Application of Doss Transformation in Approximation of Stochastic Differential Equations

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Abstract: The solution of the financial application, be it asset pricing, portfolio allocation or risk management, relies on the simulation of discretized versions of the stochastic differential equations(SDEs). The simplest way to confront SDEs in numerical situations is to discretize them and use monte carlo simulation. The Euler scheme is most often used for discretization of SDEs. This discretization involves an approximation error. In this topic at the first we recall an introduction to SDEs and Monte carlo simulation. Then, we study the asymptotic error distribution of Euler approximations of solutions of SDEs. We also study the error distribution associated with a Doss transformation of the state variables. Convergence results for Euler schemes with and without doss transformation and the comparison of them with Milshtein scheme are presented at the end.

Keywords: stochastic differential equations, Euler scheme, Doss transformation, Milshtein scheme, monte carlo estimator

1. Introduction

Suppose that one seeks to compute the conditional expection $f(t,s)=E_t(g(s_t))$, where X_T is the terminal value of the solution of the stochastic differential equation

$$ds_v=\mu(s_v)dv+\sigma(s_v)dB_v \quad ,S_t=s \tag{1}$$

To approximate the terminal value S_T, of the solution of (1), several discretization schemes can be used.1 The most popular, perhaps because of its ease of implementation, is the Euler scheme. This iterative procedure evaluates the drift and volatility functions at the value $S(t_n)$ at time t_n in order infer the value $S(t_{n+1})$ at t_{n+1}, and proceeds in this manner until $t_N=T$ .The second approximation is in the computation of the conditional expectation that is performed by averaging over a finite sample of approximated terminal values $S(T)$. Justification for this averaging rests on the law of large numbers. The combination of these two operations, labelled MCE (Monte Carlo with Euler discretization), produces an estimate of $f(t,s)$ that involves the two types of errors mentioned above. Understanding the trade-off between these errors requires the asymptotic error distribution. In an insightful paper, [6] highlighted the trade-off between the discretization error and the Monte Carlo averaging error. In this paper we study the asymptotic distributions of errors associated with discretization schemes for general diffusion processes and of Monte Carlo estimators of conditional expectations of diffusions. For the Euler discretization scheme the asymptotic error distribution was found by [11] and [9]. We extend their results by proposing a change of variables, commonly referred to as a Doss transformation (see [8]; [4]) that reduces the diffusion coefficient of the SDE to unity. This transformation has enjoyed recent popularity in financial econometrics (see, for instance, [1]; [7]) and has been used for the computation of optimal portfolios in dynamic asset allocation models ([3]). We show that a Doss transformation of the SDE can improve the speed of convergence of the discretization scheme.

This paper is organized as follows. In sections 2 and 3, we review SDE and Monte carlo simulation in briefly. In section 4 we recall Euler scheme. Section 5 introduces the Euler approximation with doss transformation. The asymptotic laws of estimators of conditional expectations are presented in section 6 and new asymptotic convergence results for the Milshtein scheme are presented in section 7.

2. An Introduction to Stochastic Differential Equations

If we allow for some randomness in some of the coefficients of a differential equation we often obtain a more realistic mathematical model of the situation.

Example 1. Consider the simple population growth model

$$\frac{dN}{dt}=a(t)N(t), \quad N(0)=N_0 \quad (\text{constant}) \tag{2}$$

Where $N(t)$ is the size of the population at time , and $a(t)$ is the relative rate of growth at time t. It might happen that $a(t)$ is not completely known, but subject to some random environmental effects, so that we have

$$a(t)=r(t)+\text{"}noise\text{"}$$

Where we do not know the exact behavior of the noise term, only its probability distribution. The function $r(t)$ is assumed to be nonrandom. How do we solve (2) in this case? Now, we consider general case of these equations where $S_t(\omega)$ is the solution of them and named the

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[1] A detailed analysis o discretization schemes available can be found in [10].

solution of stochastic differential equation

$$\frac{dS_t}{dt} = \mu(t, X_t) + \sigma(t, S_t)W_t, \quad \mu(t,S_t) \in R, \quad \sigma(t,S_t) \in R \qquad (3)$$

Where W_t is 1-dimensional "white noise". The Ito interpretation of (3) is that X_t satisfies the stochastic integral equation

$$S_t = S_\circ + \int_\circ^t \mu(S, X_s)ds + \int_\circ^t \sigma(S, x_s)dB_s$$

Or in differential form

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t \qquad (4)$$

Therefore, to get from (3) to (4) we formally just replace the white noise W_t by $\frac{dB_t}{dt}$ in (3) and multiply by dt. It is natural to ask:

Can one obtaion existence and uniqueness theorems for such equations?

