A Wavelet Operational Matrix Approach for Solving a Nonlinear Mixed Type Fractional Integro-Differential Equation

Habibollah Saeedi¹*, Mahmoud Mohseni Moghadam¹²

¹Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran, 76169-14111.
²Department of Mathematics, Islamic Azad University of Kerman, Kerman, Iran
*Corresponding author email: saeedi@uk.ac.ir.

Abstract: In this paper, an effective operational method will be used to determine the numerical solution of a specific Nonlinear Fractional Volterra-Fredholm integro-differential (NFVFID) equation. The method is based on CAS wavelets and Block Pulse Functions (BPFs) and their operational matrices. The main characteristic of this approach is to reduce a NFVFID equation to a system of algebraic equations, which greatly simplifies the problem. Also some numerical examples are provided to illustrate the accuracy and computational efficiency of the method.

Keywords: Fractional Calculus; CAS Wavelets; Block Pulse Functions; Operational Matrix.

1. Introduction

The aim of this work is to present a numerical method for approximating the solution of the following Nonlinear Fractional Volterra-Fredholm Integro-Differential (NFVFID) equation:

\[
D^\alpha f(x) - \lambda_1 \int_0^t k_1(x,t) F_1(t, f(t)) \, dt - \lambda_2 \int_0^t k_2(x,t) F_2(t, f(t)) \, dt = g(x),
\]

with supplementary conditions:

\[
f^{(i)}(0) = f_i, \quad i = 0,1,\ldots,r-1, \quad r-1 < \alpha \leq r, \quad r \in \mathbb{N},
\]

where \( D^\alpha \) is the Caputo fractional differentiation operator, parameters \( \lambda_1, \lambda_2 \) and functions \( k_1(x,t), k_2(x,t), F_1(t, f(t)), F_2(t, f(t)) \) and \( g(x) \) are given functions in \( \mathcal{L}^1[0,1] \) and \( \mathcal{L}^1[0,1] \), respectively, and \( f(x) \) is the unknown function. Here, we assume that \( F_1(t, f(t)) = f^{q_1}(t) \) and \( F_2(t, f(t)) = f^{q_2}(t) \), where \( q_1 \) and \( q_2 \) are positive integers.

In recent years, it has turned out that many phenomena in signal processing, control engineering [1, 2], electromagnetism [3], biosciences [4], fluid mechanics [5], electrochemistry [6], diffusion processes [7], dynamic of viscoelastic materials [8], continuum and statistical mechanics [9] and propagation of spherical flames [10] can be successfully modelled by the use of fractional derivatives and integrals. It is well known that the fractional order differential and integral operators are non-local operators. This is one reason why fractional differential operators provide an excellent instrument for description of memory and hereditary properties of various physical processes. Motivated by increasing number of applications of fractional differential equations, considerable attention has been given to provide efficient methods for exact and numerical solutions of fractional differential equations.

In general, most of the fractional integro-differential equations do not have exact solutions. Particularly, there is no known method for solving NFVFID equations exactly. Therefore the numerical solution of these problems is very important. However, there are a few methods for solving these equations, numerically, and most of these methods have been applied to the linear and non-fractional problems, see [11,12,13,14,15,16,17].

In this paper, we will introduce a new operational method, based on CAS wavelets and Block Pulse Functions (BPFs), to solve NFVFID equations. The method is based on reducing Eq. (1) to a system of algebraic equations by expanding the solution via CAS wavelets with unknown coefficients. The main characteristic of an operational method is to convert a differential equation into an algebraic one. This method does not only simplify the problem but also speeds up the computation. The sparse structure of the operational matrices will simplify the problem and reduce the solution procedure. It is considerable that, our method can be easily applied for solving Eq. (1), when \( \alpha \in \mathbb{N} \).

The plan of this paper is as follows. In section 2, fractional integral and differentiation operators, CAS wavelets, Block Pulse Functions and some of their properties are introduced. Section 3 introduces function approximations via CAS wavelets and BPFs in the matrix forms. In Section 4, the CAS wavelet operational matrix of fractional integration is derived. In Section 5, the approximation of CAS wavelets and BPFs and their operational matrices are used to reduce Eq. (1) to a system of algebraic equations. The error analysis of the method is discussed in section 6. In section 7 some numerical examples are presented to show the convergence, accuracy and advantages of the proposed method with a comparison to other methods. There is a conclusion of the
2. Preliminaries

In this section, we present some notations, definitions, and preliminaries that will be used in the rest of the paper.

