B-spline Collocation Approach to the Solution of Options Pricing Model

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***Abstract*:** In this paper, we construct a numerical method to the solution of the Black-Scholes partial differential equation modeling the European option pricing problem with regard to a single asset. We use an explicit spline-difference scheme which is based on using a finite difference approximation for the temporal derivative and a cubic B-spline collocation for spatial derivatives. The derived method leads to a tri-diagonal linear system. The stability of this method has been discussed and shown to be unconditionally stable. The computational performance of the proposed scheme is compared with those obtained by using a scheme based on the radial basis function.

***Keywords*:** Black-Scholes equation, Cubic B-spline method, Stability.

1. Introduction

The past few decades have witnessed a revolution in the trading of derivative securities in the world financial markets. A derivative security, or contingent claim, is a financial contract whose value at its expiry date T is completely determined by the prices of an underlying asset in a fixed range of times within the interval $[0,T]$[1].

In this paper we will concentrate on European Call and Put options. We recall that a European Call option gives the holder the right to buy the underlying asset for an agreed price E by the date T. The put option gives the holder the right to sell the underlying asset for a price E by the date T. The previously agreed price E of the contract is called the strike or exercise price. Option pricing theory has made a great leap forward since the development of the Black-Scholes option pricing model by Fischer Black and Myron Scholes in [2], and by Robert Merton in [3]. In an idealized financial market, the price of a European option can be obtained as the solution arising from the use of the celebrated Black-Scholes Equation. We consider the dividend-free Black-Scholes equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial s}-rV=0, \qquad (1)$$

where $V(S,t)$ is the option price, r is the risk free interest rate, σ is the volatility, and S is the stock price associated with a final condition $V(S,T)=V_g(S)$ and boundary conditions of the form

$$V_a(a,t)=\alpha(t), \qquad V_b(b,t)=\beta(t), \qquad (2)$$

where T is the expiration date. Following [4], a simple transformation $S=e^x$ changes the Black-Scholes equation into a constant-coefficient PDE in the domain $\Omega=[x\times t], x\in[\log(a),\log(b)],\ t\in[0,T]$,

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}-ru=0, \qquad (3)$$

with the final condition $u(x,T)=g(x)$ and boundary conditions

$$u(\log(a),t)=\alpha(t), \quad u(\log(b),t)=\beta(t). \qquad (4)$$

In the mathematical literature, only a few results can be found in terms of the numerical discretization of the Black-Scholes equation, mainly for linear Black-Scholes equations. Some numerical methods such that a finite element and a finite difference were applied in [5, 6, 7]. The numerical discretization of the Black-Scholes equations with nonlinear volatility has been performed using explicit finite-difference schemes [8, 9]. A finite-difference scheme often employed is the Crank-Nicolson (CN) scheme (see [7]). The CN scheme employs a classical trapezoidal formula for time integration and second-order central difference formulas for the discretization of asset derivatives. Chawla et al. [10] presented high-accuracy finite-difference methods for the Black-Scholes equation in which they employed the fourth-order L-stable time integration schemes (LSIMP) developed by Chawla et al. [11] and the well-known Numerov method for discretization in terms of the asset direction. Company et al. [12] constructed a finite difference scheme and the numerical analysis of its solution for a nonlinear Black-Scholes partial differential equation, modeling stock option prices in the realistic case in which transaction costs arising in the hedging of portfolios are taken into account. Hon and Mao [13] developed a radial basis function method for solving option pricing model. Kadalbajoo et al. presented a cubic B-spline collocation method for a numerical solution of the generalized Black-Scholes equation [14]. Khabir and Patidar [15] applied the spline approximation method for solving an option pricing model.

B-spline functions have some attractive properties. Due to their being a piecewise polynomial, they can easily be integrated and differentiated. Since they have compact support, numerical methods in which B-spline functions are used as a basic function lead to matrix systems including band matrices [16]. Such systems can be handled and solved with low computational cost. Therefore, the use of spline solutions of partial differential equations are suggested in many studies [17, 18].

The rest of the paper is organized as follows. In Section 2, we present a finite-difference approximation to discretize Eq. (3) in terms of the time variable. The B-spline collocation method is constructed in Section 3. This method is analyzed for stability in Section 4. Comparative numerical results are presented in Section 5.

