Is Perfect Bayesian Equilibium a Subset of Nash Equilibrium?

Jimmy Teng
Harbin Institute of Technology, School of Humanities and Social Sciences,
Corresponding addresses
jimmyteng6@yahoo.com

Abstract: The Perfect Bayesian equilibrium is not a refinement of the Nash equilibrium. The Perfect Bayesian equilibrium requires players to have beliefs that are consistent with the equilibrium strategies of other players. The Nash equilibrium does not explicitly specify the beliefs of the players. In any given Nash equilibrium, the default beliefs held by players are that the other players are playing the equilibrium strategies (or a particular subset of the equilibrium strategies in the case of multiple equilibriums). The default beliefs and equilibrium strategies in Nash equilibrium could be more narrowly defined than those allowed under Perfect Bayesian equilibrium. Consequently, the set of equilibriums under Perfect Bayesian equilibrium is not a subset of the set of equilibriums under Nash equilibrium.

***Keywords*:** Nash Equilibrium, Perfect Bayesian Equilibrium, Signaling Game, Non Existence, Refinement.

1. Introduction

The Perfect Bayesian equilibrium is widely held to be a refinement of the Nash equilibrium ([1], third paragraph, p. 52: "It is not difficult to see that a perfect Bayesian equilibrium is always subgame perfect, and therefore also a Nash equilibrium."). In Perfect Bayesian equilibrium, players hold beliefs that are consistent with the equilibrium strategies of other players. The Nash equilibrium makes no such requirement and hence purportedly generates more equilibriums than the Perfect Bayesian equilibrium. However, by not explicitly specifying the beliefs of the players, a Nash equilibrium assumes by default that the players hold certain beliefs about the other players' strategies ([2], fourth paragraph, p. 20). The default beliefs and equilibrium strategies in Nash equilibrium could be more narrowly defined than those allowed under Perfect Bayesian equilibrium, especially when one or more players are indifferent among the equilibrium strategy and alternative strategies that yields equal payoff. Consequently, the set of equilibriums under Perfect Bayesian equilibrium is not a subset of the set of equilibriums under Nash equilibrium.

In a simultaneous game, the action of a player is not observed by the other players. Since there is no observation of action, there could not be any reaction to action. Hence, the reaction functions in a simultaneous game are best understood as some kind of beliefs, or conjectures, or "virtual reaction functions" [3,4]. The concept of Perfect Bayesian equilibrium is therefore more appropriate for solving simultaneous games rather than the concept of Nash equilibrium. As Nash equilibrium does not specify the beliefs of the players, beliefs, by default, fill the voids and these beliefs need not be equilibrium consistent, as is required under Perfect Bayesian equilibrium. Consequently, the Perfect Bayesian equilibrium is not a refinement of the Nash equilibrium but an alternative equilibrium concept.

Section two gives four examples to illustrate that perfect Bayesian equilibrium is not a subset of Nash equilibrium. Section three gives a formal statement. Section four concludes the paper.

2. Examples

For the first example, consider the game represented by the payoff matrix in Table 1.

Table 1. Payoff matrix of example 1 game

Player 1\Player 2	L (Left)	R (Right)
U (Up)	2, 2	0, 2
D (Down)	0, 1	1, 1

There are two extensive form representations of this game:

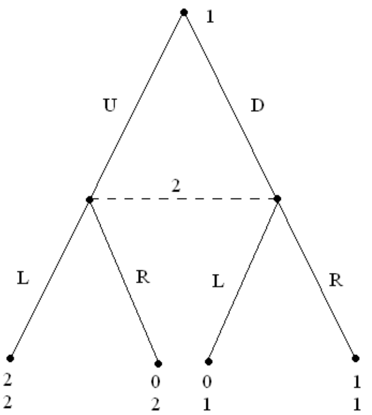

Figure 1. First extensive form presentation of example 1 game.

Therefore, the concept of Perfect Bayesian equilibrium could solve this game, as could that of Nash equilibrium.

