Application of Pseudo-Analysis on Reduction of Nonlinear Ordinary Differential Equations

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***Abstract*:** In this paper, we have presented a new method to solve second order nonlinear differential equations. Using the pseudo-operations given by monotone and continuous function g, the reduction of a nonlinear ordinary pseudo-differential equation is introduced and investigated.

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1. Introduction

Motivation for the research presented here lies both in the capability of the pseudo-analysis and generalization of the classical analysis to extend the range of possible applications. Instead of the usual field of real numbers, the pseudo-analysis, (see [3, 7, 8, 9]), is based on a semiring acting on the real interval $[a,b]\subset[-\infty,+\infty]$, denoting the corresponding operations as $\oplus$ (pseudo-addition) and $\otimes$ (pseudo-multiplication) of the following form:

$$x\oplus y=g^{-1}(g(x)+g(y)),\quad x\otimes y=g^{-1}(g(x)g(y)),$$

where g is a strictly monotone and continuous generating function. By using these pseudo-operations [4, 5, 6, 9], new solutions for the considered nonlinear equation have been obtained. This method is capable of supplying solutions that were not achieved by the classical tools.

The reduction of the order of a nonlinear ordinary differential equation (NLODE) is a basic procedure in the reduction of a NLODE to quadratures. Using the pseudo-analysis and the reduction of a second order pseudo-differential equation to a first order pseudo-differential equation, it is possible to obtain solutions that can be interpreted in the aforementioned classical method.

2. Preliminaries

We consider a particular type of algebra, by means of a strictly monotonic function $g:\mathbf{R}\to\mathbf{R}$, which we assume to be of class C^2 together with its inverse: we also require that $g(0)=0$, that $g'(x)\neq 0$ for all x, and that g is onto, so its inverse is defined on the whole real line.
Using this map, we shall introduce some new operations in the real line as follows:

$$x\oplus y=g^{-1}(g(x)+g(y)),\quad x\otimes y=g^{-1}(g(x)g(y)),\tag{1}$$

(see [2, 4, 9] for details). It is not difficult to determine that these operations are well-defined.
The operations $\oplus$ and $\otimes$ also give $\mathbf{R}$ an order relation $\leq$:

$$x\leq y\Leftrightarrow g(x)\leq g(y).$$

It is easy to determine that this kind of order always coincides with its usual type in $\mathbf{R}$, provided that g is strictly monotonic.
Now we introduce the notion of a g-derivative, as follows:

Definition 2.1 Let the function f be defined on the interval $[c,d]$ and with values in $[a,b]$, if f is differentiable on (c,d) and has the same monotonicity as the function g, then we define the g-derivative of f at the point $x\in(c,d)$ as

$$\frac{d^{\oplus}f(x)}{dx}=g^{-1}\left(\frac{d}{dx}g(f(x))\right),\tag{2}$$

Theorem 2.1 If there exists nth g-derivative of f, then we have

$$\frac{d^{(n)\oplus}f(x)}{dx^n}=g^{-1}\left(\frac{d^{(n)}}{dx^n}g(f(x))\right).\tag{3}$$

Proof. The proof follows directly from Definition 2.1 and properties of pseudo-operations [4].

Example 1 Let $g(u)=u^3$, then we have $g^{-1}(u)=(u)^{1/3}$. The g - derivative is given by

$$\frac{d^{\oplus}y}{dx}=g^{-1}\left(\frac{d}{dx}g(y)\right)=g^{-1}\left(\frac{d}{dx}y^3\right)$$

$$= g^{-1}(3y'y^2) = (3y'y^2)^{1/3}. \quad (4)$$

Also, the second g-derivative and third g-derivative are given by

$$\frac{d^{(2)\oplus}y}{dx^2} = g^{-1}(\frac{d^{(2)}}{dx^2}g(y)) = g^{-1}(\frac{d^{(2)}}{dx^2}y^3)$$
$$= g^{-1}(\frac{d}{dx}3y'y^2) = g^{-1}(3y''y^2 + 6y'^2y)$$
$$= (3y''y^2 + 6y'^2y)^{1/3}$$

and

$$\frac{d^{(3)\oplus}y}{dx^3} = g^{-1}(\frac{d^{(3)}}{dx^3}g(y))$$
$$= g^{-1}(\frac{d}{dx}(3y''y^2 + 6y'^2y))$$
$$= g^{-1}(3y'''y^2 + 18y''y'y + 6y'^3)$$
$$= (3y'''y^2 + 18y''y'y + 6y'^3)^{1/3}.$$

