Entrance Region Flow of Casson Fluid in an Annular Cylinder

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***Abstract*:** The entrance region flow in channels constitutes a problem of fundamental interest in engineering applications such as nuclear reactors, polymer processing industries, haemodialyzers and capillary membrane oxygenators. In such installations, the behavior of the fluid in the entrance region may play a significant part in the total length of the channel and the pressure drop may be markedly greater than for the case where the flow is regarded as fully developed throughout the channel. Recently, there has been an increasing interest in problems involving materials with variable viscosity such as Bingham materials, Casson fluids and Hershel-Bulkley fluids which are characterized by an yield value. The entrance region flow of a Casson fluid in an annular cylinder has been investigated numerically without making prior assumptions on the form of velocity profile within the boundary layer region. This velocity distribution is determined as part of the procedure by cross sectional integration of the momentum differential equation for a given distance z from the channel entrance. Using the macroscopic mass and momentum balance equation the entrance length has been obtained at each cross section of the entrance region of the annuli for specific values of Casson Number and various value of aspect ratio. The effects of non-Newtonian characteristics and channel width on the velocity profile, pressure distribution and the entrance length have been discussed.

***Keywords*:** Entrance Region Flow, Non-Newtonian Fluids, Casson Fluid, Boundary Layer Region, Momentum Integral Equation.

1. Introduction

The entrance region flow in channels constitutes a problem of fundamental interest in engineering applications such as nuclear reactors, polymer processing industries, haemodialyzers and capillary membrane oxygenators. In such installations, the behavior of the fluid in the entrance region may play a significant part in the total length of the channel and the pressure drop may be markedly greater than for the case where the flow is regarded as fully developed throughout the channel. Recently, there has been an increasing interest in problems involving materials with variable viscosity such as Bingham materials, Casson fluids and Hershel-Bulkley fluids which are characterized by an yield value. Various authors [1-7] have studied the entrance region flow of such non-Newtonian fluids in different geometries. Recently, Kandasamy and Pai [8] have investigated the core variation in the entrance region flow of a Casson fluid in an annuli.

The purpose of the present work is to analyze numerically the entrance region flow of a Casson fluid through annular cylinder without making assumptions in the form of velocity profile within the boundarylayer. The velocity distribution is determined as part of the procedure by cross-sectional integration of the momentum differential equation for a given distance z from the channel entrance. Using macroscopic mass and momentum balances, the change in the velocity profile and pressure gradient downstream arecalculated numerically. Considering the fluid flow being symmetric about the axis of the channel, the entrance length, velocitydistribution and the pressure drop have been calculated in the upper portion of the annuli at each cross section of the entrance region of the channel for various values of Casson number and different values of aspect ratio.

2. Analysis

We are analyzing the entrance region flow of a Casson fluid through an annular cylinder without making assumptions in the form of velocity profile within the boundary layer. Fluid enters a horizontal annular duct from a large chamber with an uniform velocity along the axial direction. The analysis has been carried out over the wide range of aspect ratios, that is, the ratio of the radius of the inner cylinder to that of the outer cylinder. The development of boundary layer is visualized when the fluid enters an annulus and the fully developed velocity profile is observed in the region starting from the point down - stream where the boundary layers meet asymptotically with the outer edge of the plug flow zone.

We consider a horizontal annular duct consisting of inner cylinder of radius r_1 and outer cylinder of radius r_2. The Casson fluid enters from a large chamber to this duct with a steady, laminar, incompressible and isothermal flow of velocity v_0. We use cylindrical polar coordinates system (r, θ, z) with axial symmetry and origin at the center of the cylinders at the inlet with z axis coinciding with the axis of the cylinders. v_r is the velocity component in r direction, v_z is the velocity in the z direction. Since,Casson fluid possesses an yield value, there is a plug core formation away from the walls. On each wall there is boundary layerformationseparated by the core. As the two regions about the axis are symmetrical, the solution of the problem will be considered in the upper half only. We have inner boundary layer region $r_1 \le r \le r_1 + \delta_1$ with thickness $\delta_1(z)$ and outer boundary layer region $r_2 - \delta_2 \le r \le r_2$ with thickness $\delta_2(z)$ and the plug flow region

$r_1 + \delta_1 \le r \le r_2 - \delta_2$ separating the two boundary layers, where the core is moving with constant velocity. In the plug core region, the shear stress τ is less than or equal to the yield stress τ_0 and the velocity at each cross section is constant. The geometry of the problem is shown in Figure1.