There are two pure strategy Nash equilibriums, (U, L) and (D, R), and a whole range of mixed strategy Nash equilibriums: $\left(x=0, y\in\left[0,\frac{1}{3}\right]\right)$; $\left(x\in[0,1], y=\frac{1}{3}\right)$ and $\left(x=1, y\in\left[\frac{1}{3},1\right]\right)$. The darker line in figure 3 represents this.

In this game, player 2 is always indifferent between L and

R whatever the action of player 1. Player 1 should choose U if player 2 plays L and should choose D if player 2 plays R. The game therefore degenerates into an economics of uncertainty exercise for player 1. The only equilibrium consistent belief for y is that $y \sim U[0,1]$. Given that $y \sim U[0,1]$, the optimal strategy for player 1 is $x = 1$. Hence, the Nash equilibriums listed earlier failed to be in Perfect Bayesian equilibrium. The set of Perfect Bayesian equilibriums is $\left(x = 1, y \in [0,1]\right)$. It is not a subset of Nash equilibriums. The lighter line in Figure 3 represents this.

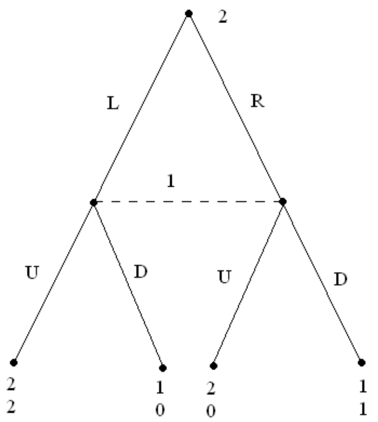

Figure 2. Second extensive form presentation of example 1 game.

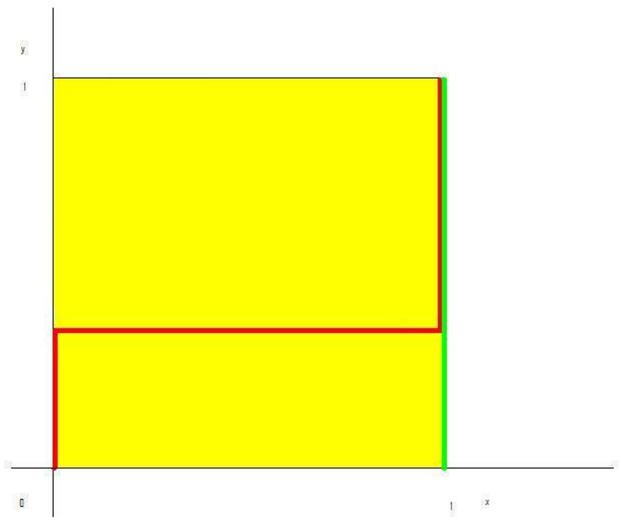

Figure 3. Nash equilibriums and perfect Bayesian equilibriums of example 1.

The second example is the matching pennies game. Table 2 represents its payoff matrix.

Table 2. Payoff matrix of example 2 matching pennies game

1\2	Head (y)	Tail (1-y)
Head (x)	1, -1	-1, 1
Tail (1-x)	-1, 1	1, -1

Let the probability that player 1 plays Head be x and the probability that player 2 plays Head be y. The unique (mixed strategy) Nash equilibrium is (0.5, 0.5). However, when player 1 plays x=0.5, player 2 is indifferent between the two strategies, and so y could be any value from 0 to 1. Similarly, when player 2 plays y=0.5, player 1 is indifferent among all values of x from 0 to 1. Consequently, the Perfect Bayesian equilibrium of this game is $x \in [0,1]$ and $y \in [0,1]$, which is a larger set than the Nash equilibrium which is x=1/2 and y=1/2.

For the third example, consider the following signaling game as represented by Figure 4.