How can one solve a given such equation?

It is the Ito formula that is the key to the solution of many stochastic differential equations.

Theorem1. Existence and uniqueness theorem for stochastic differential equations

Let $T > 0$ and $\mu(.,.) : [0,T] \times R^n \to R^n$, $\sigma(.,.) : [0,T] \times R^n \to R^{n \times m}$ be measurable functions satisfying

$$|\mu(t,s)| + |\sigma(t,s)| \le C(1 + |s|); \quad s \in R^n, t \in [0,T] \qquad (5)$$

For some constant C , (where $|\sigma|^2 = \Sigma |\sigma_{i,j}|^2$) and such that

$$|\mu(t,s) - b(t,y)| + |\sigma(t,s) - \sigma(t,y)| \le D|s - y|; \; s,y \in R^n, t \in [0,T] \quad (6)$$

For some constant D .Let Z be a random variable which is independent of the $\sigma - \text{algebra} F_\infty^{(m)}$ generated by $B_s(.), s \ge \cdot$ and such that $E[|Z|^2 < \infty.$

Then the stochastic differential equation

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t, \qquad \circ \le t \le T, S_\circ = Z \qquad (7)$$

has a unique t-continuous solution $S_t(\omega)$ with the property that $S_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(.); s \le t$ and

$$E[\int_0^T |S_t|^2 dt] < \infty \qquad (8)$$

(proof [15])

3. Stochastic differential equations and Monte Carlo methods

This method simulates the expected values E [g (X (T))] for a solution, X, of a stochastic differential equation with an unknown unknown function g we provide. In general bipartite approximation error, random error, and error is the discretization of the time. The statistical error is estimated based on the central limit theorem. Error estimates for the discretization error of the Euler method is directly associated with an approximate 1.2 times as high as the remaining extra careful around

Consider the following stochastic differential equation:

$$dX(t) = a(t, X(t)) + b(t, X(t))dW(t) \qquad (9)$$

The t_0≤t≤T, the value of E [g (X (T))] can be calculated by Monte Carlo Markov Chain we have:

$$E[g(X(T))] \cong \sum_{j=1}^{N} \frac{g(\bar{X}(T;\omega_j))}{N} \qquad (10)$$

Where X¯ an approximation of X is, the error in Monte Carlo method is:

$$E[g(X(T))] - \sum_{j=1}^{N} \frac{g(\bar{X}(T;\omega_j))}{N},$$

$$E[g(X(T)) - g(\bar{X}(T))] - \sum_{j=1}^{N} \frac{g(\bar{X}(T;\omega_j)) - E[g(\bar{X}(T))]}{N}$$

3.1. Simulating asset price dynamics

There are different possible models for asset price dynamics such as geometric Brownian motion, Ornstein-ahlenbeck, cox-ingersoll-ross and … . we show here the geometric Brownian motion model for the asset price $S(t)$, with drift μ and volatility σ:

$$dS = \mu S dt + \sigma S dz$$

Where dz is a standard Wiener process. An equivalent expression is

$$d \ln S = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz \qquad (11)$$

We also put $\upsilon = -\mu - \frac{\sigma^2}{2}$.