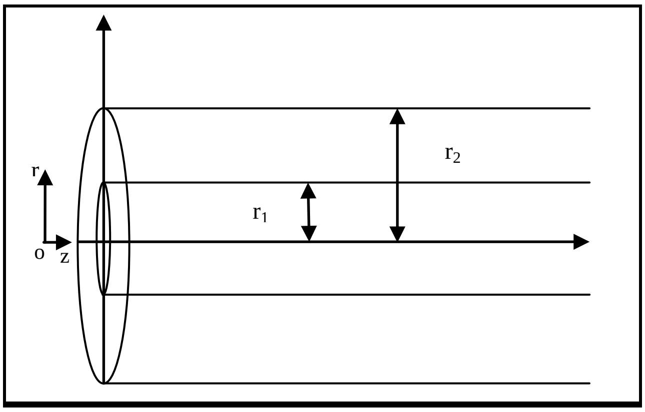

Figure 1. Geometry of an annuli

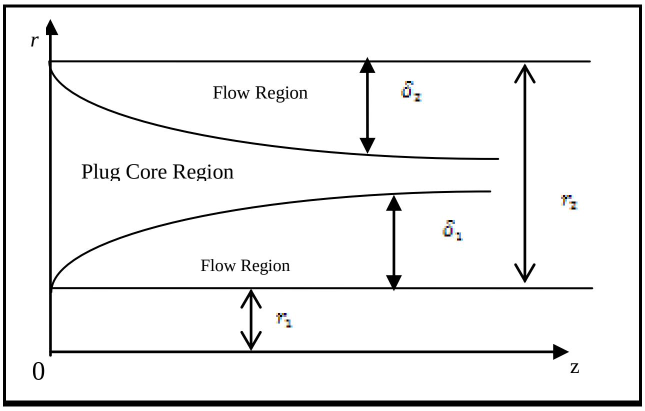

Figure 2. Fluid flow in various regions in an annuli

Under the assumptions of boundary layer theory and neglecting inertia terms, the equation of motion of an isotropic, incompressible Casson fluid can be given as

$$\frac{1}{r}\frac{\partial(\tau r)}{\partial r} = -\frac{dp}{dz} \qquad (1)$$

where 'p' is the pressure of the fluid.

The constitutive equation of a Casson fluid is given by

$$|\tau|^{1/2} = \tau_0^{1/2} + k_c\left|\frac{\partial v_z}{\partial r}\right|^{1/2}, \qquad |\tau| \ge \tau_0 \qquad (2)$$

Where k_C^2 represents Casson's viscosity.

Equations (1) and (2) will be solved under thefollowing boundary conditions:

i) The axial velocity components are zero at the wall, i.e.,

$$v_z(z,r_1) = 0\ ,\ v_z(z,r_2) = 0 \qquad (3a)$$

ii) At the edge of the boundary layers, v_z is equal to the plugcore velocity, i.e.,

$$v_z(z, r_1+\delta_1) = v_{zc_1}\ ,\ v_z(z, r_2-\delta_2) = v_{zc_2} \qquad (3b)$$

iii) The shear stress at the edge of the lower and upper boundary of the plug core region are equal to $-\tau_0$ and τ_0 respectively. Hence Eq. (2) reduces to

$$\left(\frac{\partial v_z}{\partial r}\right)_{r=r_1+\delta_1} = 0,\quad \left(\frac{\partial v_z}{\partial r}\right)_{r=r_2-\delta_2} = 0 \qquad (3c)$$

In other words, the velocity profile is assumed to be "smooth" at the edge of both regions. At the walls of the annular boundary, the axial velocity components are zero. Using equations (1),(2) and conditions (3a), (3b), (3c) we obtain the velocity profile in various regions in the non – dimensional form as

$$V_1 = \left[\begin{array}{l} \left(\frac{\varepsilon^2}{16}-\frac{R^2}{16}\right)\frac{dP}{dZ} - \frac{1}{8}\frac{dP}{dZ}(\varepsilon+\delta_{C_1})^2 \log\left(\frac{R}{\varepsilon}\right) \\ + N(\varepsilon+\delta_{C_1})\log\left(\frac{R}{\varepsilon}\right) + N(R-\varepsilon) \\ -4N\left(R^{1/2}-\varepsilon^{1/2}\right)\left(-\frac{1}{8N}\frac{dP}{dZ}(\varepsilon+\delta_{C_1})^2 + (\varepsilon+\delta_{C_1})\right)^{1/2} \\ -\frac{1}{20}\left(R^{5/2}-\varepsilon^{5/2}\right)\frac{dp}{dz}\left(-\frac{1}{8N}\frac{dP}{dZ}(\varepsilon+\delta_{C_1})^2 + (\varepsilon+\delta_{C_1})\right)^{-1/2} \end{array}\right], \quad \varepsilon \le R \le r_{C1} \qquad (4)$$