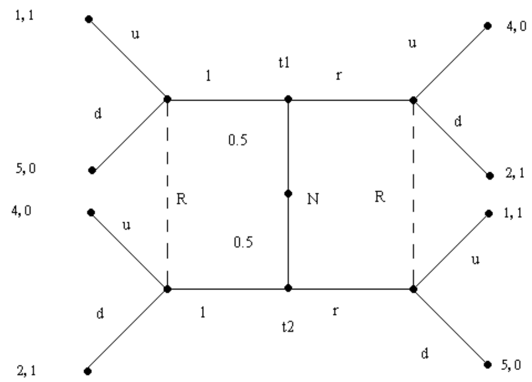

Figure 4. Signaling game of Example 3.

Table 3 gives the normal form representation.

Table 3. Normal form representation of the signaling game of Example 3

1\2	UU	UD	DU	DD
LL	2.5, 0.5	2.5, 0.5	3.5, 0.5	3.5, 0.5
LR	1, 1	3, 0.5	3, 0.5	5, 0
RL	4, 0	3, 0.5	3, 0.5	2, 1
RR	2.5, 0.5	3.5, 0.5	2.5, 0.5	3.5, 0.5

There are four pure strategy Nash equilibriums, (LL, DU), (LL, DD), (RR, UD) and (RR, DD). Among these, (L, L, D, U) and (R, R, U, D) are (pure strategy) Perfect Bayesian equilibriums. There is a mixed strategy Perfect Bayesian equilibrium, (0.5, 0.5, 0.5, 0.5). There is also another Perfect Bayesian equilibrium, ($[0,1], [0,1], [0,1], [0,1]$), which is not among the Nash equilibriums. Therefore, for this game, the set of Perfect Bayesian equilibriums is not a subset of Nash equilibriums.

For the fourth example, consider the following game: Jerry solves $\max_x U = (x - y)^2$ and Tom solves $\max_y V = -(x - y)^2$. For the simultaneous game, there is no Nash equilibrium, as the reaction functions do not intersect each other, as depicted in figure 5.

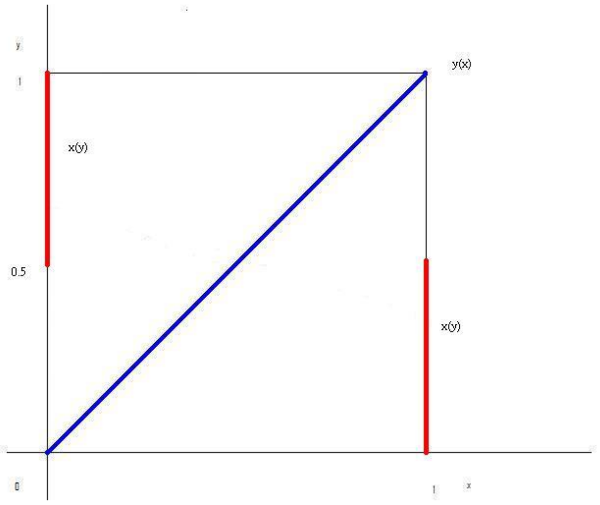

Figure 5. Reaction functions of the game of example 4.

The above game could be represented by two extensive form games of imperfect information. In one of them, Tom moves first and Jerry moves second. There is however imperfect information on the move of Tom. In the other, Jerry moves first and then Tom moves. There is however imperfect information on the move of Jerry. The game could therefore be solved by the concept of Perfect Bayesian equilibrium and there is a unique perfect Bayesian equilibrium: (x=1/2, pr(y=0)=1/2, pr(y=1)=1/2). Given the belief that x=1/2, Jerry is indifferent between y=0 and y=1; hence he plays pr(y=0)=1/2 and pr(y=1)=1/2. Alternatively, given that Jerry is indifferent between y=0 and y=1, it is optimal for Tom to set x=1/2. The beliefs are consistent with the equilibrium strategies, and the strategies and beliefs constitute the unique Perfect Bayesian equilibrium of the game. In this research, mixed strategies are not considered since the action spaces are continuous [5]. Again, for the above example, perfect Bayesian equilibrium is not a subset of Nash equilibrium.