For nth g-derivative of this pseudo-addition, we have

$$\frac{d^{(n)\oplus}y}{dx^n} = g^{-1}(\frac{d^{(n)}}{dx^n}g(y))$$
$$= g^{-1}(\frac{d^{(n)}}{dx^n}y^3)$$
$$= g^{-1}(\frac{d^{(n)}}{dx^n}y.y^2)$$
$$= g^{-1}(\sum_{k=0}^{n}\binom{n}{k}y^{(n-k)}.(y^2)^{(k)})$$
$$= (\sum_{k=0}^{n}\binom{n}{k}y^{(n-k)}.(y^2)^{(k)})^{1/3}.$$

Similarly, a notion of g-integral can be introduced as follows:

Definition 2.2 Let g be a generating function and $\oplus$ and $\otimes$ pseudo-operations given by (1). The pseudo-integral for a function $f:[c,d]\to[a,b]$ reduced on the g-integral is given by

$$\int_{[c,d]}^{\oplus} f(x)\mathrm{d}x = g^{-1}(\int_c^d g(f(x))\mathrm{d}x). \quad (5)$$

Example 2 Let $g(u) = \log_a^u$. Then we have $g^{-1}(u) = a^u$. The pseudo-integral is given by

$$\int^{\oplus} x dx = g^{-1}(\int g(x)\mathrm{d}x) = g^{-1}(\int \log_a^x \mathrm{d}x)$$
$$= g^{-1}(\int \frac{\ln x}{\ln a}\mathrm{d}x) = g^{-1}(\frac{1}{\ln a}\int \ln x \mathrm{d}x)$$
$$= g^{-1}(\frac{1}{\ln a}(x\ln x - x + c_1))$$
$$= a^{\frac{x\ln x}{\ln a} - \frac{x}{\ln a} + \frac{c_1}{\ln a}} = a^{\frac{x\ln x}{\ln a} - \frac{x}{\ln a} + c},$$

which

$$c = \frac{c_1}{\ln a}.$$

Now, we use the pseudo-analysis counterpart of the fundamental theorem of the usual calculus.

Theorem 2.2 Suppose that f has continuous g-derivative on (c,d). Then we have

$$\int_{[c,x]}^{\oplus} \frac{d^{\oplus}f}{dx}\mathrm{d}x \oplus f(c) = f(x), \quad (6)$$

for $x \in (c,d)$.

Proof. The proof is based on pseudo-operations and the fundamental theorem of the usual calculus [4].

2.1 Application

We shall apply the g-derivative and g-integral on nonlinear differential equations.

Theorem 2.3 Suppose that $f:[c,d]\times[a,b]\to[a,b]$ is continuous, that ψ is defined and continuous on $J = \{x: x_0 - h < x < x_0 + h\} \subset [c,d]$ with values in $[a,b]$ that $(x_0, y_0) \in [c,d]\times[a,b]$ with $\psi(x_0) = y_0$. Then the necessary and sufficient condition that ψ be a solution of

$$\frac{d^{\oplus}\psi}{dx} = f(x,\psi(x)), \quad (7)$$

on J is that ψ satisfies the g-integral equation

$$\psi(x) = y_0 \oplus \int_{[x_0,x]}^{\oplus} f(t,\psi(t))\mathrm{d}t, \quad (8)$$

for $x \in J$.

Proof. We apply the g-integral on both sides of (7) and pseudo-adding $\psi(x_0)$ to both sides and we obtain (8) (see [4]).

Example 3 Let $\oplus$ and $\otimes$ be pseudo-operations. Taking $g(u) = \tanh u$, we can consider the simple g-differential equation

$$\frac{d^{\oplus}y}{dx} = x, \quad i.e.$$

$$\frac{1}{2}\ln(\frac{1 + y'\sec h^2 y}{1 - y'\sec h^2 y}) = x. \quad (9)$$

In a simple way, we can obtain the general solution of this equation, only applying the corresponding g -integral on both sides (i.e. for $g(u) = \tanh u$, then $g^{-1}(u) = \frac{1}{2}\ln(\frac{1+u}{1-u})$). Then we obtain