2.1 Fractional Calculus

There are several definitions of a fractional derivative of order \( \alpha > 0 \). The two most commonly used definitions are the Riemann-Liouville and Caputo fractional derivative. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The Riemann-Liouville fractional integration of order \( \alpha \) is defined as:

\[
I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0, I^0f(x) = f(x),
\]

and the Caputo fractional derivative of order \( \alpha \) is defined as \( D^\alpha f(x) = I^{m-\alpha} D^m f(x) \), where \( D^m \) is the ordinary integer differential operator of order \( m \) and \( I^{m-\alpha} \) is the Riemann-Liouville integral operator of order \( m-\alpha \) with \( m-1 < \alpha \leq m \).

The relationship between the Riemann-Liouville operator and the Caputo operator is given by the following lemma [18]:

**Lemma 2.1** If \( m-1 < \alpha \leq m, m \in \mathbb{N} \), then:

\[
D^\alpha I^\alpha f(x) = f(x), \quad \text{and:} \quad I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.
\]

2.2 CAS Wavelets and Block Pulse Functions (BPFs)

Wavelets are special kinds of oscillatory functions with compact support that are constructed by using dilation and translation of a single function, called the mother wavelet, denoted by \( \psi(x) \) and must satisfy in certain requirements.

If the dilation parameter is \( a \) and translation parameter is \( b \), then we have the following family of wavelets:

\[
\psi_{a,b}(x) = \alpha^{-1/2} \psi\left(\frac{x-b}{a}\right), \quad a,b \in \mathbb{R}, a \neq 0.
\]

The CAS wavelets employed in this paper, are defined as:

\[
\psi_{n,m}(x) = \begin{cases} 
2^{k/2} \text{CAS}_m(2^k x - n), & \text{if } \frac{n}{2^k} \leq x < \frac{n+1}{2^k}; \\
0 & \text{otherwise},
\end{cases}
\]

where:

\[
\text{CAS}_m(x) = \cos(2m\pi x) + \sin(2m\pi x),
\]

and \( n = 0,1,\ldots,2^k-1, k \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z} \).

Let us introduce the following useful notation, corresponding to CAS wavelets:

\[
\vec{\psi}_{n,m}(x) = \begin{cases} 
2^{k/2} \text{CAS}_m(n-2^k x), & \text{if } \frac{n}{2^k} \leq x < \frac{n+1}{2^k}; \\
0 & \text{otherwise}.
\end{cases}
\]

**Definition 2.1** An \( m \)-set of Block Pulse Functions (BPFs) over the interval \([0,T]\) is defined as [19]:

\[
b_i(x) = \begin{cases} 
1, & \text{if } \frac{iT}{m} \leq x < \frac{(i+1)T}{m}, \quad i = 0,1,2,\ldots,m-1, \\
0 & \text{otherwise},
\end{cases}
\]

with a positive integer value for \( m \). In this paper, it is assumed that \( T = 1 \), so BPFs are defined over \([0,1]\). Now we explain some applicable properties of BPFs:

- **Disjointness**:

  \[
b_i(x)b_j(x) = \begin{cases} 
1, & i = j; \\
0, & i \neq j.
\end{cases}
\]

- **Orthogonality**:

  \[
\int_0^1 b_i(x)b_j(x) dx = \begin{cases} 
1/m, & i = j; \\
0, & i \neq j.
\end{cases}
\]

- **Completeness**:

  For every \( f \in L^2([0,1]) \), the sequence \( \{b_i\} \) is complete if \( \int_0^1 f(x)b_i(x) dx = 0 \) results in \( f = 0 \), almost everywhere.

