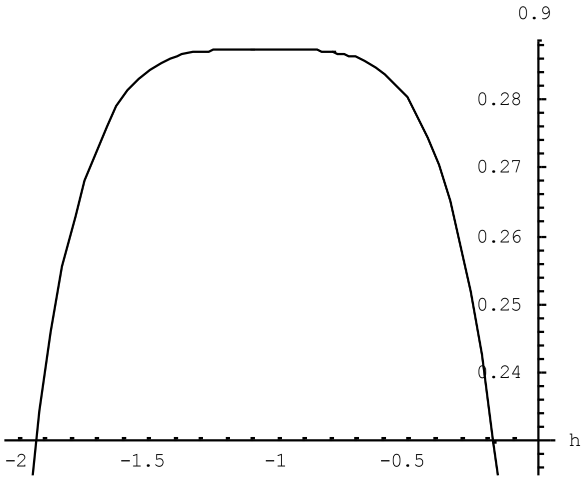

Figure 5. h - curve of u(t)

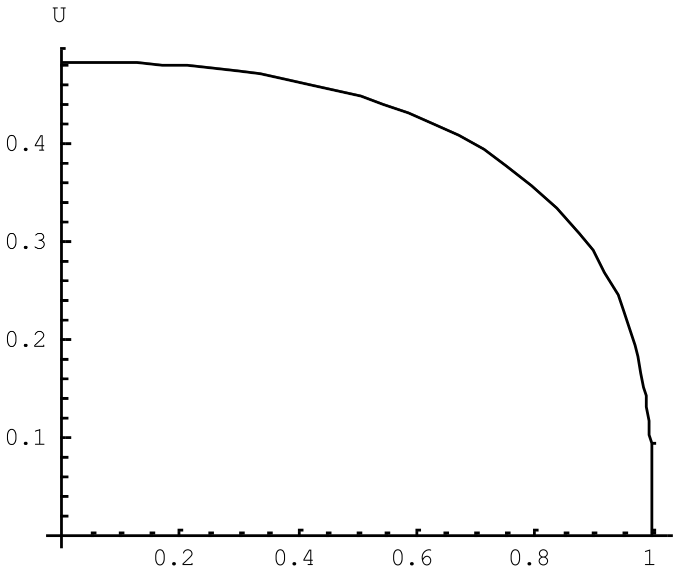

Figure 6. Approximation solution for u (t) with h=-1

6. Conclusions

In this paper, the homotopy analysis method was proposed to solve an optimal control problem of Volterra integral equations. The comparison of the results obtained from the HAM method with the other solutions shows that the present method is effective and powerful. Illustrative examples have been given to show the applicability and validity of the method. We can choose the auxiliary parameter h to be optimal to guarantee the convergence of series solution.

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