$$V_2 = -\left[\begin{array}{l} \left(\frac{1}{16}-\frac{R^2}{16}\right)\frac{dP}{dZ} + \frac{1}{8}\frac{dP}{dZ}(1-\delta_{C_2})^2 \log R \\ + N(1-\delta_{C_2})\log R + N(R-1) \\ -4N\left(R^{1/2}-1\right)\left(\frac{1}{8N}\frac{dP}{dZ}(1-\delta_{C_2})^2 + (1-\delta_{C_2})\right)^{1/2} \\ +\frac{1}{20}\left(R^{5/2}-1\right)\frac{dP}{dZ}\left(\frac{1}{8N}\frac{dP}{dZ}(1-\delta_{C_2})^2 + (1-\delta_{C_2})\right)^{-1/2} \end{array}\right], \quad r_{C_2} \le R \le 1 \qquad (5)$$

$$V_{c2} = -\left[\begin{array}{l} \left(\frac{1}{16}-\frac{(1-\delta_{C_2})^2}{16}\right)\frac{dP}{dZ} + \frac{1}{8}\frac{dP}{dZ}(1-\delta_{C_2})^2 \log(1-\delta_{C_2}) \\ + N(1-\delta_{C_2})\log(1-\delta_{C_2}) + N((1-\delta_{C_2})-1) \\ -4N\left((1-\delta_{C_2})^{1/2}-1\right)\left(\frac{1}{8N}\frac{dP}{dZ}(1-\delta_{C_2})^2 + (1-\delta_{C_2})\right)^{1/2} \\ +\frac{1}{20}\left((1-\delta_{C_2})^{5/2}-1\right)\frac{dP}{dZ}\left(\frac{1}{8N}\frac{dP}{dZ}(1-\delta_{C_2})^2 + (1-\delta_{C_2})\right)^{-1/2} \end{array}\right], \quad r_{C_1} \le R \le r_{C_2} \qquad (6)$$

Where the non – dimensional quantities are

$$V_1 = \frac{v_{z_1}}{v_0},\ V_2 = \frac{v_{z_2}}{v_0},\ P = \frac{p}{\rho v_0^2},\ R = \frac{r}{r_2},$$

$$V_{C_1} = \frac{v_{z_{c_1}}}{v_0},\ V_{C_2} = \frac{v_{z_{c_2}}}{v_0},\ Z = \frac{z\,k_c^2}{4 v_0 r_2^2 \rho},$$

$$\varepsilon = \frac{r_1}{r_2} = \text{aspect ratio},\ r_{C_1} = \frac{r_1+\delta_1}{r_2} = \varepsilon + \delta_{C_1},$$

$$r_{C_2} = \frac{r_2-\delta_2}{r_2} = 1-\delta_{C_2},\ N = \frac{\tau_0 r_2}{k_c^2 v_0} = \text{Casson number} \qquad (7)$$

The macroscopic mass and momentum balance equations are given by

$$2\int_0^1 V\,R\,dR = 1 \tag{8}$$

$$\frac{d}{dZ}\int_0^1 V^2 R dR + \frac{dP}{dZ}\int_0^1 R dR + 4\left[N + \left|\frac{\partial V}{\partial R}\right| + 2N^{1/2}\left|\frac{\partial V}{\partial R}\right|^{1/2}\right]_{R=1} = 0 \tag{9}$$

