A Generalized Stackelberg Model With Noisy Observability and Incomplete Information

Jimmy Teng

Harbin Institute of Technology, School of Humanities and Social Sciences,
Corresponding addresses
jimmyteng6@yahoo.com

Abstract: This paper analyzes a generalized Stackelberg model of market competition with noisy observation. The current prevailing Stackelberg model and Cournot market are two extreme cases of this generalized Stackelberg model. In this generalized model, as the follower relies less on the inaccurate noisy observation and more on prior information and conjecture for his statistical inference and decision, the value of moving first and strategic commitment of the Stackelberg leader decreases and his profit lowers. On the other hand, the profit of the follower increases and the equilibrium moves towards the Cournot model and away from the Stackelberg model.

Keywords: Stackelberg, Cournot, Noisy Observation, Iterative Conjectures, Commitment.

1. Introduction

This paper solves a sequential game with noisy observation and incomplete information. It is a market leadership game where the leader moves first by setting his output level. The follower then sets his output level after observing the output level of the leader. The average production cost of the leader decides his type. The follower does not know the type of the leader. The follower observes the output level of the leader inaccurately with a noise term.

The Stackelberg and Cournot models of market competition are standard economic textbook materials. The current economic literature treats them as two distinct models. The Stackelberg model is a sequential game, while the Cournot model is a simultaneous game. In the Stackelberg model, the market leader moves first by setting his output level; he sets the level of output that maximizes his profit given the reaction function of the second mover. The equilibrium solution is based on the follower's reaction function but not on the leader's reaction function. The profit of the market leader is higher than that of his in the Cournot model. The literature refers to this higher profit as the value of strategic commitment [1].

This paper shows that the two models are actually related: they are the two extreme cases of a more general model. That general model is the noisy observation Stackelberg model with incomplete information. There is noisy inaccurate observation on the output level of the market leader by the follower and the follower also does not know the type of the leader. When the follower relies totally upon the inaccurate observation for his statistical inference and decision, the game becomes the Stackelberg game. When the follower relies totally upon his prior information and conjectures for statistical inference and decision, the game becomes the Cournot game.

The literature on games with noisy observation started in 1995 with Bagwell's seminal paper on the Stackelberg game with noisy observation [2]. Since then there have been many papers studying the value of strategic pre-commitment and the relationship between the Stackelberg and Cournot games [3-5]. However, none of them propose a generalized model wherein the Stackelberg model and the Cournot model are extremes cases. Furthermore, they do not address the important issue of statistical decision theoretic foundation in making inference and decision when confronting noisy observation. This paper fills this gap by using the equilibrium concept and solution algorithm of Bayesian equilibrium by iterative conjectures [6]. Bayesian equilibrium by iterative conjectures requires the agent to make Bayesian statistical predictions and decisions, starting with first order non informative prior and keeps updating with Bayesian statistical decision theoretic reasoning and game theoretic reasoning until a convergence of conjectures is achieved.

Section two presents the generalized model. Section three studies the value of strategic pre-commitment. Section four concludes the paper.

2. The Model

There are two players: Firm 1, the market leader and Firm 2, the market follower. Firm 1 moves first by setting its output level. Firm 2 observes inaccurately the output level of Firm 1 due to a confounding noise term. Firm 2 forms iterative conjectures on the output level of Firm 1, starting with a first order uninformative prior predictive distribution function and keeps updating by Bayesian statistical decision theoretic and game theoretic reasoning until a convergence of conjectures is achieved and then sets its output level.