2. Temporal derivative

We consider a uniform mesh Δ with the grid points $\lambda_{j,n}$ to discretize the region $\Omega = [x \times t]$. Each $\lambda_{j,n}$is the vertices of the grid points (x_j, t_n), where $x_j = \log(a) + jh,\ j = 0, \dots, N$ and $t_n = \mathrm{T} - nk, n = 1,2, \dots, \mathrm{M}$.

First of all, we discretize the problem in terms of the time variable using the following finite difference approximation with uniform step size k.

$$u_t^n = \frac{u^n - u^{n-1}}{k},$$
$$u_{xx}^n = \frac{u_{xx}^n + u_{xx}^{n-1}}{2},$$
$$u_x^n = \frac{u_x^n + u_x^{n-1}}{2},$$

and

$$u^n = \frac{u^n + u^{n-1}}{2}.$$

Substituting the above approximations into the discretized form of equation (3), we have

$$\frac{u_j^n - u_j^{n-1}}{k} + \frac{1}{2}\sigma^2 \frac{(u_{xx})_j^n + (u_{xx})_j^{n-1}}{2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{(u_x)_j^n + (u_x)_j^{n-1}}{2} - r\frac{u_j^n + u_j^{n-1}}{2} = 0.$$

Rearranging and simplifying the terms we get

$$\left(-1 - \frac{kr}{2}\right)u_j^{n-1} + \frac{k}{2}\left(r - \frac{1}{2}\sigma^2\right)(u_x)_j^{n-1} + \frac{k\sigma^2}{4}(u_{xx})_j^{n-1} = R(x), \tag{5}$$

where

$$R(x) = \left(-1 + \frac{kr}{2}\right)u_j^n - \frac{k}{2}\left(r - \frac{1}{2}\sigma^2\right)(u_x)_j^n - \frac{k\sigma^2}{4}(u_{xx})_j^n,$$

with the boundary conditions,

$$u(\log(a), t_n) = \alpha(t_n), \quad u(\log(b), t_n) = \beta(t_n). \tag{6}$$

In order to start any computations using the above formula, we need the value of u at the expiration date which is known from the final condition:

$$u(x, T) = g(x). \tag{7}$$

3. Description of method

In this section we use the B-spline collocation method to solve equation (5). We define the cubic B-spline for $j = -1,0, \dots, N, N+1$ by the following relation in [18] as

$$B_{3,j} = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}], \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 & \\ -3(x - x_{j-1})^3, & x \in [x_{j-1}, x_j], \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 & \\ -3(x_{j+1} - x)^3, & x \in [x_j, x_{j+1}], \\ (x_{j+2} - x)^3, & x \in [x_{j+1}, x_{j+2}], \\ 0, & otherwise, \end{cases} \tag{8}$$

Our numerical treatment for solving equation (5) using the collocation method with cubic B-splines is to find an approximate solution $U(x)$ for the exact solution $u(x,t)$in the form,

$$U(x) = \sum_{j=-1}^{N+1} \hat{c}_j(t) B_j(x), \tag{9}$$

where $\hat{c}_j(t)$ are unknown time dependent parameters that need to be determined. Using an approximate solution (9) and cubic B-spline (8), the approximate values at the knots of $U(x)$ and its derivatives are determined in terms of the time dependent parameters $\hat{c}_j(t)$ as

$$U(x) = \hat{c}_{j-1} + 4\hat{c}_j + \hat{c}_{j+1},$$
$$hU'(x) = 3(\hat{c}_{j+1} - \hat{c}_{j-1}),$$
$$h^2U'' = 6(\hat{c}_{j-1} - 2\hat{c}_j + \hat{c}_{j+1}). \tag{10}$$

Substituting the approximate solution U and its derivatives using (10), equation (5) finally yields the following system of equations

$$r_1\hat{c}_{j-1} + r_2\hat{c}_j + r_3\hat{c}_{j+1} = R_j, \qquad 0 \le j \le N \tag{11}$$

where

$$r_1 = -1 - \frac{kr}{2} + 3\frac{k\sigma^2}{2h^2} - \frac{3k}{2h}(r - \frac{1}{2}\sigma^2)$$
$$r_2 = -4 - 2kr - \frac{3k}{h^2}\sigma^2$$
$$r_3 = -1 - \frac{kr}{2} + 3\frac{k\sigma^2}{2h^2} + \frac{3k}{2h}\left(r - \frac{1}{2}\sigma^2\right).$$