Equation (9) is particularly useful as it can be integrated exactly, yielding

$$S(t) = S(0)\exp(\upsilon t + \sigma \int_0^t dz)$$

To simulate the path of the asset price over an interval $(\circ, T)$, we must discretize time with a time step δt . From the last equation, and recalling the properties of the standard Wiener process (see [ ]) , we get

$$S(t + \delta t) = S(t)\exp(\upsilon \delta t + \sigma \sqrt{\delta t}\,\varepsilon) \qquad (12)$$

Where ε~N(0,1) is a standard normal random variable. Based on equation (10), it is easy to generate sample paths for the asset price.

```
%AssetPaths.m
function SPaths = AssetPaths(SO,mu,sigma,T,NSteps,NRep1)
SPaths = zeros(NRep1, 1+NSteps);
SPaths(:,1) = SO;
dt = T/ NSteps;
nudt = (mu-0/5*sigma^2)*dt;
sidt = sigma*sqrt(dt);
for i = 1;MRepl
    for j = 1:NSteps
        SPaths(i, j+1) = SPaths(i, j)*exp(nudt + sidt*randn);
   end
end
```

Fig1. Naive code to generate asset price paths by Monte carlo simulation

Due to the extent of contents we don't have opportunity to talk more about this subject. For more study see [2].

4. Euler Approximation without Transformation

To put the stage for the convergence results with the transformation and Milshtein scheme, we review known results for the standard Euler scheme.

Consider the $d\times 1$ random vector S_T given by the terminal value of the solution of the SDE.

$$dS_v = \mu(S_v)dv + \sum_{j=1}^{d}\sigma_j(S_v)dB_v^j$$

Where μ and σ_j are $d\times 1$ vectors such that $\in C^3(R^d)$, $\sigma_j \in C^3(R^d)$ and μ, σ_j are at most of linear growth.2 The Euler approximation of (11) is

$$S_T^N = S_0 + \sum_{n=0}^{N-1} A(S_{nh}^N)h + \sum_{n=0}^{N-1}\sum_{j=1}^{d}\sigma_j(S_{nh}^N)\Delta B_{nh}^j \qquad (14)$$

Where $h = \dfrac{T}{N}$ and $\Delta B_{nh}^j = B_{(n+1)h}^j - B_{nh}^j$.

Theorem 2. The approximation error $S_T^N - S_T$ converges weakly at the rate $\dfrac{1}{\sqrt{N}}$ (i.e. $(\sqrt{N}(S_T^N - S_T) \Rightarrow \phi_T^S)$). The asymptotic error is

$$\phi_T^S = -\frac{1}{\sqrt{2}}\gamma_T\int_0^T \gamma_v^{-1}\sum_{i,j=1}^{d}[\partial\sigma_j\sigma_t](S_v)dZ_v^{i,j} \qquad (15)$$

With $[Z^{i,j}]_{i,j\in\{1,\dots,d\}}$ a $d^2\times 1$ standard Brownian motion independent of , $\partial\sigma_j$ a $d\times d$ matrix of derivatives of B with respect to S and

[2] The space $C^k(R^d)$ denotes the space of κ times continuosly differentiable, R^d-valued functions.

$$\gamma_v = \varepsilon^R\left(\int_\circ^{\cdot}\partial\mu(S_s)ds + \sum_{j=1}^{d}\int_\circ^{\cdot}\partial\sigma_j(S_s)dB_s^j\right)_v \qquad (16)$$

(proof[5])

In this last expression $\partial\mu$ is the $d\times d$ matrix of derivatives of the vector μ with respect to the elements of S. Theorem 2 says the error converges in law at the rate $1/\sqrt{N}$ to the random variable φ_T^S, as the number of discretization points N becomes large. The asymptotic error φ_T^S depends on the coefficients of the SDE and their derivatives. Surprisingly, it also depends on new Brownian motions $([Z^{l,j}]_{l,j\in\{1,\dots,d\}})$, which are orthogonal to the original ones (w).