The structure of the game is common knowledge. The cost efficiency of Firm 1, which determines the type of Firm 1, is chosen by nature from a predetermined distribution function that is common knowledge. Once chosen, the type of Firm 1 is private knowledge. The type of Firm 2 is common knowledge. Firm 2 makes inference on both the type and action of Firm 1. The distribution function of the noise term that confounds the observation by the Firm 2 on the actual output level of Firm 1 is common knowledge.

q_1, the output level of Firm 1, is the action of Firm 1. q_2, the output level of Firm 2, is the action of Firm 2. Total level of output in the market is $Q=q_1+q_2$. The inverse demand function is $P=D-Q$. The payoff function of Firm 1 is $\pi_1=(D-q_1-q_2-c_1)q_1$. c_1 is the average and marginal cost of production of Firm 1. c_1 decides the type of Firm 1. Firm 1 knows c_1 but Firm 2 does not know c_1. c_1 has a normal distribution which is common knowledge: $c_1 \sim N\left(\bar{c}_1,\zeta\right)$. Firm 2 inaccurately observes the action of Firm 1 with a noise term: $R=q_1+\varepsilon$. ε is the noise term. ε has a normal distribution: $\varepsilon \sim N(0,\kappa)$. The above leads to the following sampling distribution on R: $R|q_1 \sim N(q_1,\kappa)$ and the likelihood distribution function: $q_1|R \sim N(R,\kappa)$.

For making statistical inference and decision, Firm 2 starts with an uninformative first order prior conjecture on q_1. That is, Firm 2 solves

$$\max_{q_2} E(\pi_2)=\int_{-\infty}^{\infty}(D-q_1-q_2-c_2)q_2 f(q_1|R)\,dq_1 \tag{1}$$

where $f(q_1|R)$ in Eq. (1) is the posterior distribution function with an uninformative prior distribution function. The optimal solution is $q_2=\frac{D-R-c_2}{2}$.

Firm 1 anticipates the response of Firm 2 and solves

$$\max_{q_1} E(\pi_1)=\int_{-\infty}^{\infty}\left(D-q_1-\frac{D-R-c_2}{2}-c_1\right)q_1 f(\varepsilon)\,d\varepsilon \tag{2}$$

The optimal solution is $q_1=\frac{D+c_2-2c_1}{2}$. Therefore, the second order prior conjecture is $q_1 \sim N\left(\bar{q}_1,\zeta\right)$ with $\bar{q}_1=\frac{D+c_2-2\bar{c}_1}{2}$. The second order posterior conjecture is $q_1|R \sim N\left(\hat{q}_1,\hat{\rho}\right)$. The posterior mean and variance are $\hat{q}_1=\frac{\zeta}{\kappa+\zeta}R+\frac{\kappa}{\kappa+\zeta}\bar{q}_1=\theta R+(1-\theta)\bar{q}_1$ and $\hat{\rho}=\frac{\zeta\kappa}{\kappa+\zeta}$ and $\theta=\frac{\zeta}{\kappa+\zeta}$.

Given the second order conjectures, Firm 2 solves

$$\max_{q_2} E(\pi_2)=\int_{-\infty}^{\infty}(D-q_1-q_2-c_2)q_2 f(q_1|R)\,dq_1 \tag{3}$$

where $f(q_1|R)$ in Eq. (3) is the second order posterior conjecture or distribution function and the optimal solution is

$$q_2=\frac{D-\hat{q}_1-c_2}{2} \text{ and}$$

$$q_2|q_1 \sim N\left(\frac{D-c_2-\left(\theta q_1+(1-\theta)\bar{q}_1\right)}{2},\frac{\theta^2}{4}\kappa\right)$$

Anticipating that Firm 1 solves

$$\max_{q_1} E(\pi_1)=\int_{-\infty}^{\infty}\left(D-q_1-\frac{D-c_2}{2}+\frac{\hat{q}_1}{2}-c_1\right)q_1 f(\varepsilon)\,d\varepsilon \tag{4}$$

The optimal solution is $q_1=\frac{D+c_2+(1-\theta)\bar{q}_1-2c_1}{2(2-\theta)}$.

Therefore, the third order prior conjecture is $q_1 \sim N\left(\bar{q}_1,\rho\right)$ with $\bar{q}_1=\frac{D+c_2-2\bar{c}_1}{3-\theta}$ and $\rho=\frac{1}{(2-\theta)^2}\zeta$. The third order posterior conjecture is $q_1|R \sim N\left(\hat{q}_1,\hat{\rho}\right)$ with $\hat{q}_1=\frac{\rho}{\kappa+\rho}R+\frac{\kappa}{\kappa+\rho}\bar{q}_1=\theta R+(1-\theta)\bar{q}_1$ and $\hat{\rho}=\frac{\rho\kappa}{\kappa+\rho}$ and $\theta=\frac{\rho}{\kappa+\rho}$.

