Numerical Solution of linear and nonlinear Periodic Physical Problems using Fourier spectral method

Mohammad Reza Mohaghegh

Department of Mechanical Engineering, University of Torbat-e-Heydarieh, Torbat-e-Heydarieh, Iran,
Corresponding addresses
mo.mohaghegh@stu-mail.um.ac.ir; mor.mohaghegh@gmail.com

***Abstract*:** This paper uses a spectral algorithm for the numerical solution of unsteady periodic problems. This algorithm which is based on discrete Fourier transforms uses a Fourier representation for periodic conditions of problems and hence solves the periodic state directly, without resolving transients because the discrete Fourier transforms have the periodic property to confirm physics of flow. The algorithm has been proposed for the fast and efficient computation of periodic flows. The algorithm has been validated with Stokes' second problem as a linear problem and Burgers' equation as a nonlinear problem. The same numerical results are compared with an analytical solution, second-order Backward Difference Formula (BDF) and finite difference method (FDM) results. By enforcing periodicity by using Fourier representation that has a spectral accuracy, a tremendous increase of accuracy has been obtained compared to using the conventional numerical methods like BDF and FDM. Results verify the small number of time intervals per oscillating cycle required to capture the flow physics in Stokes' second problem. Moreover, they show that a small number of points in a computational grid are required to capture the flow physics in Burgers' equation. Furthermore, this algorithm is more able than a finite difference method to capture shock.

***Keywords*:** Fourier Spectral Algorithm; Discrete Fourier Transform; Periodic Problems; Stokes' Second Problem; Burgers' equation.

1. Introduction

There are many actual physical phenomena with periodic boundary conditions like periodic flows that have been used extensively, including in the analysis of flow around helicopter blades, acoustic streaming around an oscillating body, analysis of problems with periodic boundary conditions, oscillation bodies etc. In these matters, flow behaviour is periodic in the time or space computational domain. The employment of algorithms capable of using the periodic property of flow can be helpful in analysing these problems. What is important in the numerical solution is the balance between computational time and accuracy of the solution. In traditional numerical schemes like BDF and FDM, solution accuracy comes down, if large time and space steps are chosen and time resolved intensity will increase when they are small. Therefore, an algorithm is suitable to reduce the time of solving and also maintain good accuracy.

In recent years, researchers have turned to using Fourier-based algorithms to reduce the computational expense and to increase the significant accuracy of numerical solutions for analysing periodic problems. Moin [1] has carried out many studies in the Fourier Spectral method. Also Hall et al. [2]; McMullen et al. [3, 4]; Gopinath and Jameson [5, 6] and Mohaghegh and Malek-Jafarian [7] have used Fourier-based methods for time periodic problems. Gopinath and Jameson [5, 6] and Mohaghegh and Malek-Jafarian [7] used a Fourier-based time integration for periodic flow solutions by using a Fourier collocation matrix for the temporal derivative term. This method avoids the transformation of dependent variables back and forth from time and frequency domain. The governing equations are now essentially solved in the physical domain. The detailed algorithm of this technique will be presented in section 2.3.

The physical problems can be periodic on the time or on the space. It depends on their boundary conditions. In this study, Stokes' second problem and Burgers' equation are investigated numerically by the Fourier spectral algorithm, as widely used and interested classical problems of linear and nonlinear problems, respectively.

In the first part of this study, the Stokes' second problem is analysed. This problem is one of the most famous and classic problems that describe the oscillatory flat plate in a semi-infinite flow domain with a specific frequency. This problem has received much attention owing to its practical applications.

The Stokes' second problem was first treated by Stokes [8] and later by Rayleigh [9]. In recent years, many searches have been made on this problem. Erdogan [10] analysed the unsteady flow of viscous fluid owing to an oscillating plane wall by using Laplace transform technique. Vajravelu and Rivera [11] discussed the hydro magnetic flow over an oscillating plate. Devakar and Iyengar [12] solved Stokes' first and second problems for an incompressible couple stress fluid, considered under isothermal conditions, through the use of Laplace transform technique. Fetecau et al. [13] introduced new and simpler exact solutions corresponding to the second problem of Stokes for Newtonian fluids. These solutions as a sum of the steady-state and transient solutions are in accordance with the previous results. Fetecau et al. [14] presented new exact solutions corresponding to the second problem of Stokes for Maxwell fluids. Srinivasan and Rajagopal [15] studied a variant of Stokes' first and second problems for fluids with pressure dependent viscosities and found that the velocity field and hence the structure of the vorticity and the shear stress at the walls for fluids with pressure dependent viscosity, are markedly different from those for the classic Navier–Stokes fluid. Prusa [16] presented an exact formulae by revisiting the numerical study carried out by Srinivasan and Rajagopal [15]. Liu [17] solved the extended Stokes' problems by integral transforms. Riccardi [18] discussed the analytical solutions of first and second Stokes' problems for infinite and finite-depth flows of a Newtonian fluid in planar geometries. By using this approach, he concluded that stagnation points have been found in infinite-depth flows.

Massoudia and Vaidya [19] studied the unsteady motion of an inhomogeneous incompressible viscous fluid, where the viscosity varies spatially according to various models.

In the second part of this study, Burgers' equation is analysed numerically by the Fourier spectral algorithm.

During the last few decades considerable efforts have been directed towards the development of a robust computational procedure to tackle nonlinear partial differential equations encountered in various fields of science and engineering. One of the most common equations involving both nonlinear propagation effects and diffusive effects is the Burgers' equation. It is a simplified form of the Navier–Stokes' equation carrying a combination of convection and diffusion terms and without the pressure term and the volume forces. Generally this equation is used in order to test numerical schematics. Also, this equation is well-known to show shock formation.

Burgers' equation was proposed as a model of turbulent fluid motion. It is important in a variety of applications because nonlinear phenomena play a crucial role in applied mathematics and physics and in mechanics and biology.

Bateman [20] in 1915 first introduced Burgers' equation (1). Later, describing a mathematical model of turbulence, it was proposed by J.M. Burgers [21, 22].

In the context of gas dynamics, the problem was first solved and discussed by Hop [23] and Cole [24]. They also illustrated independently that the Burgers' equation can be solved exactly for an arbitrary initial condition. Also, Cole [24] studied the general properties of the Burgers' equation and pointed out that it shows the typical features of shock wave theory.