For instance, in the case of a CIR3 process

$$dS_v = K(\bar{S} - S_v)ds + \sigma\sqrt{S_v}\,dB_v$$

One finds that

$$\phi_T^S = -\frac{\sigma^2}{2\sqrt{2}}\gamma_T\int_0^T \gamma_v^{-1}dZ_v = Z_{\frac{\sigma^4}{8}\int_0^T \gamma_v^{-2}dv}$$

The law of the mixing random variable (13) $(\sigma^4/8)\gamma_T^2\int_\circ^T\gamma_v^{-2}$ is unknown as γ satisfies the equation $d\gamma_v = (-k\,dv + ((2\sigma/\sqrt{\ })_v / S_v$

where $\gamma_\circ = 1$, whose solution depends on the path S. The dependence of the asymptotic distribution on an independent Brownian motion, that does not exist on the original probability space, shows that it is not possible to approximate the distribution of the approximation error in a finite simulation experiment using a simulated benchmark for the true value S_T.

5. Euler Approximation with Doss Transformation

Let us first introduce the doss transformation. Consider the transformed volatility coefficient, $\sigma^{\hat\sigma}(x) \equiv \sigma(x)\hat\sigma^{-1}$,where $\hat\sigma$ is an arbitrary matrix of constants. Suppose that the rotated volatility coefficient $\sigma^{\hat\sigma}$ satisfies the rank condition,

$$\mathrm{rank}(\sigma^{\hat\sigma}) = d, \qquad \text{a.s.,}$$

And the commutativity condition,

$$\partial\sigma_j^{\hat\sigma}\sigma_i^{\hat\sigma} = \partial\sigma_i^{\hat\sigma}\sigma_j^{\hat\sigma} \quad \text{for all } i,j = 1,\dots,d. \qquad (19)$$

[3] Cox-Ingersoll-Ross

Then, there exists a function $G^{\hat{B}}: R^d \to R^d$ solution of the total ODE

$$\partial_Z G^{\hat{B}}(Z) = \sigma^{\hat{\sigma}}(G^{\hat{\sigma}}(Z)) \quad , \quad G^{\hat{\sigma}}(0) = 0 \qquad (20)$$

Such that $S_t = G^{\hat{\sigma}}(\hat{S}_t)$, where

$$d\hat{S}_v = \hat{\mu}(\hat{S}_v)dv + \sum_{j=1}^{d} \hat{\sigma}_j dB_v^j \text{ with } G^{\hat{B}}(\hat{S}_0) = S_0, \qquad (21)$$

And

$$\hat{\mu}(S) \equiv \sigma^{\hat{\sigma}}(S)^{-1} \mathcal{A}_t G^{\hat{\sigma}}(S) \qquad (22)$$

Where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} G^{\hat{\sigma}} \equiv A(G^{\hat{\sigma}}) - \frac{1}{2}\sum_{j=1}^{d} \partial\sigma_j^{\hat{\sigma}}(G^{\hat{\sigma}}) \qquad (23)$$

The Euler approximation of the transformed state variables satisfying (9) is

$$\hat{S}_T^N = \hat{S}_\circ + \sum_{n=\circ}^{N-1} \hat{\mu}(\hat{S}_{nh}^N)h + \sum_{n=\circ}^{n-1}\sum_{j=1}^{d} \hat{\sigma}_j \Delta B_{nh}^j$$

The error distribution of this approximation of the d-vector $\hat{S}_T$ is given next.

Theorem3. Suppose that the rank and commutativity conditions (16) and (17) are satisfied. The approximation error $\hat{S}_T^N - \hat{S}_T$ converges weakly at the rate $\frac{1}{N}$ (i.e. $(N(\hat{S}_T^N - \hat{S}_T) \Rightarrow \phi_T^{\hat{S}}$). The asymptotic error is

$$\phi_T^{\hat{S}} = -\hat{\gamma}_T \int_\circ^T \hat{\gamma}_v^{-1} \partial\hat{\mu}(\hat{S}_v)$$
$$\times(\frac{1}{2}d\hat{S}_v + \frac{1}{\sqrt{12}}\sum_{j=1}^{d}\hat{\sigma}_j dZ_v^j + \frac{1}{2}\sum_{j,k,\ell=1}^{d} \partial_{l,k}\hat{\mu}(\hat{S}_v)\hat{\sigma}_{k,j}\hat{\sigma}_{l,j}dv) \qquad (24)$$