Given the third order conjectures, Firm 2 solves

$$\max_{q_2} E(\pi_2)=\int_{-\infty}^{\infty}(D-q_1-q_2-c_2)q_2 f(q_1|R)\,dq_1 \tag{5}$$

where $f(q_1|R)$ in Eq. (5) is the third order posterior distribution function. The optimal solution is

$$q_2=\frac{D-\hat{q}_1-c_2}{2} \text{ and}$$

$$q_2|q_1 \sim N\left(\frac{D-c_2-\left(\theta q_1+(1-\theta)\bar{q}_1\right)}{2},\frac{\theta^2}{4}\kappa\right)$$

Foreseeing that, Firm 1 solves

$$\max_{q_1} E(\pi_1)=\int_{-\infty}^{\infty}\left(D-q_1-\frac{D-c_2}{2}+\frac{\hat{q}_1}{2}-c_1\right)q_1 f(\varepsilon)\,d\varepsilon \tag{6}$$

The optimal solution is $q_1=\frac{D+c_2+(1-\theta)\bar{q}_1-2c_1}{2(2-\theta)}$.

Therefore, the fourth order prior conjecture is $q_1 \sim N\left(\bar{q}_1,\rho\right)$ with $\bar{q}_1=\frac{D+c_2-2\bar{c}_1}{3-\theta}$, $\rho=\frac{1}{(2-\theta)^2}\zeta$ and $\theta=\frac{\rho}{\kappa+\rho}$. At this point, the conjectures converge ([6] for an earlier treatment).

At the Bayesian equilibrium by iterative conjectures, Firm 2 produces

$$q_2=\frac{D-c_2}{2}-\frac{\theta(q_1+\varepsilon)+(1-\theta)\bar{q}_1}{2} \tag{7}$$

and Firm 1 produces

$$q_1=\frac{D+c_2}{3-\theta}-\frac{1}{2-\theta}\left(c_1+\frac{1-\theta}{3-\theta}\bar{c}_1\right) \tag{8}$$

In equilibrium, the two equations that simultaneously determine ρ and θ are $\rho=\frac{1}{(2-\theta)^2}\zeta$ and $\theta=\frac{\rho}{\kappa+\rho}$. For analytical convenience, they are rewritten as

$$G:\rho(2-\theta)^2-\zeta=0$$

$$H:\theta(\kappa+\rho)-\rho=0$$

The following are derived:

$$\frac{\partial G}{\partial \rho}=(2-\theta)^2>0$$

$$\frac{\partial G}{\partial \theta}=-2\rho(2-\theta)<0$$

$$\frac{\partial H}{\partial \rho}=\theta-1<0$$

$$\frac{\partial H}{\partial \theta}=(\kappa+\rho)>0$$

$$J=\begin{vmatrix}\frac{\partial G}{\partial \rho} & \frac{\partial G}{\partial \theta}\\ \frac{\partial H}{\partial \rho} & \frac{\partial H}{\partial \theta}\end{vmatrix}$$

$$|J|=\frac{\partial G}{\partial \rho}\frac{\partial H}{\partial \theta}-\frac{\partial G}{\partial \theta}\frac{\partial H}{\partial \rho}=(2-\theta)\kappa+\theta\rho>0 \quad \frac{\partial G}{\partial \zeta}=-1<0$$

$$\frac{\partial H}{\partial \zeta}=0$$

$$\frac{\partial G}{\partial \kappa}=0$$

$$\frac{\partial H}{\partial \kappa}=\theta>0$$

$$\frac{\partial \rho}{\partial \zeta}=\frac{\begin{vmatrix}-\frac{\partial G}{\partial \zeta} & \frac{\partial G}{\partial \theta}\\ -\frac{\partial H}{\partial \zeta} & \frac{\partial H}{\partial \theta}\end{vmatrix}}{|J|}=\frac{(\kappa+\rho)}{|J|}>0 \tag{9}$$

$$\frac{\partial \theta}{\partial \zeta}=\frac{\begin{vmatrix}\frac{\partial G}{\partial \rho} & -\frac{\partial G}{\partial \zeta}\\ \frac{\partial H}{\partial \rho} & -\frac{\partial H}{\partial \zeta}\end{vmatrix}}{|J|}=\frac{(1-\theta)}{|J|}>0 \tag{10}$$