Because of the completeness of \( \{b_i(x)\} \), Parseval’s identity holds, i.e. we have \( \int_0^1 f(x)^2 dx = \sum_{i=0}^{m-1} |\mathbb{P}_i|^2 \), for every real bounded function \( f(x) \in L^2([0,1]) \), and:

\[
f_i = m \int_0^1 b_i(x)f(x) dx.
\]

- **BPFs have compact supports, i.e.**:

  \[
\text{Supp}(b_i(x)) = \left[ \frac{i}{m}, \frac{i+1}{m} \right].
\]

3. Function Approximation

The set of CAS wavelets forms an orthonormal basis for \( L^2([0,1]) \). This implies that any function \( f(x) \) defined over \([0,1]\), can be expanded as:

\[
f(x) = \sum_{n=0}^{\infty} \sum_{m=-M}^{M} c_{n,m} \psi_{n,m}(x)
\]

where:

\[
c_{n,m} = \int f(x) \psi_{n,m}(x) dx, \quad \text{and} \quad < f, g > \text{ is the inner product of the functions } f \text{ and } g.
\]

and \( \Psi \) are \( 2^k(2M+1) \times 1 \)-vectors given by:
\[ c = [c_{0,M}, c_{0,M+1}, \ldots, c_{0,M}, c_{1,M}, \ldots, c_{2,M}, \ldots, c_{2,M}], \]
\[ \Psi(x) = [\psi_{0,M}, \psi_{0,M+1}, \ldots, \psi_{0,M}, \psi_{1,M}, \ldots, \psi_{2,M}, \ldots, \psi_{2,M}]^T. \]

Also from the orthogonality property of BFs, it is possible to expand functions into their Block Pulse series [19]. This means that for every \( f(x) \in L^2([0,1]) \), we have:
\[ f(x) \approx \sum_{i=0}^{m-1} f_i b_i(x) = \mathbf{f}^T \mathbf{B}_m(x), \]
where:
\[ \mathbf{f} = [f_0, f_1, \ldots, f_{m-1}]^T, \]
\[ \mathbf{B}_m(x) = [b_0(x), b_1(x), \ldots, b_{m-1}(x)]^T, \]
such that \( f_i \)'s, \( i = 0, 1, \ldots, m-1 \), are obtained by Eq. (6).

From the above representation and disjointness property, it follows that:
\[ \mathbf{B}_m(x) \mathbf{B}_m^T(x) = \begin{bmatrix} b_0(x) & 0 & \cdots & 0 \\ 0 & b_1(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{m-1}(x) \end{bmatrix}, \]
\[ \mathbf{B}_m^T(x) \mathbf{B}_m(x) = 1, \]
and:
\[ \mathbf{B}_m(x) \mathbf{B}_m^T(x) \mathbf{a} = \tilde{\mathbf{a}} \mathbf{B}_m(x), \]
where \( \mathbf{a} \) is an \( m \)-vector and \( \tilde{\mathbf{a}} = \text{diag}(\mathbf{a}) \).

Moreover, we can clearly conclude that for every \( m \times m \) matrix \( \mathbf{A} \):
\[ \mathbf{B}_m^T(x) \mathbf{A} \mathbf{B}_m(x) = \tilde{\mathbf{A}}^T \mathbf{B}_m(x), \]
where \( \tilde{\mathbf{A}} \) is an \( m \)-vector with elements equal to the diagonal entries of matrix \( \mathbf{A} \).

**Notation.** From now on, we define \( m' = 2^k(2M+1) \), where \( k, M \in \mathbb{N} \cup \{0\} \).

4. **Operational Matrix of Fractional Integration**

There is a relationship between BFs and CAS wavelets. Providing some calculations outlined in appendix A, we have found:
\[ \Psi(x) = \mathbf{F}_{m \times m} \mathbf{B}_m(x). \]

The Block Pulse operational matrix of fractional integration \( F^\alpha \) is given as follows [20]:
\[ (F^\alpha \mathbf{B}_m)(x) \equiv \mathbf{F}_m^\alpha \mathbf{B}_m(x), \]
where:
\[ F^\alpha = \frac{1}{m^{\alpha}} \Gamma(\alpha + 2) \]
\[ \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \]
and \( \xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1} \).