With $\left[Z^j\right]_{j\in\{1,\dots,d\}}$ a $d\times 1$ standard Brownian motion independent of B and $Z^{h,j}$, $\partial\hat{\mu}(\hat{S}_v) = \left[\partial_1\hat{\mu}(\hat{S}_v)\right],\dots,d_d\hat{\mu}(\hat{S}_v)$ the $d\times d$ matrix with columns given by the derivatives of the vector $\hat{\mu}(\hat{S}_v)$,and $\partial_{l,k}\hat{\mu}(\hat{S}_s)$ the $d\times 1$ vector of cross derivatives of $\hat{\mu}(\hat{S}_v)$ with respect to arguments k,ℓ .

The $d\times d$ matrix $\hat{\gamma}_v$ is

$$\hat{\gamma}_v = \varepsilon^R(\int_0^\cdot \partial\hat{\mu}(\hat{S}_s)ds)_v \qquad (25)$$

(proof[5])

Theorem 3 shows that the speed of convergence increases after application of the transformation.

6. Asymptotic Laws of Estimators of Conditional Expectations

We now derive the asymptotic error of the estimate of the conditional expectation of a function of the terminal value of an SDE, S_T . When the distribution of S_T is unknown, an estimator of the expected value is obtained by sampling independent replications of the numerical solution of the SDE and averaging over the sampled values. The approximation error of this scheme has two components. The first is the error due to the discretization of the SDE. The second is the error in the approximation of the conditional expectation by a sampled average.

Section 6.1 presents our central result, namely the asymptotic error distributions associated with estimators of conditional expectations. Auxiliary results concerning the error componenet associated with the discretization scheme are described in section 6.2. The second-order biases of these estimators are discussed in section 6.3.

6.1. Asymptotic error distributions

Suppose that we wish to calculate

$$E\left[g(S_T)|F_0\right] = E_0\left[g(S_T)\right] = E_0\left[\hat{g}(\hat{S}_T)\right]$$

where S solves (1) or (8). The estimators with out and with transformation are

$$g^{N,M} \equiv \frac{1}{M}\sum_{i=1}^{M} g(S_T^{i,N}) \qquad (26)$$

$$\hat{g}^{N,M} \equiv \frac{1}{M}\sum_{i=1}^{M} \hat{g}(\hat{S}_T^{i,N}) \qquad (27)$$

These estimators of the conditional expectation draw independent replications $S_T^{i,N}(resp.\hat{S}_T^{i,N})$ of the terminal points $S_T^N (resp.\hat{S}_T^N)$ of the Euler discretized diffusion without (resp.with) doss transformation. Our next theorem describes their asymptotic laws.

Theorem4. Let $\in C^3(R^d)$, $\hat{g} \in C^1(R^d)$, and suppose that $g(X_T) \in D^{1,2}$ .4 Also suppose that the assumptions of theorems 1 and 2 hold. For the schemes without and with transformation, we have, as $M \to \infty$,

$$\sqrt{M}(g^{N_M,M} - E_0[g(S_T)]) \Rightarrow \varepsilon\frac{1}{2}K_T(S_0) + L_T(S_0) \qquad (28)$$

$$\sqrt{M}(\hat{g}^{N_M,M} - E_0[g(S_T)]) \Rightarrow \varepsilon\frac{1}{2}\hat{K}_T(S_0) + L_T(S_0) \qquad (29)$$

Where $\lim_{M\to\infty} N_M = +\infty$ and $\varepsilon = \lim_{M\to\infty} \sqrt{M}/N_M$.(see[5])

The theorem shows that the asymptotic laws of the estimators have two parts. The first, , corresponds to the

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[4] The space $D^{1,2}$ is defined as the domain of Malliavin derivative operator (see [14],[3])

discretization bias; the second, L , results from the montecarlo estimation of the expectation. Note that L would not vanish, even if samples where taken from the law of S_T . This is because the conditional expectation can not be calculated in closed form. The theorem also shows that the estimators converge at the same rate. This follows from the fact that the convergence rate of the expected approximation error, described in theorems 4 and 5 in the next section is the same.