To obtain a unique solution for $\hat{C} = (\hat{c}_{-1}, \hat{c}_0, \dots, \hat{c}_{N+1})$ we need to use the boundary conditions. Using the first boundary condition we have

$$u(\log(a), t_n) = \alpha(t_n) = \hat{c}_{-1} + 4\hat{c}_0 + \hat{c}_1,$$

By eliminating $\hat{c}_{-1}$ from the above equation and from equation (11) for $j = 0$ we have

$$\left(-\frac{9k\sigma^2}{h^2}+\frac{6k}{h}\left(r-\frac{1}{2}\sigma^2\right)\right)\hat{c}_0+\left(\frac{3k}{h}\left(r-\frac{1}{2}\sigma^2\right)\right)\hat{c}_1 = R_0-r_1\alpha(t_n). \quad (12)$$

Similarly, using the boundary condition

$$u(\log(b),t_n)=\beta(t_n)=\hat{c}_{N-1}+4\hat{c}_N+\hat{c}_{N+1}, \quad (13)$$

and eliminating $\hat{c}_{N+1}$ from the above equation and from equation (11) for $j = N$ we have

$$-\frac{3k}{h}\left(r-\frac{1}{2}\sigma^2\right)\hat{c}_{N-1}+\left(-\frac{9k\sigma^2}{h^2}-\frac{6k}{h}\left(r-\frac{1}{2}\sigma^2\right)\right)\hat{c}_N = R_N-r_3\beta(t_n). \quad (14)$$

Associating (14) and (12) with (11), we obtain a linear $(N+1)\times(N+1)$ system of equations which can be written in the matrix form as

$$\boldsymbol{A}\hat{\boldsymbol{c}}=\widehat{\boldsymbol{b}} \quad (15)$$

where

$$\boldsymbol{A}=\begin{pmatrix} w_1 & z & \cdots & 0\\ r_1 & r_2 & r_3 \ldots & 0\\ \ddots & \ddots & \ddots & \vdots\\ 0 & r_1 & r_2 & r_3\\ 0 & \cdots & -z & w_2\end{pmatrix},$$

$$\widehat{\boldsymbol{b}}=\begin{pmatrix} R_0-r_1\alpha(t_n)\\ R_1\\ R_2\\ \vdots\\ R_{N-1}\\ R_N-r_3\beta(t_n)\end{pmatrix},$$

$$\hat{\boldsymbol{c}}=\begin{pmatrix}\hat{c}_0\\ \hat{c}_1\\ \vdots\\ \hat{c}_N\end{pmatrix},$$

and

$$w_1=\left(-\frac{9k\sigma^2}{h^2}+\frac{6k}{h}\left(r-\frac{1}{2}\sigma^2\right)\right),$$

$$w_2=\left(-\frac{9k\sigma^2}{h^2}-\frac{6k}{h}\left(r-\frac{1}{2}\sigma^2\right)\right),$$

$$z=-\frac{3k}{h}\left(r-\frac{1}{2}\sigma^2\right).$$

Finally the approximate solution $U(x,t)$ will be obtained from (9).

4. Stability analysis

Following [19, 20] we established the Von-Neumann stability analysis of the proposed method. For stability analysis we should consider the equation (5) as follows

$$\left(-1-\frac{kr}{2}\right)u_j^{n-1}+\frac{k}{2}\left(r-\frac{1}{2}\sigma^2\right)(u_x)_j^{n-1} +\frac{k\sigma^2}{4}(u_{xx})_j^{n-1}=R(x_j), \quad (16)$$

using the relations (10) and also using equation (16) we can obtain

$$r_1\hat{c}_{j-1}^{n-1}+r_2\hat{c}_j^{n-1}+r_3\hat{c}_{j+1}^{n-1}+(r_1+2)\hat{c}_{j-1}^n+(r_2+8)\hat{c}_j^n +(r_3+2)\hat{c}_{j+1}^n=0 \quad (17)$$