A greater uncertainty about the production cost of Firm 1 increases the variance of the prior on the output level of Firm 1. Firm 2 therefore relies more on the inaccurate noisy observation and less on the prior for making inference on the output level of Firm 1.

$$\frac{\partial \rho}{\partial \kappa}=\frac{\begin{vmatrix}-\frac{\partial G}{\partial \kappa} & \frac{\partial G}{\partial \theta}\\ -\frac{\partial H}{\partial \kappa} & \frac{\partial H}{\partial \theta}\end{vmatrix}}{|J|}=\frac{-2\rho\theta(2-\theta)}{|J|}<0 \tag{11}$$

$$\frac{\partial \theta}{\partial \kappa}=\frac{\begin{vmatrix}\frac{\partial G}{\partial \rho} & -\frac{\partial G}{\partial \kappa}\\ \frac{\partial H}{\partial \rho} & -\frac{\partial H}{\partial \kappa}\end{vmatrix}}{|J|}=\frac{-\theta(2-\theta)^2}{|J|}<0 \tag{12}$$

A greater variation in the noise that clouds the observation leads to a lesser reliance on the observed data and a greater reliance on the prior. This leads to lesser variations in the output level of Firm 2 as Firm 2 responds less to the observed output changes of Firm 1. As Firm 2 responds less to changes in Firm 1's output level, Firm 1 makes less use of changes in its output level to affect the production level of Firm 2 and the price level. Therefore, the prior has a smaller variance.

3. Perfect and Complete Information and Indeterminacy

Now let $\rho\to 0$ and $\kappa\to\infty$ and hence $\lim_{\rho\to 0}\lim_{\kappa\to\infty}\theta=0$. In this case, at BEIC,

$$q_1=\frac{D+c_2+(1-\theta)\bar{q}_1-2c_1}{2(2-\theta)}=\frac{D+c_2-2c_1}{3} \tag{13}$$

$$q_2=\frac{D-c_2}{2}-\frac{\left(\theta(q_1+\varepsilon)+(1-\theta)\bar{q}_1\right)}{2}=\frac{D-2c_2+c_1}{3} \tag{14}$$

This is the Cournot solution for the complete and imperfect information game or simultaneous game.

Now let the variance of the type distribution function (ζ) and variance of the noise term (κ) both tend to zero. ρ, the variance of the prior conjectural distribution function on q_1 therefore tends to zero as well. That is, $\zeta\to 0, \kappa\to\infty$ and $\rho\to 0$. Consequently, $c_1\to\bar{c}_1$, $q_1\to\bar{q}_1$ and $\varepsilon\to 0$. In this case, at BEIC,

$$q_1=\frac{D+c_2-2c_1}{(3-\theta)} \tag{15}$$

and

$$q_2=\frac{D-c_2-q_1}{2}=\frac{(2-\theta)D-(4-\theta)c_2+2c_1}{2(3-\theta)} \tag{16}$$

The equilibrium q_1 and q_2, when all the three variances tend to zero, depends upon the value of $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta$ which could take on any value from 0 to 1. If $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta=0$, then the BEIC has the Cournot solution. If $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta=1$, then the BEIC has the Stackelberg solution. That is,

$$q_1=\frac{D+c_2-2c_1}{2} \tag{17}$$

$$q_2=\frac{D-3c_2+2c_1}{4} \tag{18}$$

If $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta=0.5$, then in the BEIC,

$$q_1=\frac{D+c_2-2c_1}{2.5} \tag{19}$$

$$q_2=\frac{1.5D-3.5c_2+2c_1}{5} \tag{20}$$

These cases are illustrated in Fig. 1. In Fig. 1, C is the solution when $\theta=0$, S is the solution when $\theta=1$ and G is the solution when $\theta=0.5$. This indeterminacy arises from given perfect information and complete information. It begs the question: which is more accurate, the information on action that is perfect or the information on type that is complete?