6.2. Expected approximation errors

We now provide auxiliary results concerning the error component associated with the discretization scheme. Let $e_T^N\ (\hat{e}_T^N)$ be the expected approximation error for the scheme with out (with) transformation. By definition

$$e_T^N \equiv E_0\left[g(S_T^N)\right] - E_0\left[g(S_T)\right] \quad (30)$$

$$\hat{e}_T^N \equiv E_0\left[\hat{g}(\hat{S}_T^N)\right] - E_0\left[g(S_T)\right] \quad (31)$$

Where $g(S_T^N)$ is an approximation of $g(S_T)$ based on the Euler discretization of S , and $\hat{g}(\hat{S}_T^N)$ is an approximation of $g(S_T)$ based on the Euler discretization of the transformed state variables $\hat{S}$. Next we study the convergence properties of these errors.

6.3. Euler scheme on the original state variables

Our first result describes the convergence of the expected approximation error e_T^N in (28). Define the random variables

$$V_{1,T} \equiv -\gamma_T \int_\circ^T \gamma_s^{-1}(\partial\mu(S_s)dS_s + \sum_{j=1}^d [\partial\sigma_j \mu](S_s)db_s^j - \sum_{l,j=1}^d [\partial\sigma_j \partial\sigma_j \sigma_l](S_s)dB_s^l)$$
$$+\gamma_T \int_\circ^T \gamma_s^{-1}\left[\sum_{j=1}^d [\partial\sigma_j\partial\sigma_j\mu] + \sum_{j,k,l=1}^d [\partial_k(\partial_l \mu\sigma_{k,j})]\right](S_s)ds$$
$$+\gamma_T \int_\circ^T \gamma_s^{-1} \sum_{i,j=1}^d ([\partial[\partial\sigma_j\partial\sigma_j\sigma_i]\sigma_i - \partial\sigma_i\partial\sigma_j\partial\sigma_j\sigma_i](S_s)ds \quad (32)$$

$$V_{2,T} \equiv -\int_\circ^t \sum_{i,j=1}^d \nu_{i,j}(s,T)ds \quad (33)$$

Where $\partial\mu$, $\partial\sigma_j$ are $d\times d$ matrices of Jacobians, γ is defined in theorem 1 and $\nu_{i,j}(s,T)$ in [5]. With this notation we have:

Theorem 5. Suppose that, $\sigma_j \in C^2(R^2)$. Let $g\in C^3(R^d)$ be such that

$$\lim_{r\to\infty}\lim_N SUP\ E_0\left[1_{\{|N(g(S_T^N)-g(S_T))|>r\}} |\ N(g(S_T^N)-g(S_T)|\right]=0 \quad (34)$$

(P-a.s). Then,

$$Ne_T^N \to \frac{1}{2}K_T(S_0) \equiv \frac{1}{2}E_0\left[\partial g(S_T)V_{1,T} + V_{2,T}\right] \quad (35)$$

Where $V_{1,T}, V_{2,T}$ are given by (30) and (31), and e_T^N is defined in (28). (see [5])

Theorem 4 provides a probabilistic characterization of the asymptotic expected error. The expressions in (32) depend on random variables V_1 and V_2 that are determined in closed form by the derivatives of the drift and volatility coefficients of the SDE.

Implementation, in numerous applications, requires the computation of conditional expectations of path-dependent functionals of diffusion processes, such as the Riemann integral $\int_\circ^T g(S_s)ds$ .A convergent estimator for this integral, based on the Euler scheme, is $\sum_{n=\circ}^{N-1} g(S_{nh})h$, where $h = T/N$ and S^N is the solution of the Euler-discretized SDE starting at $S_\circ$. Theorem 5 can be used to deduce the asymptotic expected approximation error in these cases.