Now, it is necessary to assume that the solution of the scheme (17) at the mesh point (x_j,t_n) may be written as $\hat{c}_j^n=\xi^n\exp(i\theta j)$, where ξ is, in general, complex, θ is real, and $i=\sqrt{-1}$. Thus, using $\hat{c}_j^n=\xi^n\exp(i\theta j)$ in (17), we obtain the characteristic equation

$$\xi=\frac{X_1-iY}{X_2+iY},$$

where

$$X_1=(-2-kr+\frac{3k\sigma^2}{h^2})\cos(\theta)-\left(4+2kr+\frac{3k\sigma^2}{h^2}\right),$$

$$X_2=(-2+kr-\frac{3k\sigma^2}{h^2})\cos(\theta)+\left(-4+2kr+\frac{3k\sigma^2}{h^2}\right),$$

$$Y=-(\frac{3k}{h}\left(r-\frac{1}{2}\sigma^2\right))\sin(\theta).$$

We can prove that $|\xi|\le 1$, so that the presented scheme is unconditionally stable.

5. Numerical examples

In this section, we present some numerical results for the solution of the Black-Scholes equation describing European call and put options.

Example 1. To indicate the numerical accuracy of the B-spline method as a spatial approximation for the option values, we consider a European call option with $T=0.5$, $E=10, \sigma=0.2$ and $r=0.05$ for a domain $S\in[1,30]$. The appropriate boundary conditions for this problem are

$$\begin{cases}\alpha(t)=0\\ \beta(t)=S-Ee^{r(T-t)}\end{cases}$$

and the final condition

$$g(S)=max(S-E,0).$$

Since $S\in[1,30]$, hence $x\in[\log 1,\log 30]$. The analytical solution is given in [13]. The comparisons resulting from the spline solution [15] and the RBF approximation [13] for some S is given in Table (1) for $h = 0.006$, $k = 0.0004$, and $h = 0.003, k = 0.0002$. Furthermore, the numerical order of convergence can be computed as

$$p = \frac{\log(E_{h_1}) - \log(E_{h_2})}{\log(h_1) - \log(h_2)}$$

where $E_{h_1}$and E_{h_2} are maximum absolute errors for two uniform mesh widths h_1 and h_2, respectively.

Table 1. Comparison of results for European call options

S	Exact solution	RBF[13]	spline method[15]	present method h=0.006 k=0.0004	present method h=0.003 k=0.0002
S=2	7.7531	7.7531	7.7531	7.7531	7.7531
S=4	5.5731	5.5731	5.5731	5.5731	5.5731
S=6	3.7532	3.7532	3.7532	3.7532	3.7532
S=8	1.7987	1.7988	1.7987	1.7988	1.7987
S=10	0.4420	0.4420	0.4420	0.4420	0.4420
S=12	0.0483	0.0483	0.0483	0.0484	0.0483
				Order of convergence: 0.9634	

Table 2. Comparison of results for European put options

S	Exact solution	present method h=0.001 k=0.005	present method h=0.0005 k=0.0025
S=5	9.26859	9.26859	9.26859
S=10	4.47423	4.47421	4.47423
S=15	1.40312	1.40310	1.40312
S=20	0.32806	0.32806	0.32806
S=25	0.06720	0.06720	0.06720
		Order of convergence: 0.9701	

Example 2. In this example we want to price a put option with T = 1, σ=0.3, E = 15 and r = 0.05 for a domain S ϵ [1,30]. The boundary conditions for this problem are

$$\begin{cases} \alpha(t) = 0 \\ \beta(t) = Ee^{r(T-t)} - S \end{cases}$$

and the final condition is

$$g(S) = max(S - E, 0).$$

The comparison results in the solution for some S and the numerical order of convergence are given in Table (2) for $h = 0.001$, k=0.005 and $h = 0.0005$, $k = 0.0025$.

The numerical results in Table (1) and Table (2) indicate that the B-spline collocation method provides a reasonable approximation to the solution of European options.

6. Conclusion

For the linear Black-Scholes equation, a two-level spline difference scheme has been discussed in this work. This method is based on a B-spline collocation. Finite difference approximations for the time derivatives and spline for the spatial derivative are used. During the computation, we found that the proposed difference scheme is unconditionally stable. To examine the accuracy and efficiency of the proposed algorithm, we tested our approach on two examples. These computational results show that our proposed algorithm is effective and accurate.

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