$$\int \frac{d^{\oplus}y}{dx}\mathrm{d}x = \int x\mathrm{d}x.$$

Therefore, by theorem (2.3) we have

$$y = \int^{\oplus} x dx = g^{-1}\int g(x)\mathrm{d}x = g^{-1}\int \tanh x\mathrm{d}x$$

$$= g^{-1}(\ln\cosh x + c_1)$$

$$= \frac{1}{2}\ln(\frac{1+\ln\cosh x + c_1}{1-\ln\cosh x + c_2}).$$

Hence, $y = \frac{1}{2}\ln(\frac{1+\ln\cosh x + c_1}{1-\ln\cosh x + c_2})$ is a general solution of the nonlinear differential equation (9).

3. Main Result

3.1 Application on reduction of order in pseudo-differential equations

There are certain second order pseudo-differential equations that reduce to first order pseudo-differential equations. We will describe some of these here.

TYPE 1 When the variable y is missing from the right side, we have

$$\frac{d^{(2)\oplus}y}{dx^2} = f(x, \frac{d^{\oplus}y}{dx}).$$

So we proceed as follows.

Set $p = \frac{d^{\oplus}y}{dx}$. We obtain

$$\frac{d^{(2)\oplus}y}{dx^2} = \frac{d^{\oplus}p}{dx} = f(x, p).$$

Thus, we obtain a first order d.e. for p. We can solve this and obtain y by reduction of $\frac{d^{(2)\oplus}y}{dx^2}$.

Example 4 Let us solve the following nonlinear second order differential-equation in the usual way:

$$-\ln(-y''e^{-y} + y'^2e^{-y}) = -x. \quad (10)$$

This can be done by choosing $g(u) = e^{-u}$ and $g^{-1}(u) = -\ln u$. By means of the formula concerning first and second order pseudo-derivatives, we obtain

$$\frac{d^{\oplus}y}{dx} = y - \ln(-y'),$$

$$\frac{d^{(2)\oplus}y}{dx^2} = -\ln(-y''e^{-y} + y'^2e^{-y}).$$

So the equation (10) can be written as

$$\frac{d^{(2)\oplus}y}{dx^2} = -x. \quad (11)$$

Now, by reduction of the pseudo-derivative, we have

$$\frac{d^{\oplus}y}{dx} = p, \quad \frac{d^{(2)\oplus}y}{dx^2} = \frac{d^{\oplus}p}{dx}. \quad (12)$$

Hence, our pseudo equation reduces to

$$\frac{d^{\oplus}p}{dx} = -x. \quad (13)$$

We have

$$-\ln(\frac{d}{dx}e^{-p}) = -x \Rightarrow \frac{d}{dx}e^{-p} = e^{x}$$

$$\Rightarrow de^{-p} = e^{x}dx.$$

Now, applying the integral on both sides, we obtain

$$\int \mathrm{d}e^{-p} = \int e^{x}\mathrm{d}x \Rightarrow e^{-p} = e^{x} + c_1$$

$$\Rightarrow -p = \ln(e^{x} + c_1).$$

Alternatively, we have

$$p = \frac{d^{\oplus}y}{dx},$$

therefore,

$$-\ln(\frac{d}{dx}e^{-y}) = -\ln(e^{x} + c_1)$$

$$\Rightarrow \frac{d}{dx}e^{-y} = e^{x} + c_1 \Rightarrow -y'e^{-y} = e^{x} + c_1,$$

applying the integral on both sides, we obtain

$$\int -y'e^{-y}\mathrm{d}x = \int (e^{x} + c_1)\mathrm{d}x$$

$$\Rightarrow e^{-y} = e^{x} + c_1x + c_2,$$

and

$$y = -\ln(e^{x} + c_1x + c_2).$$

Hence, y is the solution to the equation (10).