During the past decade, the use of high-order numerical algorithms in the context of finite difference method has become relatively popular in computational fluid dynamics. In recent years, many researchers have introduced new and various methods to solve the Burger's equation. Darvishi and Javidi [25, 26] studied a numerical solution of Burgers' equation by a pseudo spectral method and Darvishi's preconditioning. Abdou and Soliman [27] used a variational iteration method for solving Burgers and coupled Burgers' equations. Also, this can be referred to recent studies in this field inclusive of recent works of [28] [29] [30] and [31]. The non linearity and the shock formation are the main difficulties for correct solving of the problems. For the numerical computation the scheme should be accurate enough to be able to render the strong gradients over a limited set of points.

In this paper Fourier spectral algorithm has been used for the numerical solution of the linear periodic problem (Stokes' second problem) and nonlinear periodic problem (Burgers' equation) by using Fourier collocation matrix for temporal and space derivative terms respectively.

Also, the same obtained numerical results are compared with an analytical solution with BDF and FDM results.

2. Model Problem Governing Equations.

A. Stokes' Second Problem:

An infinitely large plate which is oscillating back and forth at a constant frequency is shown in Fig. 1.

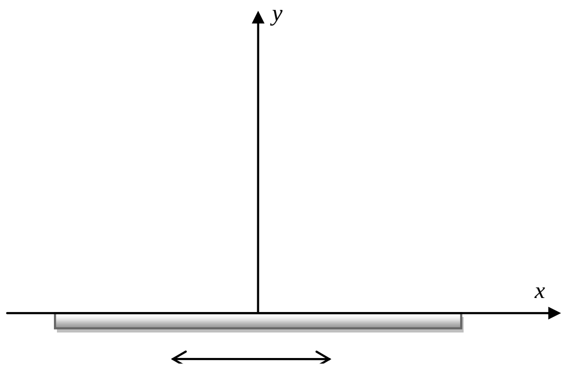


Figure 1. The oscillating plate.

Consider a viscous fluid near this plate. The motion of a viscous fluid caused by the sinusoidal oscillation of a flat plate was termed as Stokes' second problem by Schlichting [32]. Initially, both the plate and fluid are assumed to be at rest. At time t = 0+, the plate suddenly starts oscillating with the velocity Ucos(ωt). The study of the flow of a viscous fluid over an oscillating plate is not only of fundamental theoretical interest but it also occurs in many applied problems such as acoustic streaming around an oscillating body, an unsteady boundary layer with fluctuations etc [33].

2.1 Differential Form of the Flow Field Equations:

A two-dimensional and unstable flow is considered parallel to the plate. The volume forces have been ignored and a constant pressure field is considered. The only non-zero momentum equation of unsteady Navier-Stokes reduces to:

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

The boundary conditions are:

$$\begin{cases} y=0 & : \ u=U\cos(\omega t) \\ y\to\infty & : \ u=0 \end{cases}$$

where ρ is the density, μ is viscosity, u is component of velocity in the x direction and U is the maximum of the velocity. Analytical solution of this equation is in the following form ([34]):

$$u(y,t)=Ue^{-\beta y}\cos(\omega t-\beta y) \tag{2}$$

where

$$\beta=\frac{1}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}} \tag{3}$$

where ω and ν are frequency fluctuations and kinematic viscosity, respectively.

2.2 The BDF scheme

The semi-discrete form of the unsteady equation (1) in the Cartesian coordinates can be written as follows:

$$\frac{\partial u}{\partial t}+R(u)=0 \qquad (4)$$

where $R(u)=-\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$

The time derivative term can be approximated by the Taylor series. One of these approximations in the backward form is BDF scheme. Time discretisation form of the governing equation by explicit BDF scheme is as:

$$\frac{3u^n-4u^{n-1}+u^{n-2}}{2\Delta t}+R(u^n)=0 \qquad (5)$$

While time discretisation form contains only a few sentences in BDF scheme, this form in the Fourier Spectral Algorithm is as a product of matrices at a vector that is more studied.

2.3 Spectral Method for time derivative term:

The core of this method is based on discrete Fourier transform for solving periodic unsteady partial differential equations. The discrete Fourier transform of u, for a time period T, is given by

$$\hat{u}_k=\frac{1}{N}\sum_{n=0}^{N-1}u^n e^{-ik\frac{2\pi}{T}n\Delta t} \qquad (6)$$

And its inverse transform is:

$$u^n=\begin{cases}\displaystyle\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\hat{u}_k e^{ik\frac{2\pi}{T}n\Delta t} & : \text{N is even}\\ \displaystyle\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}}\hat{u}_k e^{ik\frac{2\pi}{T}n\Delta t} & : \text{N is odd}\end{cases} \qquad (7)$$

The Fourier transform of the derivative approximations at nth time interval is computed by [1] (for more detail see appendix):

$$D_t u^n=\frac{2\pi}{T}\sum_{j=0}^{N-1}d_n^j u^j \qquad (8)$$

where d_n^j is defined by:

$$d_n^{j\,even}=\begin{cases}\frac{1}{2}(-1)^{n-j}\cot(\frac{\pi(n-j)}{N}) & : n\neq j\\ 0 & : n=j\end{cases} \qquad (9)$$

and

$$d_n^{j\,odd}=\begin{cases}\frac{1}{2}(-1)^{n-j}\operatorname{cosec}(\frac{\pi(n-j)}{N}) & : n\neq j\\ 0 & : n=j\end{cases}$$

(10)

Since here flow is periodic in time, flow variables (the velocity vector u) vary periodically by time, too. Therefore, its derivative can be expressed using (8).Thus, the governing equation (4) in semi-discrete form for a computational grid is:

$$D_t(u_i^n)+R(u_i^n)=0 \qquad (11)$$

Introducing pseudo time, τ, to (11) in the same manner as the explicit dual time stepping scheme:

$$\frac{du_i^n}{d\tau}+D_t u_i^n+R(u_i^n)=0 \qquad (12)$$

The solver used for this purpose is a conservative cell-centered finite volume scheme. A pseudo-time step with a two-stage Runge-Kutta [35] scheme is performed at each level.

B. Burgers' equation:

The Burgers' equation has a convective term, a diffusion term and a time-dependent term, as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2} \qquad (13)$$

where u is velocity, $\frac{\partial u}{\partial t}$ is unsteady term, $u\frac{\partial u}{\partial x}$ is convective term and $\nu\frac{\partial^2 u}{\partial x^2}$ is diffusion term.