Using the velocity expressions in the equations (8) and (9), the momentum balance equation in an algebraic form is given by

$$\frac{dF(r_{C_2})}{dZ} = G(r_{C_2}) \tag{10}$$

where

$$F(r_{C_2}) = \int_{\varepsilon}^{r_{C_1}} V_1^2 R dR + \int_{r_{C_1}}^{r_{C_2}} V_{c2}^2 R dR + \int_{r_{C_2}}^{1} V_2^2 R dR \tag{11}$$

$$G(r_{C_2}) = -\left\{\begin{aligned} &\frac{1}{2}\frac{dP}{dZ} + 4N + 4\left[-\frac{1}{8}\frac{dP}{dZ} + N r_{C_2} + \frac{r_{C_2}^2}{8}\frac{dP}{dZ} + N\right] \\ &+4\left[-\frac{N}{2}\left(r_{C_2} + \frac{r_{C_2}^2}{8N}\frac{dP}{dZ}\right)^{1/2} + \frac{1}{8}\left(r_{C_2} + \frac{r_{C_2}^2}{8N}\frac{dP}{dZ}\right)^{-1/2}\right] \\ &+8N^{1/2}\left[-\frac{1}{8}\frac{dP}{dZ} + N r_{C_2} + \frac{r_{C_2}^2}{8}\frac{dP}{dZ} + N\right]^{1/2} \\ &+8N^{1/2}\left[-\frac{N}{2}\left(r_{c2} + \frac{r_{C_2}^2}{8N}\frac{dP}{dZ}\right)^{1/2} + \frac{1}{8}\left(r_{c2} + \frac{r_{C_2}^2}{8N}\frac{dP}{dZ}\right)^{-1/2}\right]^{1/2} \end{aligned}\right\} \tag{12}$$

Rewriting Eq. (10) can be written as

$$dZ = \frac{\dfrac{d\,F(r_{C_2})}{d\,r_{C_2}}}{G(r_{C_2})}\,dr_{C_2} \tag{13}$$

Integrating the above equation w.r.t r_{C_2} we get

$$Z(r_{C_2}) = \int_{0.99}^{r_{C_2}} \frac{\dfrac{d}{dr_{C_2}}F(r_{C_2})}{G(r_{C_2})}\,dr_{C_2} \tag{14}$$

The solution of Eq. (14) gives the relation between the core thickness and the dimensionless axial distance Z. Here, $\left(-\frac{dr_{C_2}}{dZ}\right)_{r_{C_2}=1}$ is not defined. Hence, without making significant error, the lower limit of the integration of Eq.(14) corresponding to the value Z= ΔZ =0 may be assumed to have the value r_{C_2} = 0.99 instead of 1, and the upper limit of the integration corresponds to the dimensionless core thickness r_{C_2} . Eq. (14) can be integrated numerically and the variation of entrance length along the dimensionless axial distance can be obtained for different values of Casson number and aspect ratios. From Eq.(8), we can say that the pressure gradient is a function of r_{C_2} , i.e. $\frac{dP}{dZ} = f(r_{c_2})$. Then, using the condition that the pressure is constant and equal to P_0 at the entrance, the pressure drop can be expressed as

$$\Delta P = \int_{0.99}^{r_{C_2}} f(r_{C_2}) \frac{\dfrac{d}{dr_{C_2}}F(r_{C_2})}{G(r_{C_2})}\,dr_{C_2} \tag{15}$$

Again, the above equation can be numerically integrated for various values of Casson number and aspect ratios for the pressure drop.

3. Results and Discussion

The solution of the entrance region flow of a Casson fluid through an annular cylinder has been investigated numerically without prior assumptions on the form of the velocity profile in the developing boundary layer. Initially, a nonlinear algebraic equation for determining the pressure gradient as a function of core thicknesshas been derived using the mass balance equation.This equation has been solved using an iterative procedure to obtain dP/dZ numerically. Using the values of pressure gradient so obtained, in the momentum balance equation,we obtain the entrance length for various values of Cassonnumber and aspect ratios of the annuli. The entrance length has been defined as the distance at which 99 percent of fully developed velocity is reached.Figures 3 to 6 show graphically the variation of entrance length along the axial distance for various Casson numbers when the annuli aspect ratio $\varepsilon = 0.4$ and $\varepsilon = 0.6$. It is observed that at any cross section Z, the entrance length decreases with the increase in aspect ratio of the annuli. Further, it is noted that as the Casson number increases, the value of entrance length diminishes, that is, thick viscous fluids attains the fully developed state at early stage compared to that of Newtonian fluid like water. Expressing pressure gradient as a function of r_{C_2} and integrating from 0.99 to fully developed value, the pressure drop along the axial distance has been determined for different values of Casson number and aspect ratios. Figures 7 to10 show the variation of pressure drop for different values of Casson number when the annuli aspect ratio $\varepsilon = 0.4$ and $\varepsilon = 0.6$. We observe that at any cross section, pressure drop is greater for materials with larger values of Casson Number for a particular aspect ratio. For a particular value of Casson number and at a particular cross section pressure drop decreases with the increase of aspect ratio.The velocity profile along the radial direction is shown in Figures 11 to 14. It is found that with the increase in Casson number, the velocity decreases and its profile takes parabolic form for smaller values of Casson number which can be approximated to that of Newtonian fluid.