**Remark 4.1** For \( \alpha = 1 \), \( F^\alpha \) is the BPF’s operational matrix of integration.

Let:
\[ (I^{\alpha} \Psi_m)(x) \equiv \mathbf{P}_{m \times m}^\alpha \Psi_m(x), \]
where matrix \( \mathbf{P}_{m \times m}^\alpha \) is called the CAS wavelet operational matrix of fractional integration. Using Eqs. (11) and (12), we have:
\[ (I^{\alpha} \Psi_m)(x) \equiv (I^{\alpha} \mathbf{F}_{m \times m} \mathbf{B}_m)(x) = \mathbf{F}_{m \times m}^\alpha \mathbf{B}_m(x) = \mathbf{F}_{m \times m}^\alpha \mathbf{F}_m \Psi_m(x). \]

By Eqs. (13) and (14), we get:
\[ \mathbf{P}_{m \times m}^\alpha \Psi_m(x) \equiv \mathbf{F}_{m \times m}^\alpha \mathbf{F}_m \Psi_m(x). \]
Therefore, the CAS wavelet operational matrix of fractional integration \( \mathbf{P}_{m \times m}^\alpha \) is given by:
\[ \mathbf{P}_{m \times m}^\alpha = \mathbf{F}_{m \times m} \mathbf{F}_m \mathbf{F}_m^{-1}. \]

5. **Analysis of the Method**

Consider Eq. (1), two variables functions \( k_i(x,t) \in L^2([0,1]^2), r = 1, 2 \), can be approximated as:
\[ k_i(x,t) = \sum_{l=0}^{2^k-1} \sum_{j=1}^{2^k-M} \sum_{l=1}^{2^k-M} k_{l, j} \psi_{M,l}(x) \psi_{M,l}(t), \]
for \( i = n(2^k+1)+l+1, j = m(2^k+1)+l_2+1 \), or in the matrix form:
\[ k_i(x,t) \approx \mathbf{K}_i \Psi(t), \]
where \( \mathbf{K}_i = [k_{l, j}] \) and \( k_{l, j} = <\psi_{M,l}(x), \psi_{M,l}(t)>, k_{l, j} >> \), for \( r = 1, 2 \). The right hand side of Eq. (1) can also be written as:
\[ g(x) \approx g \Psi(x). \]

**Remark 5.1** CAS wavelets cannot be used directly for solving differential equations because of their discontinuities. There are two possible ways to overcome this problem. The first possibility is converting the underlying differential equation into an equivalent integral equation and approximating the solution by truncated orthogonal CAS series and using operational matrices of integration to eliminate the integral operators. The second possibility is to expand the highest derivative term, appearing in the differential equation, into the CAS wavelet series.
So, let:

\[
D^2 f(x) \equiv c^T \Psi(x).
\]

By using lemma 2.1, Eqs. (13) and (17), we have:

\[
f(x) \equiv (c^T \mathbf{P}_{m'\omega} + c_1^T) \Psi(x) + \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}.
\]

Hence, substituting the supplementary conditions (2) in (17) and approximating it via CAS wavelets, we get:

\[
f(x) \equiv (c^T \mathbf{P}_{m'\omega} + c_1^T) \Psi(x),
\]

where \(c_1\) is an \(m'\)-vector. According to Eq. (11):

\[
f(x) \equiv (c^T \mathbf{P}_{m'\omega} + c_1^T) \Phi_{m'\omega}.
\]

Define:

\[
a = [a_0, a_1, \ldots, a_{m' - 1}]
\]

By some calculations which are mentioned in details in appendix B, we will have:

\[
\int_0^1 k_i(x, t)[f(t)]^{q_i} dt \geq \frac{1}{m'} \mathbf{N} \mathbf{B}_{m'}(x),
\]

and:

\[
\int_0^1 k_i(x, t)[f(t)]^{q_2} dt \geq \tilde{A} \mathbf{B}_{m'}(x),
\]

where:

\[
\mathbf{N} = (\Phi_{m'\omega}^T \mathbf{K}_2 \Phi_{m'\omega} \tilde{\mathbf{a}}_q)^T,
\]

\[
\tilde{\mathbf{a}}_q = [a_0^q, a_1^q, \ldots, a_{m' - 1}^q],
\]

and \(\tilde{A}\) is an \(m\)-vector whose elements are equal to the diagonal entries of the following matrix:

\[
\mathbf{A} = \Phi_{m'\omega}^T \mathbf{K}_2 \Phi_{m'\omega} \text{diag}(\tilde{\mathbf{a}}_q) \mathbf{F}^1.
\]