3. A Formal Statement

From the above examples, it is clear that the Perfect Bayesian equilibrium is not a subset of the Nash equilibrium. By extension, the Perfect Bayesian equilibrium is not a subset of a subgame perfect equilibrium as well. The reason is that a Nash equilibrium, by default, assigns some implicit beliefs about the moves of the other players. If the assumed action of the players is not uniquely optimal while the assigned belief under Nash equilibrium states that it is so, or if the assumed action of the players is not uniquely optimal and there is no Nash equilibrium, then the set of Perfect Bayesian equilibrium is not a subset of the Nash equilibrium. The following proposition formalizes this argument.

Definition 1:

In a game with M players, and each player with N_h types, where $h=1,2,3,...,M$, there are $\sum_1^M N_h = Q$ sets of strategies, A_i, where $i=1,2,3,...,Q$. Each A_i has K_i number of elements, where K_i need not be finite or countable. The i type player chooses action $a_{k_i} \in A_i$, where $1 \le k_i \le K_i$. There are Q utility functions, $U_i\left(a_{k_1}, a_{k_2}, ..., a_{k_Q}\right)$.

A set of beliefs in this game specifies the probability of each $a_{k_i} \in A_i$ being selected by p_{k_i}, where $\sum p_{k_i} = 1$ if the strategy space is discrete and $\int p_{k_i} = 1$ if the strategy space is continuous.

Proposition 1:

Perfect Bayesian equilibrium is not a refinement of Nash equilibrium.

Proof:

When there is more than one optimal action for i type player, that is, for a subset of A_i, $\bar{A}_i \subset A_i$, is such that for all elements $a_{k_i} \in \bar{A}_i \subset A_i$ and for all elements $\hat{a}_{k_i} \notin \bar{A}_i \subset A_i$,

$$U_i\left(a^*_{k_1}, a^*_{k_2}, ..., \bar{a}_{k_i}, ..., a^*_{k_Q}\right) > U_i\left(a^*_{k_1}, a^*_{k_2}, ..., \hat{a}_{k_i}, ..., a^*_{k_Q}\right),$$

a strategy profile and belief system that satisfies the requirements of perfect Bayesian equilibrium is

$$U_i\left(a^*_{k_1}, a^*_{k_2}, ..., \bar{a}_{k_i}, ..., a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right)$$
$$> U_i\left(a^*_{k_1}, a^*_{k_2}, ..., \hat{a}_{k_i}, ..., a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right),$$

and $\sum \Pr\left(\bar{a}_{k_i} \in \bar{A}_{k_i}\right) = 1$ or $\int f\left(\bar{a}_{k_i} \in \bar{A}_{k_i}\right) = 1$ (where f is the probability density function) and $\Pr\left(\bar{a}_{k_i} \in \bar{A}_{k_i}\right) = \Pr\left(\bar{a}_{l_i} \in \bar{A}_{k_i}\right)$

or $f\left(\bar{a}_{k_i} \in \bar{A}_{k_i}\right) = f\left(\bar{a}_{l_i} \in \bar{A}_{k_i}\right)$

In a Nash equilibrium,

$U_i\left(a^*_{k_1}, a^*_{k_2}, ..., a^*_{k_i}, ..., a^*_{k_Q}\right) \ge U_i\left(a^*_{k_1}, a^*_{k_2}, ..., a_{k_i}, ..., a^*_{k_Q}\right)$ holds for all i. However, in the case when there is imperfect information or simultaneous moves, the players need to form beliefs by default about the moves of the other player. It should be reminded that this is a requirement made explicit by perfect Bayesian equilibrium but not by Nash equilibrium, subgame perfect equilibrium and Bayesian Nash equilibrium [6]. Effectively, in a Nash equilibrium the player assigns probability one to the assumed optimal actions of the other players that he is responding optimally to. Then, the above condition makes sense only if, by default, it means

$$U_i\left(a^*_{k_1}, a^*_{k_2}, ..., a^*_{k_i}, ..., a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right)$$
$$\ge U_i\left(a^*_{k_1}, a^*_{k_2}, ..., a_{k_i}, ..., a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right).$$