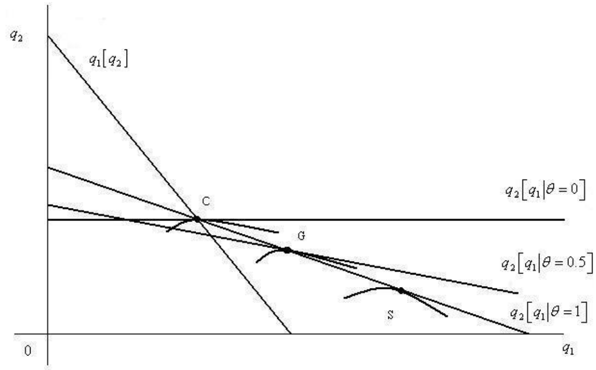

Figure 1. Indeterminacy in market leadership game

As the follower relies less upon the inaccurate observation and more upon the prior, the equilibrium moves away from the Stackelberg solution and tends towards the Cournot solution.

$$\frac{\partial q_1}{\partial \theta} = \frac{\theta\left(D + c_2 - 2c_1\right)}{\left(3-\theta\right)^2} > 0 \tag{21}$$

$$\frac{\partial q_2}{\partial \theta} = -\frac{1}{2}\frac{\partial q_1}{\partial \theta} < 0 \tag{22}$$

Proposition 1:

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Proof:

$$\begin{aligned}\frac{\partial \pi_1}{\partial \theta} &= \left(P - c_1\right)\frac{\partial q_1}{\partial \theta} + \frac{\partial P}{\partial \theta} q_1 \\ &= \left(P - c_1\right)\frac{\partial q_1}{\partial \theta} - \frac{1}{2}\frac{\partial q_1}{\partial \theta} q_1 \\ &= \left(P - c_1 - \frac{q_1}{2}\right)\frac{\partial q_1}{\partial \theta} > 0\end{aligned} \tag{23}$$

as

$$\begin{aligned}\frac{\partial P}{\partial \theta} &= \frac{\partial P}{\partial q_1}\frac{\partial q_1}{\partial \theta} + \frac{\partial P}{\partial q_2}\frac{\partial q_2}{\partial \theta} \\ &= (-1)\left(\frac{\partial q_1}{\partial \theta} - \frac{1}{2}\frac{\partial q_1}{\partial \theta}\right) \\ &= -\frac{1}{2}\frac{\partial q_1}{\partial \theta} < 0\end{aligned} \tag{24}$$

and

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= \left(P - c_1\right) + \frac{\partial P}{\partial q_1} q_1 \\ &= P - c_1 - q_1 = 0\end{aligned} \tag{25}$$

and therefore

$$P - c_1 - \frac{q_1}{2} > 0 \tag{26}$$

Using the Envelope theorem,

$$\frac{\partial \pi_2}{\partial \theta} = \frac{\partial \pi_2}{\partial q_1}\frac{\partial q_1}{\partial \theta} < 0 \tag{27}$$

□

4. Conclusions

The value of first mover strategic pre-commitment and advantage depends critically on θ, the weight given to noisy observation in statistical inference. For Bagwell (1995), where there is observational noise but no incomplete information on type, $\theta = 0$ and statistical inference depends totally on prior information and absolutely not on the noisy observation. The noisy observation Stackelberg game therefore has a Cournot solution and there is no value of strategic pre-commitment. That causes the Bagwell (1995) paradox. If there is incomplete information on type, then there need not be the disappearance of the value of first mover commitment and advantage as θ could be any value from zero to 1 and both the noisy observation and prior information affect statistical inference and decision [7].

In summation, the relative sizes of the variance of the noise term and the variance of the prior distribution function of the type of Firm 1 decide the value of commitment and being the first mover.

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