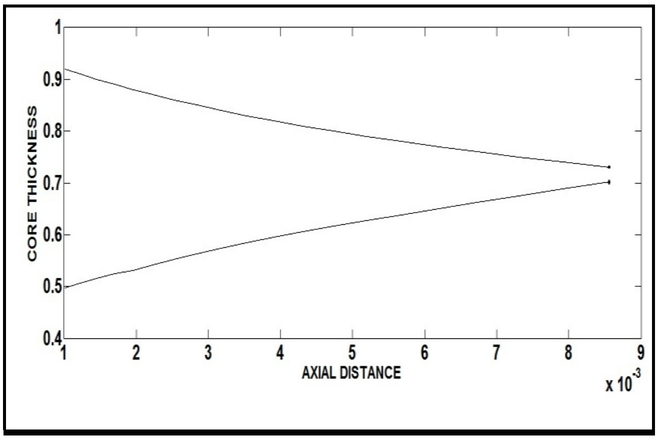

Figure 3. The variation of entrance length with core thickness for N = 0.06, ε = 0.4

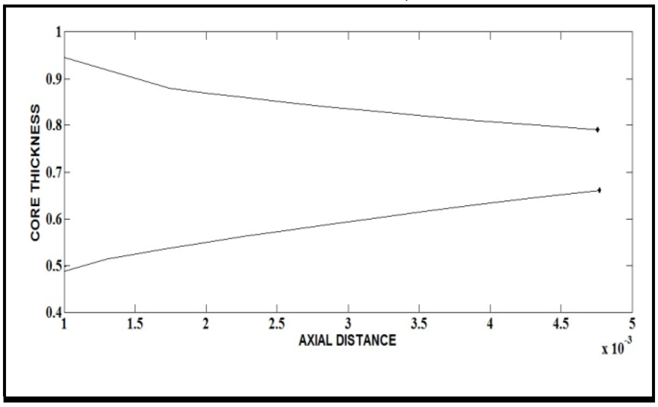

Figure 4. The variation of entrance length with core thickness for N = 0.1, ε = 0.4

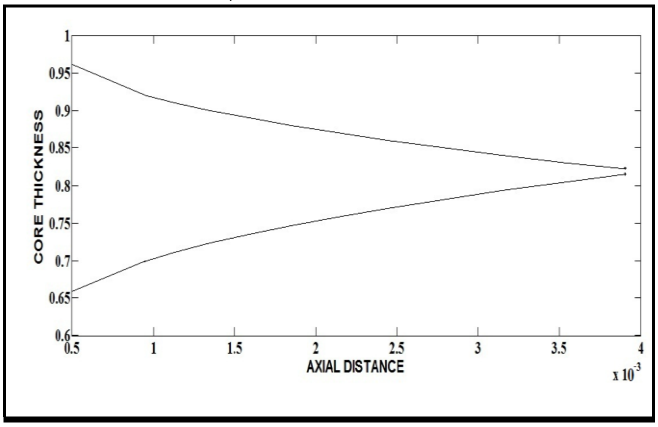

Figure 5. The variation of entrance length with core thickness for N = 0.06, ε = 0.6

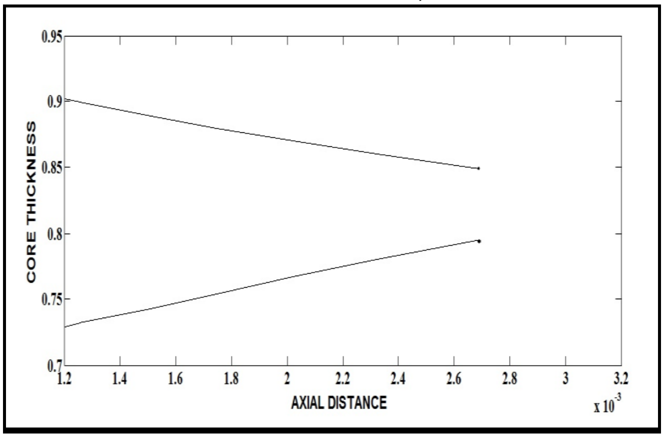

Figure 6. The variation of entrance length with core thickness for N = 0.08, ε = 0.6