An interesting test case with shock formation is provided by the time evolution of a sinusoidal wave profile. So, initial conditions of above equation are so:

$$u(x,0)=u_0\sin\left(\frac{2\pi x}{L}\right) \qquad (14)$$

where L is solution area. This initial condition is chosen in order to observe a shock. Boundary conditions are periodic, thus space derivative terms can be discrete using Fourier spectral method. Choosing variables dimensionless as: $x^*=\frac{x}{L}$, $u^*=\frac{u}{u_0}$, $t^*=t\frac{u_0}{L}$, Eq. (13) in dimensionless form is achieved as follows:

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial x^{*2}} \qquad (15)$$

where $\mathrm{Re}=\frac{u_0 L}{\nu}$ is the Reynolds number. Also, dimensionless initial condition is achieved as follows:

$$u^*(x^*,0)=\sin\left(2\pi x^*\right) \qquad (16)$$

2.4 Spectral Method for Space derivative term:

Similarity to section 2.3, the discrete Fourier transform of u, for a space period T, is given by:

$$D_x u^n=\frac{2\pi}{T}\sum_{j=0}^{N-1}d_n^j u^j \qquad (17)$$

where D_x is a spacial derivative. As a result, second-order derivative is calculated as:

$$D_{xx}u^n=\left(\frac{2\pi}{T}\right)^2\sum_{j=0}^{N-1}d_n^j . d_j^n u^j \qquad (18)$$

Therefore Eq. (13) is written as follows:

$$\frac{\partial u^*}{\partial t^*} = \frac{1}{\mathrm{Re}} D_{xx}\left(u^{*^n}\right) - u^{*^n} D_x\left(u^{*^n}\right) \qquad (19)$$

3. Numerical Results and Discussions.

3.1 The numerical results of Stokes' Second Problem:

The solution of equation (1) shows the velocity field of a liquid with infinite length that has been located in the domain y≥0 and is limited by the plate y=0. This plate produces simple harmonic oscillations with amplitude U and frequency ω. Thus transverse waves with the length 2π/β are propagated from the boundary into the liquid. Also the phase velocity of the waves is equal to ω/β and their amplitudes are decreasing. In the present work, the value of ω and U is set to 1 (ω=U=1).

The numerical results related to equation (1) are computed by the Fourier Spectral Algorithm for time derivative term that we named time spectral method (TSM) and are shown in Figure 2. As can be seen, the effect of viscosity is limited to only a short distance from the plate.

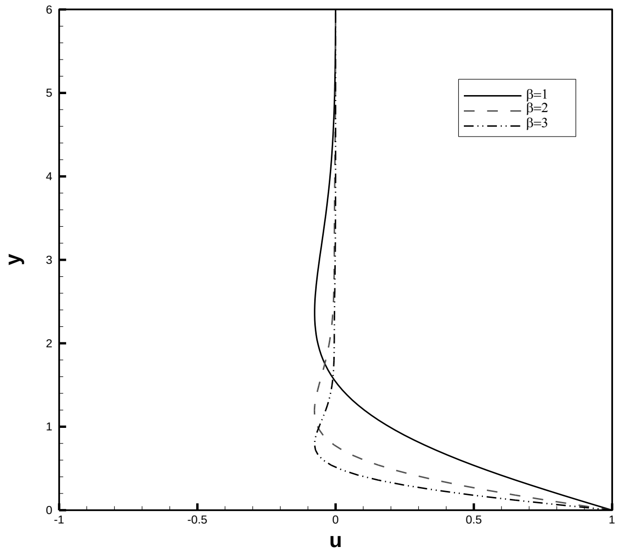


Figure 2. Velocity profiles as a function of β at t=0.

In all cases the velocity decays to zero as one moves away from the oscillating plate. For larger values of β (corresponding to a high frequency oscillation of the plate) the flow is dominated by inertial effects and the perturbation to the flow field is restricted to a shallow layer near the plate. The fluid's inertia wants to keep the fluid at rest. For smaller values of β (corresponding to low frequency oscillations or large viscosity) viscous effects dominate and the velocity perturbation caused by the moving plate is felt further inside the bulk of the fluid. Therefore the velocity perturbation decays more slowly as one moves away from the plate. After this, other results are provided for β=1

Figure 3 shows the velocity profiles as a function of time and space in each quarter of a period of oscillation. Periodic behaviour of the velocity is obvious in this figure.

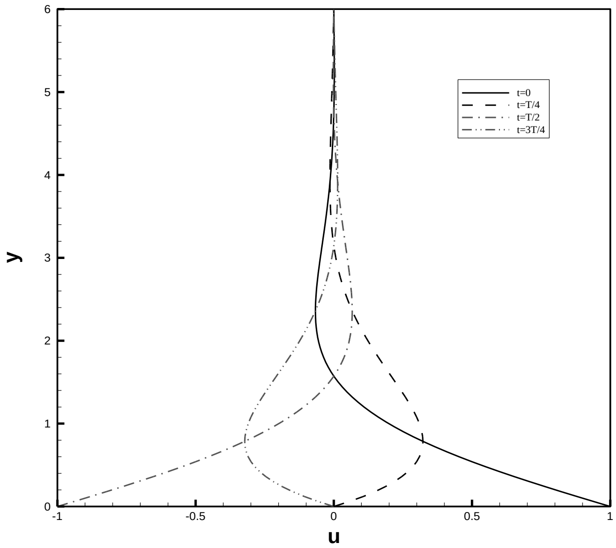


Figure 3. Velocity profiles as a function of time (t) and space (y) for β =1.

Figure 4 presents a comparison between the numerical results of TSM and analytical results in each quarter of the oscillation period. As can be seen in this figure, the results of TSM are match to the analytical results.

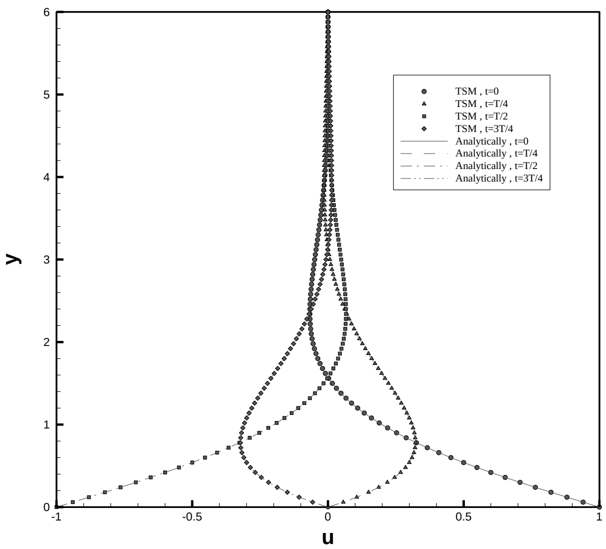


Figure 4. Comparison of TSM results with analytical results.