Now by substituting approximations (16), (17), (22) and (23) into Eq. (1), we obtain:

\[
c^T \Phi_{m'\omega} \mathbf{B}_{m'}(x) - \lambda_1 \frac{1}{m'} \mathbf{N} \mathbf{B}_{m'}(x) - \lambda_2 \tilde{A} \mathbf{B}_{m'}(x) \equiv g^T \Phi_{m'\omega} \mathbf{B}_{m'}(x).
\]

According to orthogonality of BPFs, we have the following nonlinear system of algebraic equations:

\[
c^T \Phi_{m'\omega} - \lambda_1 \frac{1}{m'} \mathbf{N} - \lambda_2 \tilde{A} \equiv g^T \Phi_{m'\omega}.
\]

By solving this system, with respect to the unknown vector \(c^T\), the approximate solution of Eq. (1) will be obtained according to Eq. (17).

6. Error Analysis

**Theorem 6.1** [21] A function \(f(x) \in L^2[0,1]\), with bounded second derivative, say \(\|f''(x)\| \leq N\), can be expanded as an infinite sum of CAS wavelets, and the corresponding series converges uniformly to \(f(x)\), that is:

\[
f(x) = \sum_{n=-\infty}^{\infty} \sum_{m=2}^{\infty} c_{n,m} \Phi_{n,m}(x).
\]

We can easily check the accuracy of the method. Since the truncated CAS wavelet series is an approximate solution of Eq. (1), the resulting equation, (24), must be satisfied approximately in the main equation, that is for \(x \in [0,1)\):

\[
R_m(x) = c^T \Phi_{m'\omega} \mathbf{B}_{m'}(x) - \lambda_1 \frac{1}{m'} \mathbf{N} \mathbf{B}_{m'}(x) - \lambda_2 \tilde{A} \mathbf{B}_{m'}(x) \equiv 0.
\]

Set \(x = x_i\), our aim is to have \(R_m(x_i) \leq 10^r\), where \(r\) is any positive integer. If we prescribe \(\text{Max}[r] = 10^r\), then we increase \(m'\) as long as the following inequality holds at each point \(x_i\):

\[
R_m(x_i) \leq 10^r.
\]

In other words, by increasing \(m'\) the error function \(R_m(x_i)\) approaches to zero.

**Figure 1.** The comparison between approximate and exact solutions of Example 7.1 for some \(k\) and \(M\), with \(\alpha = 1\).

7. Numerical Examples

To illustrate the effectiveness of the proposed method, several test examples are carried out in this section. Note that:

\[
\|e_m(x)\|_2 = \left(\int e_m^2(x) dx\right)^{1/2} \leq \left(\frac{1}{N} \sum_{i=0}^{N} e_m^2(x_i)\right)^{1/2},
\]

where \(e_m(x_i) = f(x_i) - f_m(x_i)\), \(i = 0,1,\ldots,N\), \(f(x)\) is the exact solution and \(f_m(x_i)\) is the approximate solution which is obtained by Eq. (17).

The computations associated with the examples were performed using Matlab on a Personal Computer.
Consider a NFVFID equation, as follows:

\[ D^\alpha f(x) - \frac{1}{4} \int_0^1 f^{(3)}(t) dt + \frac{1}{2} \int_0^1 f^{(2)}(t) dt = g(x), \quad (25) \]

where \( g(x) = \frac{x^6}{10} + 2x - \frac{1}{10} \), with the initial condition \( f(0) = 0 \), and the exact solution \( f(x) = x^2 \), in the case \( \alpha = 1 \). A comparison between the CAS wavelets solutions with some \( k \) and \( M \), beside the approximate solutions and the exact solutions, are shown in Figure 1. Also, Figure 2 shows the numerical results for \( k = 4, M = 1 \) and various \( 0 < \alpha \leq 1 \). The comparisons show that as \( \alpha \to 1 \), the approximate solutions tend to \( f(x) = x^2 \), which is the exact solution of the equation in the case \( \alpha = 1 \). Table 2 presents the 2-norm of the absolute error in the case of \( \alpha = 1 \).