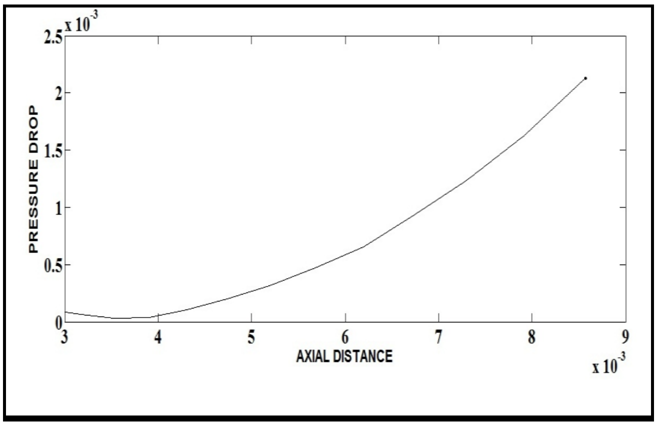

Figure 7. The variation of pressure drop with core thickness for N = 0.06, ε = 0.4

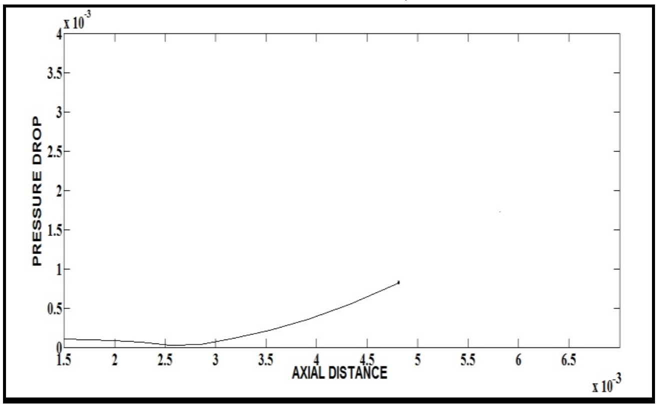

Figure 8. The variation of pressure drop with core thickness for N = 0.1, ε = 0.4

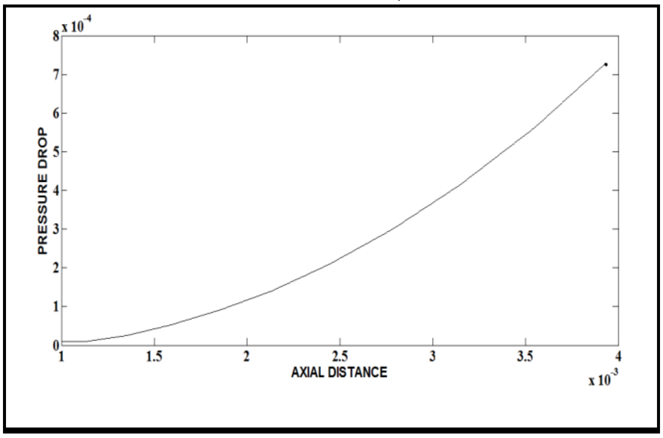

Figure 9. The variation of pressure drop with core thickness for N = 0.06, ε = 0.6

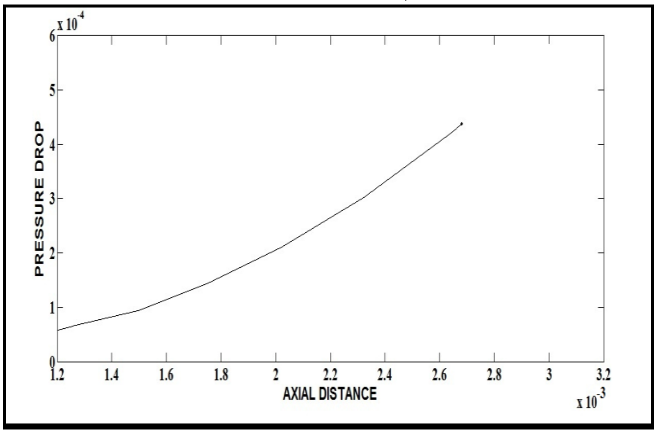

Figure 10. The variation of pressure drop with core thickness for N = 0.08, ε = 0.6