Figure 5 presents a comparison between the numerical results of BDF method and analytical results in each quarter of the oscillation period. Comparison of Fig. 4 and Fig. 5 show that the BDF method has less accuracy than the TSM.

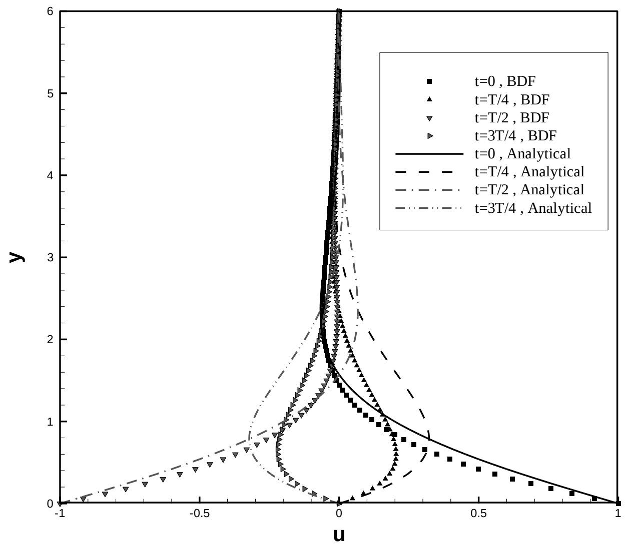


Figure 5. Comparison of BDF method results with analytical results.

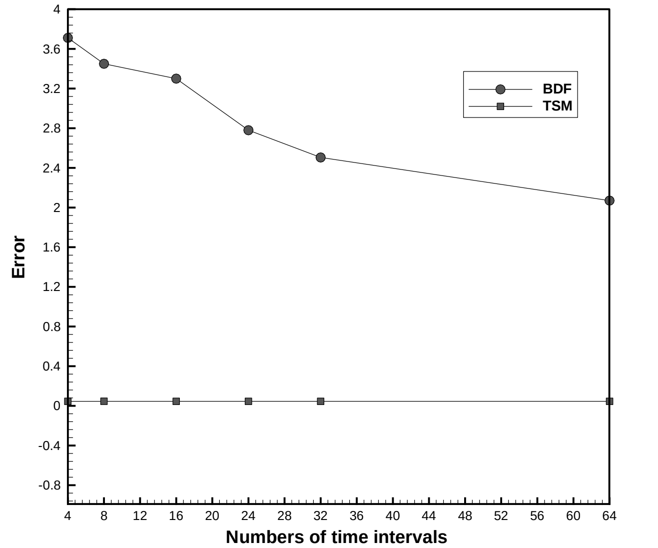


Figure 6. Comparison of value of absolute error as a function of numbers of time intervals for the BDF and TSM.

Figure 6 shows comparison of the values of absolute error for different time intervals using TSM and BDF. The value of absolute error is computed as the following:

$$Error = \frac{1}{N}\frac{1}{nx}\sum_{n=0}^{N}\sum_{i=0}^{nx}\left|v^{\,n}_{numerical}(i) - v^{\,n}_{analitycal}(i)\right| \quad (20)$$

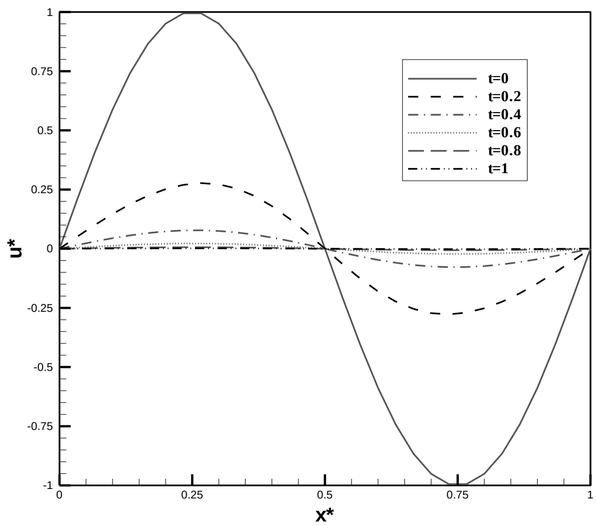
where N is the number of time intervals in each period and nx is the number of nodes in the computational grid Fig. 6 shows for the TSM result that increasing the number of time intervals, does not much change the accuracy of the solution (or the value of absolute error), so the TSM is able to capture the physics of the problem well with small numbers of time intervals. This is one of the major advantages of using the TSM. But using the BDF method, the value of errors is large in small number time intervals .The errors have been decreased by increasing the number of time intervals. But it is clear that the computational costs will increase with increasing the number of time intervals.

By comparing the absolute error in Fig. 6, it can be concluded that the Time Spectral method is more accurate in contrast to the BDF method. So the Time Spectral method is a powerful solution to analysing the periodic problems.

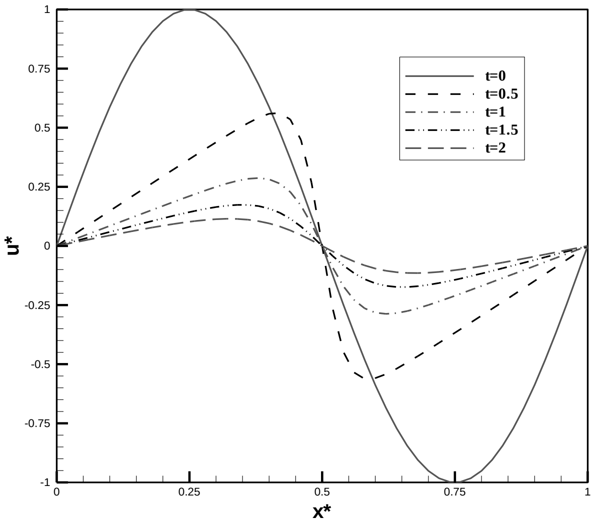
3.2 The numerical results of Burgers' equation:

This section surveys the results of Burgers' equation by the Fourier Spectral method. Also, some results using the second-order central difference method are obtained and have been compared with the results of Fourier Spectral method. Throughout this paper, we have used the values of L=1 and u_0=1, so $\mathrm{Re} = \frac{u_0 L}{\nu} = \nu^{-1}$.