6.4. Euler scheme on the transformed state variables

To derive the expected approximation error for the estimator with transformation define the random variable

$$\hat{V}_T = -\hat{\gamma}_T \int_0^T \hat{\gamma}_v^{-1}\partial\hat{\mu}(\hat{S}_v)(d\hat{S}_v + \sum_{j,k,\ell=1}^d \partial_{l,k}\ \hat{\mu}(\hat{S}_v)\hat{\sigma}_{K,j}\hat{\sigma}_{\ell,j}dv) \quad (36)$$

With $\hat{\gamma}_T = \varepsilon^R(\int_\circ^\nu \partial\hat{\mu}(\hat{S}_s)ds$. we obtain:

Theorem 6. Suppose that $\hat{\mu}\in C^{-1}(R^d)$, and that the conditions of theorem 3 hold. For $\hat{g}\in C^{-1}(R^d)$ such that:

$$\lim_{r\to\infty}\lim_N SUP\ E_0[1_{\{|N(\hat{g}(\hat{S}_T)-\hat{g}(\hat{S}_T))|>r\}} N\ |\ \hat{g}(\hat{S}_T^N)-\hat{g}(\hat{S}_T)|]=0 \quad (37)$$

P-a.s. we have, P-a.s., as $N\to\infty$,

$$N\hat{e}_T^N \to \frac{1}{2}\hat{K}_T(S_0) \equiv \frac{1}{2}E_0[\partial\hat{g}(\hat{S}_T)\hat{V}_T] \quad (38)$$

Where $\hat{V}_T$ is defined in (34), and $\hat{e}_t^N$ is defined in (29). (proof[5])

A comparison of (33) with (33) suggests that it will be difficult, in general, to establish the dominance of one method over the other on the basis of the asymptotic expected error. Indeed, the formulas reveal that both methods converge at the same speed $1/N$, and for comparing two procedures we shoud use other criteria such as the computational cost.

7. A Comparison with Milshtein's second-order Approximation

While Euler schemes for SDEs are appealing from a computational point of view, they might be judged insufficiently accurate. Second-order schemes such as

Milshtein's scheme (see [12],[13],[16],[17],[18]) have in fact been proposed to provide improved approximations.

The Milshtein approximation of S_T in (11) is

$$\tilde{S}_T^N = S_\circ + \sum_{n=0}^{N-1}(\mu(\tilde{S}_{nh}^N)h + \sum_{j=1}^{d}\sigma_j(\tilde{S}_{nh}^N)\Delta B_{nh}^j + \sum_{j,\ell=1}^{d}\left[\partial\sigma_l\sigma_j\right](\tilde{S}_{nh}^N)\Delta F^{\ell j}) \quad (39)$$

Where

$$\Delta F^{l,j} \equiv \int_{nh}^{(n+1)h}\int_{nh}^{s} dB_v^l dB_s^i \quad (40)$$

With $h = \dfrac{T}{N}$ and $\Delta B_{nh}^j = B_{(n+1)h}^j - B_{nh}^j$. This scheme is obtained using a stochastic Taylor expansion for the diffusion coefficient. Our next result describes the asymptotic distribution of the approximation error associated with (37). It will enable us to find an explicit expression for the Monte carlo estimator of a conditional expectation based in the Milshtein scheme.

Theorem 7. The approximation error $\tilde{S}_T^N - S_T$ converges weakly at the rate $\dfrac{1}{N}$. The asymptotic error is

$$\begin{aligned}\tilde{\varphi}_T^S = &-\frac{1}{2}\gamma_T\int_\circ^T \gamma_s^{-1}(\partial\mu(S_s)dS_s - \sum_{j=1}^{d}[(\partial\sigma_j)(\partial\mu)\sigma_j](S_s)ds)\\ &-\frac{1}{2}\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{j=1}^{d}[(\partial[(\partial\mu)\sigma_j])\sigma_j - (\partial\sigma_j)(\partial\sigma_j)\mu](S_s)ds\\ &-\frac{1}{2}\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{j=1}^{d}[(\partial\sigma_j)\mu](S_s)dB_s^i\\ &-\frac{1}{\sqrt{12}}\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{j=1}^{d}[(\partial\mu)\sigma_j - (\partial\sigma_j)\mu](S_s)dZ_s^j\\ &-\frac{1}{\sqrt{6}}\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{i,l,j=1}^{d}[\partial\sigma_i\partial\sigma_l\sigma_j](S_s)d\tilde{Z}_s^{l,j,i}\end{aligned} \quad (41)$$

Where the process $((Z^j)_{j\in\{1,\dots,d\}}, (\tilde{Z}^{l,j,i})_{i,l,j=1,\dots,d})$ is a $d + d^3 \times 1$ standard Brownian motion independent of W. The process γ_T is given in Theorem 2. (proof[5])

Theorem 7 shows that the speed of convergence increases when one uses the stochastic Taylor of the diffusion term. It follows that the error for the Milshtein scheme converges at the same speed as the error for the Euler scheme with transformation.