**Example 7.2** [22-23] For the following nonlinear integro-differential equation:

\[ D^\alpha f(x) + \int_0^1 \cos(x-t)f^2(t)dt = \sin(2x), \quad (26) \]

with the initial condition \( f(0) = 0 \), and in the case \( \alpha = 1 \) the exact solution is \( f(x) = \cos(x) \). Table 2 presents the comparison of the numerical solutions with the presented solution and figure 3 gives a comparison between the approximate solutions and the exact solution with some \( k \) and \( M \).

Figure 2. The comparison between approximate solutions of Example 7.1 for some \( 0.5 \leq \alpha \leq 1 \), with \( k = 5 \) and \( M = 1 \).

**Example 7.3** Consider the following nonlinear integro-differential equation:

\[ D^2 f(x) - \int_0^1 \abs{f(t)}^4 dt - \int_0^1 \abs{f(t)}^2 dt = g(x), \quad (27) \]

where \( g(x) = \frac{1}{2} \Gamma \left( \frac{3}{2} \right) \left( \frac{8}{3} \sqrt{x} - 2\sqrt{x} \right) - \frac{x}{1260} - \frac{x^3(6x^2 - 15x + 10)}{30} \)

with the initial condition \( f(0) = 0 \) and the exact solution \( f(x) = x^2 - x \). Table 1 and figure 4 show the comparison between the approximate solutions and the exact solution.

**Figure 3.** The comparison between approximate and exact solutions of Example 7.2 for some \( k \) and \( M \), with \( \alpha = 1 \).

**Figure 4.** The comparison between approximate and exact solutions of Example 7.3 for some \( k \) and \( M \).

**Table 1.** The comparison between the approximate solutions and the exact solution (Example 7.3)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Presented method ((k=4,M=1))</th>
<th>Presented method ((k=5,M=1))</th>
<th>BPFs method ([22]) ((m=16))</th>
<th>Adomian's method ([23]) ((m=16))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>0.999210</td>
<td>0.999803</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.995040</td>
<td>0.994216</td>
<td>0.995507</td>
<td>0.99514</td>
<td>0.994951</td>
</tr>
<tr>
<td>0.2</td>
<td>0.980067</td>
<td>0.982054</td>
<td>0.981213</td>
<td>0.975784</td>
<td>0.980303</td>
</tr>
<tr>
<td>0.3</td>
<td>0.955336</td>
<td>0.952894</td>
<td>0.953794</td>
<td>0.960386</td>
<td>0.955685</td>
</tr>
<tr>
<td>0.4</td>
<td>0.921061</td>
<td>0.925488</td>
<td>0.919364</td>
<td>0.918443</td>
<td>0.921165</td>
</tr>
<tr>
<td>0.5</td>
<td>0.877582</td>
<td>0.861828</td>
<td>0.869898</td>
<td>0.862193</td>
<td>0.877048</td>
</tr>
<tr>
<td>0.6</td>
<td>0.825336</td>
<td>0.819314</td>
<td>0.827871</td>
<td>0.828963</td>
<td>0.822596</td>
</tr>
<tr>
<td>0.7</td>
<td>0.764842</td>
<td>0.770616</td>
<td>0.768348</td>
<td>0.752929</td>
<td>0.755333</td>
</tr>
<tr>
<td>0.8</td>
<td>0.696707</td>
<td>0.690310</td>
<td>0.692816</td>
<td>0.710418</td>
<td>0.667739</td>
</tr>
<tr>
<td>0.9</td>
<td>0.621610</td>
<td>0.630059</td>
<td>0.618108</td>
<td>0.617232</td>
<td>0.547241</td>
</tr>
<tr>
<td>1.0</td>
<td>0.540302</td>
<td>0.549002</td>
<td>0.542915</td>
<td>0.566917</td>
<td>0.364798</td>
</tr>
</tbody>
</table>