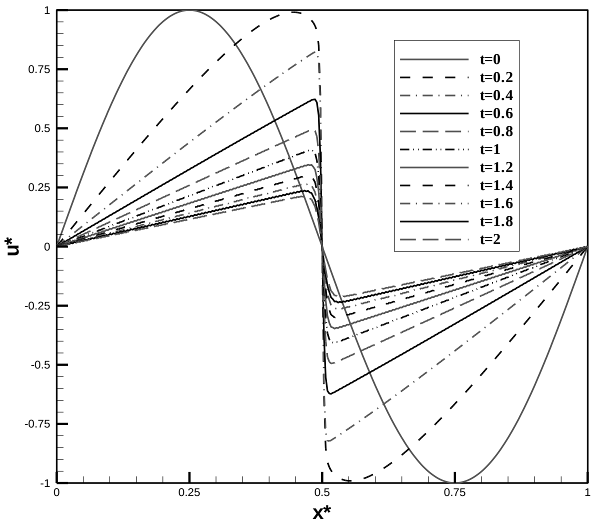
Figs. 7-(a), 7-(b) and 7-(c) show the numerical results at different times and for different viscosity. The viscosity tends to erase the disturbance in the flow. The effect of the viscous term is apparent in these figures. Viscosity reduces the amplitude of the wave for an increasing time. The bigger the viscosity is, the faster the disturbance reduces the effect. In Fig. 7-(a), kinematic viscosity is 1 (ν=1), so viscosity is too big to let the shock appear. But by reduced viscosity the possibility of shock formation increases. For ν = 0.01 as in Fig. 7-(c) as can be seen, shock occurs. The fastest fluid catches up with the slowest one so that creates a velocity break. This phenomenon is called shock.



(a)



(b)



(c)

Figure 7. Dimensionless velocity profiles at different times for (a) ν = 1, (b) ν = 0.1, (c) ν = 0.01.

Figure 8 shows the effect of viscosity changing in the velocity profiles. By increasing the viscosity the expected shock fades. The following graph shows the solution obtained for different viscosities at time t=0.3 s.

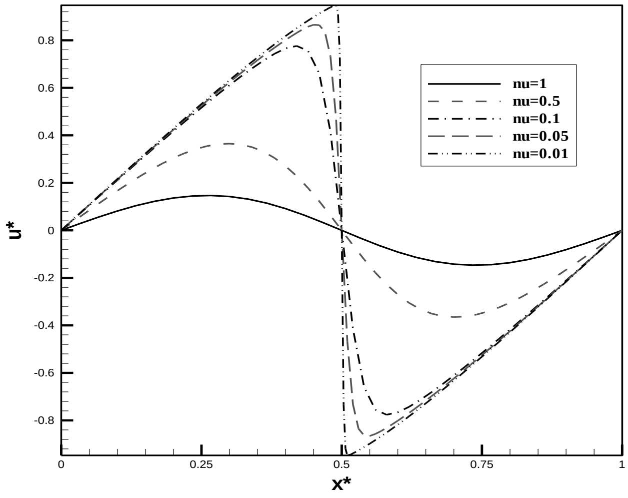
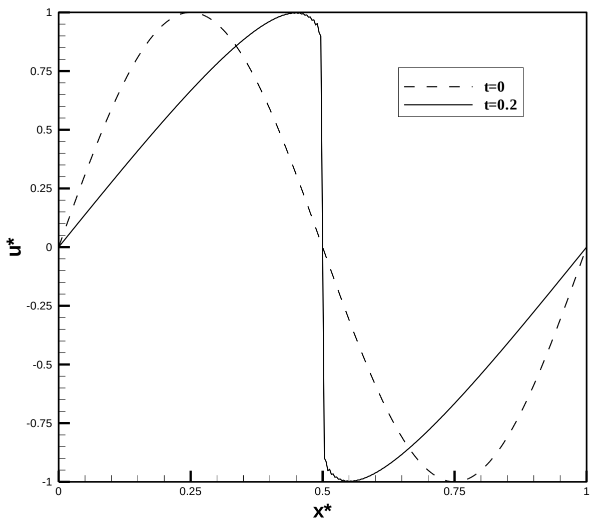
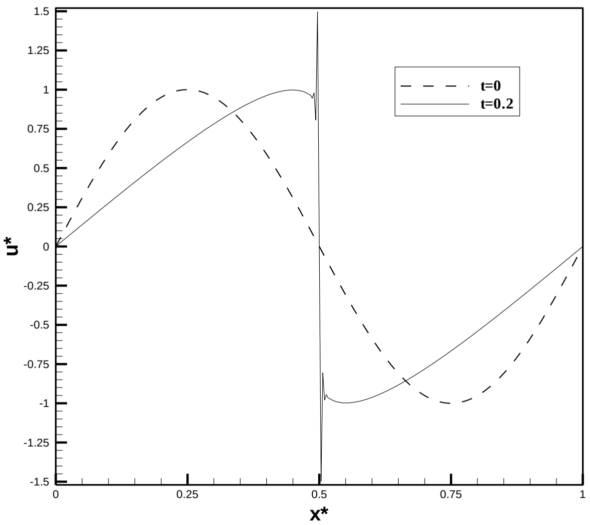


Figure 8. Dimensionless velocity profiles at different viscosities at t=0.3 s.

A numerical scheme should be accurate enough to be able to render the strong gradients over a limited set of points. Figs. 9-(a) and 9-(b) show the solution results for ν = 0.005 using Fourier Spectral method and finite difference method, respectively. The results of both methods are reached with the equal number of grid points (N=40). For small values of ν, (Here ν < 0.001) oscillations in the numerical solution near the shock location can be seen (Fig. 9-(b)). These oscillations cannot be seen in Fig. 9-(a). Therefore it is noted that for a similar computational grid to solve the equation, the Fourier spectral method is more accurate than the finite difference method for shock capturing.



(a)



(b)

Figure 9. Shock formation in velocity profiles for ν=0.005. (a) Fourier Spectral method results (b) central difference method results.