The asymptotic error distribution of estimators of conditional expectations is described next.

Theorem 8. Suppose that the assumption of theorem 9 below hold. Let $g \in C^{-1}(R^d)$ and suppose that $g(S_T) \in D^{1,2}$. For the Milshtein scheme, we have, as $M \to \infty$,

$$\sqrt{M}\,(\frac{1}{M}\sum_{i=1}^{M} g(\tilde{S}_T^{i,N_M}) - E_0[g(S_T)] \Rightarrow \varepsilon\frac{1}{2}\tilde{K}_T(S_0) + L_T(S_\circ) \quad (42)$$

Where $\lim_{M\to\infty} N_M = +\infty$ and $\varepsilon = \lim_{M\to\infty}\dfrac{\sqrt{M}}{N_M}$. The deterministic function $\tilde{K}_T$ is defined in theorem9. (proof [5])

For estimates of conditional expectations the rate of convergence of the Milshtein scheme is identical to the rates of the Euler schemes with and with out transformation. The three schemes differ only in their second-order biases. The second-order bias $\tilde{K}$ can be found explicity, as shown in theorem 9 below. It size relative to the second-order biases of Euler schemes depends on the coefficients of the underlying processes . A global ordering of the three schemes in terms of second-order asymptotic properties is not readily apparent.

As for the estimator with transformation, the expected approximation error

$$\tilde{e}_t^N \equiv E_\circ[g(\tilde{S}_T^N) - g(S_T)]$$

Can be directly deduced from theorem 7 under an additional uniform integrability condition. In order to do this, define the random variable

$$\begin{aligned}\tilde{V}_T \equiv &-\gamma_T\int_\circ^T \gamma_s^{-1}(\partial\mu(S_s)dS_s - \sum_{j=1}^{d}[(\partial\sigma_j)(\partial\mu)\sigma_j](S_s)ds\\ &-\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{j=1}^{d}[(\partial[(\partial\mu)\sigma_j])\sigma_j] - (\partial\sigma_j)(\partial\sigma_j)\mu(S_s)ds\\ &+\gamma_T\int_\circ^T \gamma_s^{-1}\sum_{j=1}^{d}[(\partial\sigma_j)\mu](S_s)dB_s^j\end{aligned} \quad (44)$$

The equivalent of theorems 5 and 6, is

Theorem 9. For $g \in C^1(R^d)$ such that

$$\lim_{r\to\infty}\lim\ \sup_N\ E_\circ[1_{\{|N(g(\tilde{S}_T^N)-g(S_T))|>r\}}N\left|g(\tilde{S}_T^N) - g(S_T)\right|] = 0 \quad (45)$$

We have, P-a.s , as

$$N\tilde{e}_T^N \to \frac{1}{2}\tilde{K}_T(S_0) \equiv \frac{1}{2}E_0\left[\partial g(S_T)\tilde{V}_T\right] \quad (46)$$

Where $\tilde{V}_T$ is defined in (42) and $\tilde{e}_T^N$ is defined in (41). (see [5])

A comparison of (33), (34) and (44), showes that these three methods converge at the same speed $1/N$.

8. Conclusion

- We saw that the doss transformation of SDE can develop the speed of convergence of discretization.
- We introduced second-order Milshtein scheme which is used for bias-reduction and we provided it's asymptotic error distribution, showed that it could not dominate Euler scheme with transformation with respect to its convergence behavior.
- Also we saw that for the conditional expectation estimators the rate of convergence of Milshtein scheme is the same as Euler schemes with or without transformation.

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