**Table 2.** The 2-norm of the absolute error.

<table>
<thead>
<tr>
<th>Examples</th>
<th>( E_1 ), ((k = 3, M = 1))</th>
<th>( E_2 ), ((k = 4, M = 1))</th>
<th>( E_3 ), ((k = 5, M = 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 7.1 (( \alpha = 1 ))</td>
<td>1.693401e-005</td>
<td>2.689175e-006</td>
<td>9.968712e-008</td>
</tr>
<tr>
<td>Example 7.2 (( \alpha = 1 ))</td>
<td>1.001033e-005</td>
<td>2.498805e-006</td>
<td>6.243634e-007</td>
</tr>
<tr>
<td>Example 7.3</td>
<td>1.814125e-004</td>
<td>9.614030e-005</td>
<td>4.942423e-005</td>
</tr>
<tr>
<td>Example 7.4</td>
<td>1.184921e-004</td>
<td>6.636244e-005</td>
<td>3.507248e-005</td>
</tr>
</tbody>
</table>
some examples. In these examples the approximate solutions are briefly compared with the exact and approximate solutions obtained by the other methods.

Increasing the number of CAS wavelets over $[0,1)$, decreases the error of the solution rapidly. To show its convergence and stability, the current method can be run with increasing $m$, until the computed results have appropriate accuracy. The advantage of this method is the low cost of setting up the equations without using any projection or collocation method and integration. The most interesting ability of this method is that, it can be simply applied to the cases where $\alpha$ is integer which is observed in the numerical examples.

Appendix A: Expanding CAS Wavelets Via BPFs

First, let us introduce the following useful notation, corresponding to CAS wavelets:

$$\psi_{m,n}(x) = \begin{cases} 2^{j/2} \text{CAS}_m(n - 2^j x), & \text{if } n \frac{2^j}{2} \leq n + 1 \frac{2^j}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Eq. (7) implies that CAS wavelets can be also expanded into an $m^2$-term BPF as:

$$f(x) = \sum_{i=0}^{m-1} f_i(x).$$

By using the properties of CAS wavelets and Eq. (6), we get:

$$f_i = m \int_{0}^{1} \psi_{m,n}(x) dx = \frac{m}{2^{k+1} \pi} \left[ \psi_{n,m} \left( \frac{i}{m} \right) - \psi_{n,m} \left( \frac{i+1}{m} \right) \right],$$

for:

$$i=n(2M+1),...n(2M+1)+(M-1),n(2M+1)+(M+1),...,(n+1)(2M+1)-1.$$ 

Note that for $i=n(2M+1)+M$, we have $f_i = 2^{k/2}$ and otherwise $f_i = 0$. Therefore, we get:

$$f_{i,n} = \frac{m}{2^{k+1} \pi} \left[ 0,0,...,0 \psi_{n,m} \left( \frac{i}{m} \right) - \psi_{n,m} \left( \frac{i+1}{m} \right),...\psi_{n,m} \left( \frac{i+2M}{m} \right) - \psi_{n,m} \left( \frac{i+2M+1}{m} \right),\ldots \right] 0,0,...,0 \text{B}_{n,m}(x),$$

where $i=n(2M+1)$, $n=0,1,...,2^k-1$ and $m=-M,...,M$.

Hence:

$$\Phi(x) = \Phi_m \Phi_n \text{B}_{n,m}(x).$$

such that $\Phi_{m,n} = \text{Diag}(\Phi_0, \Phi_1, \ldots, \Phi_m)$, and $\Phi_n$ for $n=0,1,...,2^k-1$ is a $(2M+1) \times (2M+1)$ matrix which is introduced as:

$$\Phi_n = \lambda \begin{bmatrix} \psi_{n,m} \left( \frac{i}{m} \right) & \psi_{n,m} \left( \frac{i+1}{m} \right) & \ldots & \psi_{n,m} \left( \frac{i+2M}{m} \right) & \psi_{n,m} \left( \frac{i+2M+1}{m} \right) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \psi_{n,m} \left( \frac{i}{m} \right) & \psi_{n,m} \left( \frac{i+1}{m} \right) & \ldots & \psi_{n,m} \left( \frac{i+2M}{m} \right) & \psi_{n,m} \left( \frac{i+2M+1}{m} \right) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \psi_{n,m} \left( \frac{i}{m} \right) & \psi_{n,m} \left( \frac{i+1}{m} \right) & \ldots & \psi_{n,m} \left( \frac{i+2M}{m} \right) & \psi_{n,m} \left( \frac{i+2M+1}{m} \right) \end{bmatrix}$$

here:

Figure 6. The comparison between approximate and exact solutions of Example 7.4 for some $k$ and $M$.