Oscillations generated in the numerical solution are called numerical scattering. This phenomenon appears for small values of the viscosity. It produces oscillations around the shock. Smaller amounts of ν (here ν <0.005) expands the amount of numerical error. Fourier Spectral method results for ν = 0.001 are presented in Fig. 10. The values of solution in ν = 0.001 for the central difference method are divergent.

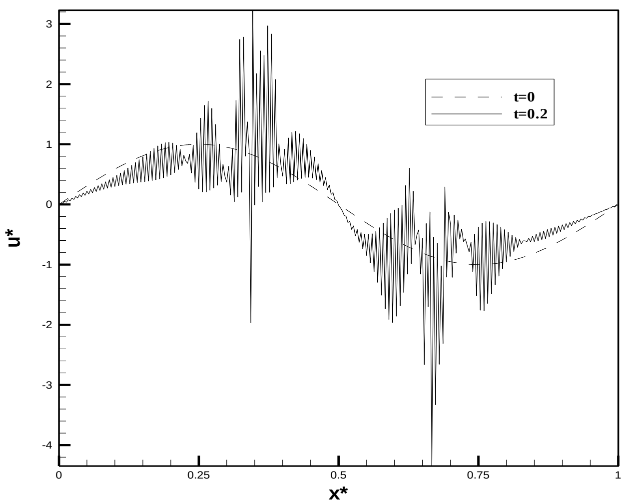
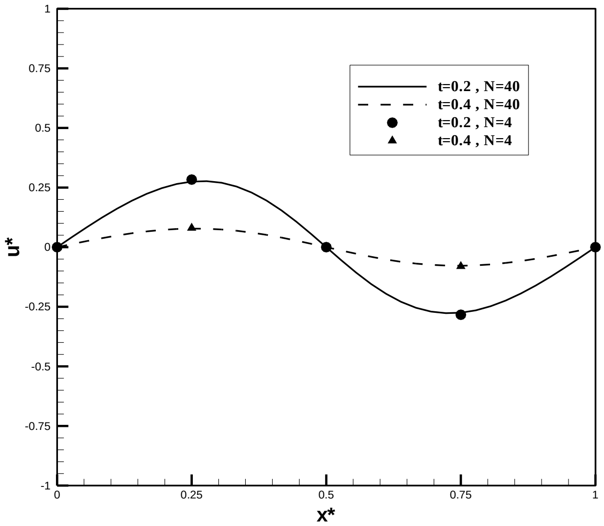
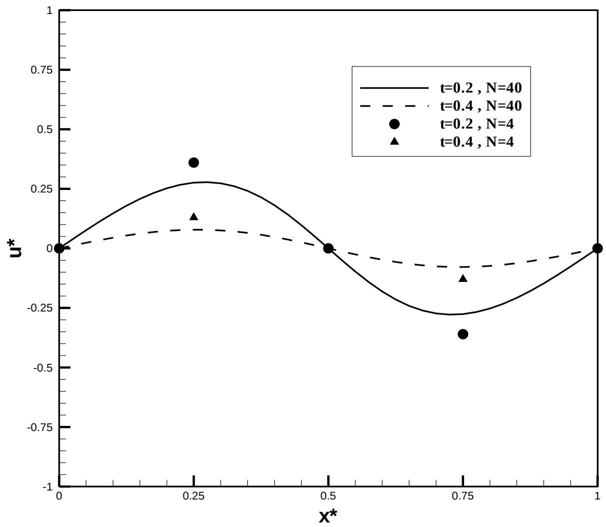


Figure 10. Numerical scattering in solution for ν=0.005 at t=0.02s.

As previously mentioned in the abstract, one of the advantages of the Fourier Spectral method is to capture periodic problems physics using a hard grid for the numerical solution. Fig. 11-(a) and 11-(b) show numerical results for number 4 and 40 nodes in the computational grid (N = 4, N=40) and ν=1, using spectral methods and the central difference method. As it can be seen, by reducing the number of nodes (hard grid), errors and changes occur in the solution of the central difference method. But these errors and changes are not observed in the results of the Fourier Spectral method. This is one of the main advantages of the Fourier Spectral method. Thus, for hard computational grids, the Fourier Spectral method is far more accurate than for the central differential method.



(a)



(b)

Figure 11. Dimensionless velocity profiles for hard computational grid (N=4) and fine computational grid (N=40) (a) Fourier Spectral method results, (b) central difference method results.

4. Conclusion:

In this paper, Stokes' second problem as linear problem and Burgers' equation as nonlinear problem is analysed by the Fourier spectral method. Also the results were compared with the BDF and FDM results. The most important conclusions of this paper are as follows:

The effect of viscosity is limited only during a short distance of (from) the plate.

The TSM (Time Spectral method) is an intense method in time-periodic unsteady flow analysis. So, that presents a proper accuracy of the solution and a low time for convergence. This method presents appropriate accuracy and needs low computational time for solving the periodic problems in comparison with typical traditional schemes like BDF.

For hard grids, the Fourier spectral algorithm presents a better solution than does a differential method. It means that an accurate solution can be reached with fewer nodes in a computational grid. Moreover, with a finer grid, there is not much change in the solution. On the other hand, more grid nodes must be used to achieve good results by differential methods and the finer the grid the more accurate the results, but computational costs increase. So, the Fourier spectral method has a higher convergence rate.

Viscosity causes depreciation in disorder created in flow, and with the passage of time the velocity of fluid flow tends toward zero. A reduction in viscosity can cause the appearance of sudden changes in velocity (shock formation).

At the shock location where changes of velocity are very high, the Fourier spectral method is more careful than the central difference method in numerical analysis of problem physics.

APPENDIX

Fourier Collocation Matrix.

This section studies the spectral Fourier derivative operator formulation and how to calculate it in physical space. The derivation is based on lecture notes of Moin [1].