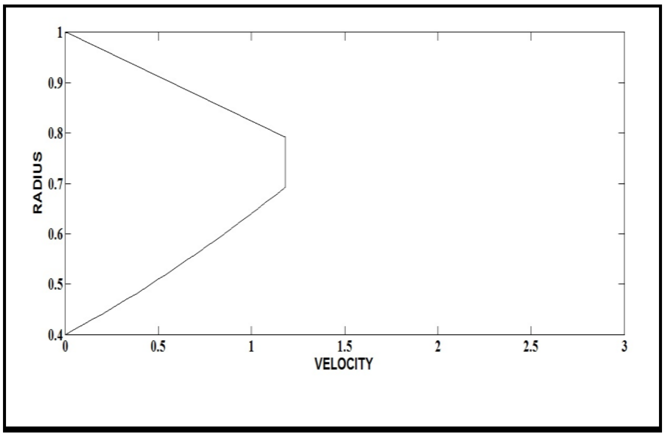

Figure 11. The velocity profile in the entrance region for N = 0.06, ε = 0.4

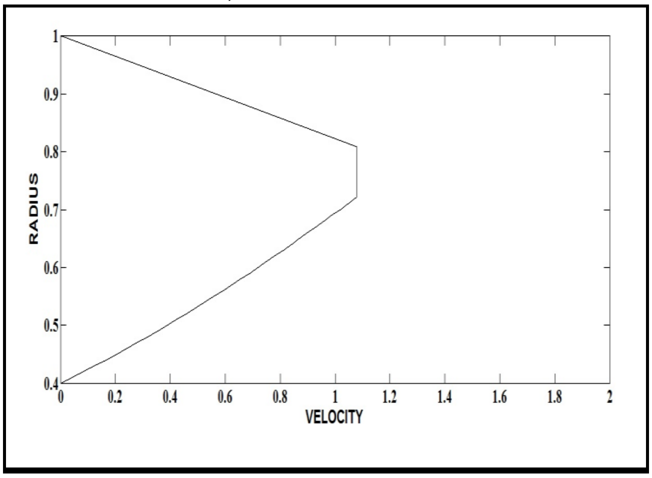

Figure 12. The velocity profile in the entrance region for N = 0.1, ε = 0.4

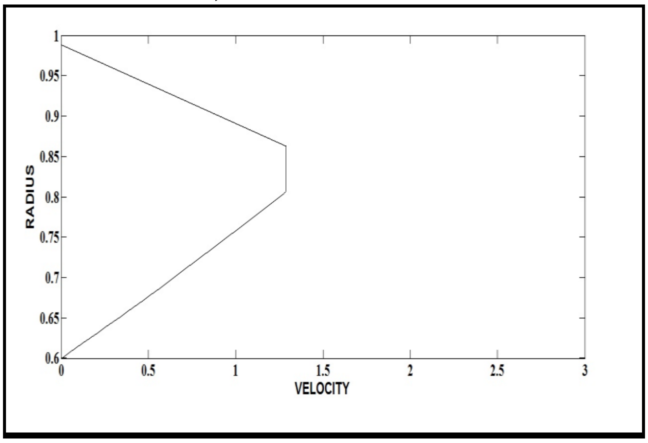

Figure 13. The velocity profile in the entrance region for N = 0.06, ε = 0.6

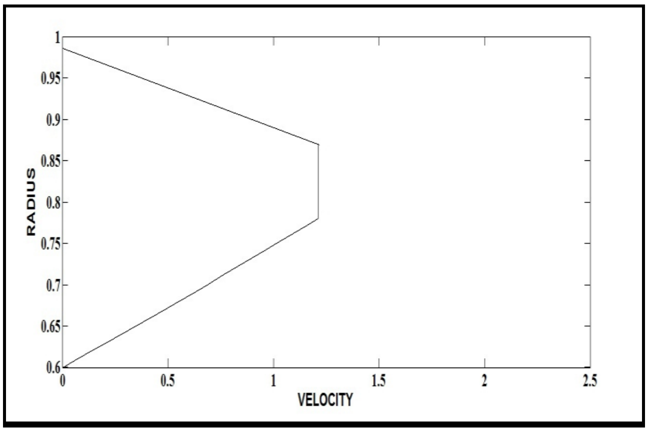

Figure 14. The velocity profile in the entrance region for N = 0.08, ε = 0.6

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