8. Conclusion

A numerical scheme, based on operational matrices of integration for CAS wavelets and BPFs, transforms a fractional nonlinear Volterra-Fredholm integro-differential equation to a set of algebraic equations without applying any projection method. Solving this system by an iterative method gives an approximate solution which is a linear combination of $m = 2^k(2M+1)$ CAS wavelets. The applicability and accuracy of the method are checked by
\[
\Lambda = \frac{m'}{2^{k+1} \pi}
\]

\[
\Psi_{n}(x) = \left[ \tilde{\Psi}_{n,-M}(x) \quad \tilde{\Psi}_{n,-M+1}(x) \quad \ldots \quad \tilde{\Psi}_{n,M}(x) \right]^T.
\]

Appendix B: Expanding the Integral Part of the Main Equation via CAS Wavelets

According to Eqs. (19) and (20), we have \( f(x) \equiv \mathbf{a} \mathbf{B}_m(x) \). From the disjoint property of the BPFs, we get:

\[
[f(x)]^2 \equiv [\mathbf{a} \mathbf{B}_m(x)]^2
= [a_0 b_0(x) + a_1 b_1(x) + \ldots + a_{n-1} b_{n-1}(x)]^2
= a_0^2 b_0^2(x) + a_1^2 b_1^2(x) + \ldots + a_{n-1}^2 b_{n-1}^2(x)
= \mathbf{\bar{a}}_n^T \mathbf{B}_m(x),
\]

where \( \mathbf{\bar{a}}_n = [a_0^2, a_1^2, \ldots, a_{n-1}^2] \).

It is easy to show by induction that:

\[
[f(x)]^2 \equiv [a_0^2, a_1^2, \ldots, a_n^2] \mathbf{B}_m(x) = \mathbf{\bar{a}}_n \mathbf{B}_m(x),
\]

where:

\[
\mathbf{\bar{a}}_n = [a_0^2, a_1^2, \ldots, a_{n-1}^2].
\]

Using Eqs. (11), (15) and (31), we will have:

\[
\int_{\phi}^{k}(x,t)[f(t)]^n dt = \int_{\phi}^{\Psi}(x) \mathbf{K}_n \mathbf{\Psi}(t) \mathbf{B}^T_m(t) \mathbf{\bar{a}}_n^T \mathbf{\Phi} dt
\]

\[
= \Psi(x) \mathbf{K}_n \mathbf{\Phi}_{n,m}^T \mathbf{B}_m(t) \mathbf{\bar{a}}_n^T dt.
\]

Using Eq. (8), we simplify the integral part of (36) as:

\[
\int_{\phi}^{\Psi}(x) \mathbf{K}_2 \mathbf{\Phi}_{n,m}^T \mathbf{B}_m(t) \mathbf{\bar{a}}_n^T dt = \int_{\phi}^{\mathbf{diag}(\mathbf{\bar{a}}_2)} \mathbf{B}_m(t) dt
\]

Thus in (36), we get:

\[
\int_{\phi}^{\Psi}(x) \mathbf{K}_2 \mathbf{\Phi}_{n,m}^T \mathbf{diag}(\mathbf{\bar{a}}_2) \mathbf{F}^T \mathbf{B}_m(x)
= \mathbf{B}_m(x) \mathbf{\Phi}_{n,m}^T \mathbf{K}_2 \mathbf{\Phi}_{n,m} \mathbf{diag}(\mathbf{\bar{a}}_2) \mathbf{F}^T \mathbf{B}_m(x) = \mathbf{\tilde{A}}^T \mathbf{B}_m(x),
\]

where \( \mathbf{\tilde{A}} \) is an \( m \)-vector with elements equal to the diagonal entries of the following matrix:

\[
\mathbf{A} = \mathbf{\Phi}_{n,m}^T \mathbf{K}_2 \mathbf{\Phi}_{n,m} \mathbf{diag}(\mathbf{\bar{a}}_2) \mathbf{F}.
\]
Reference