Assume that f(x) is a function with a period of T defined on the grid,

$$x_j = \frac{Tj}{N} \quad \text{where } j=0,1,2,...,N\text{-}1$$

A.1 Even Formulation:

The discrete Fourier transform of f(x) is introduced by

$$\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f(x_j) e^{-ik\frac{2\pi}{T}x_j} \qquad \text{(A.1)}$$

and its inverse transform is as

$$f(x_j) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k e^{ik\frac{2\pi}{T}x_j}$$

The spectral derivative of f(x) at point j is given by

$$\left.\frac{df(x)}{dx}\right|_j = \frac{2\pi}{T}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} ik\hat{f}_k e^{ik\frac{2\pi}{T}x_j}$$

Substitute for $\hat{f}_k$ from equation (A.1), to obtain

$$\left.\frac{df(x)}{dx}\right|_n = \frac{1}{N}\frac{2\pi}{T}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1}\sum_{j=0}^{N-1} ikf(x_j) e^{-ik\frac{2\pi}{T}x_j} e^{ik\frac{2\pi}{T}x_n}$$

Note that:

$$x_j = \frac{Tj}{N}, \quad x_n = \frac{Tn}{N}$$

$$Df\big|_n = \frac{1}{N}\frac{2\pi}{T}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1}\sum_{j=0}^{N-1} ikf(x_j) e^{ik\frac{2\pi}{N}(n-j)}$$

Let:

$$d_n^j = \frac{1}{N}\frac{2\pi}{T}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} ike^{ik\frac{2\pi}{N}(n-j)} \qquad \text{(A.2)}$$

Then

$$Df\big|_n = \frac{2\pi}{T}\sum_{j=0}^{N-1} d_n^j f(x_j)$$

in which $d_n^j f(x_j)$ is a multiplication of a matrix at a vector where the elements of the matrix are d_n^j and the vector is $f(x_j)$. The elements d_n^j of the matrix can be computed analytically. First, the sum in equation (A.2) is computed. Let

$$s = \sum_{k=\frac{N}{2}+1}^{\frac{N}{2}-1} e^{ikx} = e^{i(-\frac{N}{2}+1)x} + e^{i(-\frac{N}{2}+2)x} + ... + e^{i(\frac{N}{2}-1)x}$$

using the geometric series, the summation can be evaluated as,

$$\begin{aligned} s &= e^{i(-\frac{N}{2}+1)x}\left[1+e^{ix}+e^{2ix}+...+e^{i(N-2)x}\right] \\ &= e^{i(-\frac{N}{2}+1)x}\frac{1-e^{i(N-1)x}}{1-e^{ix}} \end{aligned} \qquad \text{(A.3)}$$

Multiplying the numerator and denominator in equation (A.3) by $e^{-i\frac{x}{2}}$ term, to obtain

$$S = \frac{e^{i(-\frac{N}{2}+\frac{1}{2})x} - e^{i(\frac{N}{2}-\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} = \frac{\sin(\frac{N-1}{2}x)}{\sin\frac{x}{2}}$$

This expression can be differentiated to yield the desired sum,

$$\frac{dS}{dx} = \sum_{k=\frac{N}{2}+1}^{\frac{N}{2}-1} ike^{ikx} = \frac{\frac{N\text{-}1}{2}\cos(\frac{N\text{-}1}{2}x)\sin(\frac{x}{2}) - \frac{1}{2}\cos(\frac{x}{2})\sin(\frac{N\text{-}1}{2}x)}{(\sin\frac{x}{2})^2}$$

This expression can be simplified using trigonometric identities and noting that $x = \frac{2\pi}{N}(n-j)$:

$$\sin(\frac{N\text{-}1}{2}x) = \sin(\frac{N}{2}x - \frac{x}{2}) = -(-1)^{n-j}\sin(\frac{x}{2})$$

$$\cos(\frac{N\text{-}1}{2}x) = \cos(\frac{N}{2}x - \frac{x}{2}) = (-1)^{n-j}\cos(\frac{x}{2})$$

Therefore

$$\frac{dS}{dx}=\frac{\frac{N-1}{2}(-1)^{n-j}\cos(\frac{x}{2})\sin(\frac{x}{2})+\frac{1}{2}(-1)^{n-j}\sin(\frac{x}{2})\cos(\frac{x}{2})}{(\sin\frac{x}{2})^2}$$

Thus:

$$d_n^j=\begin{cases}\frac{1}{2}(-1)^{n-j}\cot(\frac{\pi(n-j)}{N}) & : n\neq j\\ 0 & : n=j\end{cases}$$

A.2 Odd Formulation:

For Odd Formulation, same calculation has been done. Finally

$$d_n^j=\begin{cases}\frac{1}{2}(-1)^{n-j}\operatorname{cosec}(\frac{\pi(n-j)}{N}) & : n\neq j\\ 0 & : n=j\end{cases}$$

Acknowledgements.

The author would like to acknowledge and offer appreciation to the Department of Mechanical Engineering, University of Torbat-e-Heydarieh which has supported this work.

References

[1] P. Moin, "Spectral Methods in Computational Physics,Supplementary notes," Stanford University, Stanford, CA, ME 408 2003.

[2] K. C. Hall, J. P. Thomas, and W. S. Clark, "Computation of Unsteady Nonlinear Flows in Cascades Using a Harmonic Balance Technique," *AIAA Journal*, vol. 40, pp. 879-886, 2002.
doi: 10.2514/2.1754

[3] M. McMullen, A. Jameson, and J. Alonso, "Application of a non-linear frequency domain solver to the Euler and Navier-Stokes equations," in *40th AIAA Aerospace Sciences Meeting & Exhibit*, *Aerospace Sciences Meetings*: American Institute of Aeronautics and Astronautics, 2002.
doi: 10.2514/6.2002-120

[4] M. McMullen, Jameson, A., "The Computational Efficiency of Non-linear Frequency Domain Methods," *Journal of Computational Physics*, vol. 212, pp. 637-661, 2006.
doi:10.1016/j.jcp.2005.07.021

[5] A. Gopinath and A. Jameson, "Time Spectral Method for Periodic Unsteady Computations over Two- and Three-Dimensional Bodies," in *43rd AIAA Aerospace Sciences Meeting and Exhibit*, *Aerospace Sciences Meetings*: American Institute of Aeronautics and Astronautics, 2005.
doi: 10.2514/6.2005-1220

[6] A. Gopinath and A. Jameson, "Application of the Time Spectral Method to Periodic Unsteady Vortex Shedding," in *44th AIAA Aerospace Sciences Meeting and Exhibit*, *Aerospace Sciences Meetings*: American Institute of Aeronautics and Astronautics, 2006.
doi:10.2514/6.2006-449

[7] M. Mohaghegh and M. Malek-Jafarian, "Periodic transonic flow simulation using fourier-based algorithm," *Journal of Mechanical Science and Technology*, vol. 28, pp. 4109-4119, 2014.
doi: 10.1007/s12206-014-0950-4

[8] G. G. Stokes, "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," *Cambridge Philos*, vol. 9, pp. 8-106, 1851.