Given that there is more than one optimal action for player i, the default belief is narrower than that under Perfect Bayesian equilibrium as it means

$$U_i\left(a^*_{k_1}, a^*_{k_2}, \ldots, \bar{a}^*_{k_i}, \ldots, a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right)$$
$$= U_i\left(a^*_{k_1}, a^*_{k_2}, \ldots, \bar{a}_{k_i}, \ldots, a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right)$$ and
$$> U_i\left(a^*_{k_1}, a^*_{k_2}, \ldots, \hat{a}_{k_i}, \ldots, a^*_{k_Q}; \Pr\left(a_{k_j} = a^*_{k_j}\right) = 1\right)$$
$$\Pr\left(a_{k_i} = \bar{a}^*_{k_i}\right) = 1.$$

Note that if there is no $a^*_{k_j}$ such that $U_i\left(a^*_{k_1}, a^*_{k_2}, \ldots, a^*_{k_j}, \ldots, a^*_{k_Q}; \Pr\left(a_{k_i} = \bar{a}^*_{k_i}\right) = 1\right)$, then there is no $> U_i\left(a^*_{k_1}, a^*_{k_2}, \ldots, a_{k_j}, \ldots, a^*_{k_Q}; \Pr\left(a_{k_i} = \bar{a}^*_{k_i}\right) = 1\right)$ Nash equilibrium though the Perfect Bayesian equilibrium exists.

In sum, a Perfect Bayesian equilibrium, by explicitly requiring that the players hold beliefs that are consistent with the equilibrium strategies of the other players, produces equilibriums that are not a subset of the Nash equilibrium. Therefore, the Perfect Bayesian equilibrium is not a refinement of the Nash equilibrium but an alternative equilibrium concept.

4. Conclusions

What is the relationship between the Nash equilibrium and its many refinements? Why are there are so many refinements of Nash equilibrium?[6] The Nash equilibrium, by leaving the beliefs of agents undefined, results in beliefs by default: believing that the other player will play the Nash equilibrium strategy given that one is playing one's own corresponding Nash equilibrium strategy. However, as shown in the examples given, this assumption could at times be unreasonable. One of the problems it creates is that there might be multiple equilibriums. The myriad refinements and alternative equilibrium concepts are formulated to solve these problems. The refinements try to tighten up the definition and rule out unreasonable equilibriums and their default beliefs. They typically have smaller sets of equilibriums than the Nash equilibrium, as they have more stringent definitions. The Perfect Bayesian equilibrium explicitly requires players to hold beliefs about other players' strategies that are consistent with the equilibrium strategies being played. However, these equilibrium-consistent beliefs need not be a subset of the default beliefs implicit in Nash equilibriums, especially when the player is indifferent among courses of actions. Consequently, The Perfect Bayesian equilibrium is not a refinement of the Nash equilibrium but an alternative equilibrium concept.

Acknowledgements

I thank P. Y. Lai, D. Schoch and R. Baskaran for comments.

References

[1] P. Hans, Game Theory: A Multi-Leveled Approach. *Springer-Verlag*, 2008.
[2] M. J. Osborne, An Introduction to Game Theory. *Oxford University Press*, 2003.
[3] C. Daniel, Consistency of Conjectures in the Conventional Models of Monopoly, Monopolistic Competition, and Perfect Competition, *Journal of Economics and Business*, 42(3), pp.195-200, 1990.
[4] L.M.B. Cabral, Conjectural Variations as a Reduced Form, *Economics Letters*, 49(4), 397-402, 1995.
[5] R.J. Aumann, What is game theory trying to accomplish? in: K. Arrow and S. Honkapohja (Eds.), Frontiers of Economics, Basil Blackwell, Oxford, 1985.
[6] R. Gibbons, Game Theory for Applied Economists, Princeton University Press, 1992.