[9] L. Rayleigh, "On the motional of solid bodies through viscous liquid," *Phil. Mag.*, vol. 21, pp. 697-711, 1911.

[10] M. E. Erdogan, "A note on an unsteady flow of a viscous fluid due to an oscillating plane wall," *Int. J. Non-Linear Mech*, pp. 1–6, 2000.
doi:10.1016/S0020-7462(99)00019-0

[11] K. Vajravelu and J. Rivera, "Hydromagnetic flow at an oscillating plate," *Int. J. Non-Linear Mech*, pp. 305–312, 2003.
doi:10.1016/S0020-7462(01)00063-4

[12] M. Devakar and T. K. V. Iyengar, "Stokes' Problems for an Incompressible Couple Stress Fluid Iyengar," *Nonlinear Analysis: Modelling and Control*, vol. 1, pp. 181–190, 2008.

[13] C. Fetecau, D. Vieru, and C. Fetecau, "A note on the second problem of Stokes for Newtonian fluids," *International Journal of Non-Linear Mechanics*, vol. 43, pp. 451-457, 2008.
doi:10.1016/j.ijnonlinmec.2007.12.022

[14] C. Fetecau, M. Jamil, C. Fetecau, and I. Siddique, "A note on the second problem of Stokes for Maxwell fluids," *International Journal of Non-Linear Mechanics*, vol. 44, pp. 1085-1090, 2009.
doi:10.1016/j.ijnonlinmec.2009.08.003

[15] S. Srinivasan and K. R. Rajagopal, "Study of a variant of Stokes' first and second problems for fluids with pressure dependent viscosities," *International Journal of Engineering Science*, vol. 47, pp. 1357-1366, 2009.
doi:10.1016/j.ijengsci.2008.11.002

[16] V. Prusa, "Revisiting Stokes first and second problems for fluids with pressure-dependent viscosities," *International Journal of Engineering Science*, vol. 48, pp. 2054-2065, 2010.
doi:10.1016/j.ijengsci.2010.04.009

[17] C.-M. Liu, "Complete Solutions to Extended Stokes' Problems," *Mathematical Problems in Engineering*, vol. 2008, pp. 18, 2008.
doi:10.1155/2008/754262

[18] G. Riccardi, "Remarks on the solution of extended Stokes' problems," *International Journal of Non-Linear Mechanics*, vol. 46, pp. 958-970, 2011.
doi:10.1016/j.ijnonlinmec.2011.04.010

[19] M. Massoudi and A. Vaidya, "Analytical solutions to Stokes-type flows of inhomogeneous fluids," *Applied Mathematics and Computation*, vol. 218, pp. 6314-6329, 2012.
doi:10.1016/j.amc.2011.11.110

[20] H. Bateman, "Some recent researches on the motion of fluids," *Weather Review Monthly* vol. 43 pp. 163–170, 1915.

[21] J. M. Burgers, "Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion," *Transactions on Royal Netherlands Academic Science, Amsterdam*, vol. 17, pp. 1–53, 1939.
doi:10.1007/978-94-011-0195-0_10

[22] J. M. Burgers, "A Mathematical Model Illustrating the Theory of Turbulence," in *Advances in Applied*

Mechanics, vol. Volume 1, K. Richard Von Mises and Theodore Von, Ed.: Elsevier, 1948, pp. 171-199. doi:10.1016/S0065-2156(08)70100-5

[23] E. Hopf, "The partial differential equation $u_t + uu_x = \mu u_{xx}$," *Communications on Pure and Applied Mathematics*, vol. 3, pp. 201–230, 1950. doi:10.1002/cpa.3160030302

[24] J. D. Cole, "On a Quasi-Linear Parabolic Equation Occurring in Aerodynamics," *Quarterly of Applied Mathematics*, vol. 9, pp. 225-236, 1951.

[25] S. Abbasbandy and M. T. Darvishi, "A numerical solution of Burger's equation by modified Adomian method," *Applied Mathematics and Computation*, vol. 163 pp. 1265–1272, 2005. doi:10.1016/j.amc.2004.04.061

[26] M. T. Darvishi and M. Javidi, "A numerical solution of Burger's equation by pseudospectral method and Darvishi's preconditioning," *Applied Mathematics and Computation*, vol. 173, pp. 421-429, 2006. doi:10.1016/j.amc.2005.04.079

[27] M. A. Abdou and A. A. Soliman, "Variational iteration method for solving Burger's and coupled Burger's equations," *J. Comput. Appl. Math.*, vol. 181, pp. 245-251, 2005. doi:10.1016/j.cam.2004.11.032

[28] Bülent Saka and İ. Dağ, "A numerical study of the Burgers equation," *Journal of the Franklin Institute*, vol. 345, pp. 328-348, 2008. doi:10.1016/j.jfranklin.2007.10.004

[29] S. Dhawan, S. Kumar, and S. Kapoor, "APPROXIMATION OF BURGERS' EQUATION USING B-SPLINE FINITE ELEMENT METHOD," *Int. J. of Appl. Math. and Mech*, vol. 7, pp. 61-86, 2011.

[30] S. Dhawan, S. Kapoor, S. Kumar, and S. Rawat, "Contemporary review of techniques for the solution of nonlinear Burgers equation," *Journal of Computational Science*, vol. 3, pp. 405-419, 2012. doi:10.1016/j.jocs.2012.06.003

[31] S. Dhawan, S. Kapoor, and S. Kumar, "Numerical method for advection diffusion equation using FEM and B-splines," *Journal of Computational Science*, vol. 3, pp. 429-437, 2012. doi:10.1016/j.jocs.2012.06.006

[32] H. Schlichting and K. Gersten, *Boundary Layer Theory*, 8th ed. Berlin: Springer, 2000.

[33] N. Tokuda, "On the impulsive motion of a flat plate in a viscous fluid," *J. Fluid Mech.*, vol. 33, pp. 657–672, 1968. doi: http://dx.doi.org/10.1017/S0022112068001606

[34] W. M. Lai, D. Rubin, and E. Krempl, *Introduction to continuum mechanics*, 3rd ed ed. Oxford ; New York . Pergamon Press, 1933. doi: 10.1002/zamm.19800600118

[35] S. W. Jameson A., Turkel E. January, "Numerical solutions of the Euler equations by finite volume methods with Runge-Kutta time stepping schemes," *AIAA paper*, pp. 81-1259, 1981. doi: 10.2514/6.1981-1259