Example 5 Now we also present an equation of the second order, the solution of which is similar to the previous example

$$-\ln(-y''e^{-y} + y'^2e^{-y}) + y - \ln(-y') = -x. \quad (14)$$

Again, we shall choose $g(u) = e^{-u}$. Hence, we have

$$\frac{d^{\oplus}y}{dx} = y - \ln(-y'),$$

$$\frac{d^{(2)\oplus}y}{dx^2} = -\ln(-y''e^{-y} + y'^2e^{-y}).$$

Therefore, the equation (14) can be written as

$$\frac{d^{(2)\oplus}y}{dx^2}+\frac{d^{\oplus}y}{dx}=-x. \tag{15}$$

Now, by reduction of the pseudo-derivative, the pseudo equation reduces to

$$\frac{d^{\oplus}p}{dx}+p=-x. \tag{16}$$

We have

$$-\ln(\frac{d}{dx}e^{-p})=-(x+p)$$

$$\Rightarrow\frac{d}{dx}e^{-p}=e^{x+p}=e^{x}.e^{p}$$

$$\Rightarrow\frac{de^{-p}}{e^{p}}=e^{x}dx,$$

now, applying the integral to both sides, we have

$$\int e^{-p}\mathrm{d}e^{-p}=\int e^{x}\mathrm{d}x$$

$$\Rightarrow\frac{e^{-2p}}{2}=e^{x}+c_1\Rightarrow e^{-2p}=2e^{x}+2c_1$$

$$-2p=\ln(2e^{x}+2c_1)\Rightarrow p=-\frac{1}{2}\ln(2e^{x}+2c_1).$$

Alternatively, we have

$$p=\frac{d^{\oplus}y}{dx},$$

Hence, we must have

$$-\ln(\frac{d}{dx}e^{-y})=-\frac{1}{2}\ln(2e^{x}+2c_1)$$

$$\Rightarrow\frac{d}{dx}e^{-y}=(2e^{x}+2c_1)^{1/2}$$

$$-y'e^{-y}=(2e^{x}+2c_1)^{1/2},$$

applying the integral on both sides, we obtain

$$\int -y'e^{-y}\mathrm{d}x=\int(2e^{x}+2c_1)^{1/2}\mathrm{d}x$$

$$\Rightarrow e^{-y}=\int(2e^{x}+2c_1)^{1/2}\mathrm{d}x,$$

by variation of variable for the right side, we have;

$$2e^{x}+2c_1=u^2\Rightarrow 2e^{x}dx=2udu,$$

by integration we have

$$e^{-y}=\int\frac{2u^2}{u^2-2c_1}\mathrm{d}u=2\int\frac{u^2-2c_1+2c_2}{u^2-2c_1}\mathrm{d}u$$

$$=2\int(1+\frac{2c_1}{u^2-2c_1})\mathrm{d}u$$

$$=2\int(1+\frac{\frac{1}{2}\sqrt{2c_1}}{u-\sqrt{2c_1}}-\frac{\frac{1}{2}\sqrt{2c_1}}{u+\sqrt{2c_1}})\mathrm{d}u$$

$$=2(u+\frac{1}{2}\sqrt{2c_1}(\ln\frac{u-\sqrt{2c_1}}{u+\sqrt{2c_1}})+c_2).$$

Therefore, we have

$$e^{-y}=2(\sqrt{2e^{x}+2c_1}+\frac{1}{2}\sqrt{2c_1}(\ln\frac{\sqrt{2e^{x}+2c_1}-\sqrt{2c_1}}{\sqrt{2e^{x}+2c_1}+\sqrt{2c_1}})+c_2),$$

then

$$y=-\ln(2\sqrt{2}\sqrt{e^{x}+c_1}+\sqrt{2c_1}(\ln\frac{\sqrt{2e^{x}+2c_1}-\sqrt{2c_1}}{\sqrt{2e^{x}+2c_1}+\sqrt{2c_1}})+c_2),$$

which is the solution to the equation (14).

Example 6 Let $g(u)=\ln(1-u)$. We solve the following equation using the characteristic method:

$$1-\exp\frac{-y''(1-y)-(y')^2}{(1-y)^2}=-x. \tag{17}$$

We have

$$\frac{d^{\oplus}y}{dx}=1-\exp\frac{d}{dx}\ln(1-y),$$

$$\frac{d^{(2)\oplus}y}{dx^2}=1-\exp\frac{-y''(1-y)-(y')^2}{(1-y)^2}.$$

Therefore, the equation (17) can be written as,

$$\frac{d^{(2)\oplus}y}{dx^2}=-x. \tag{18}$$

Now, by reduction of the pseudo-derivative, the pseudo equation reduces to

$$\frac{d^{\oplus}p}{dx}=-x. \tag{19}$$

We have

$$1-\exp\frac{d}{dx}\ln(1-p)=-x$$

$$\Rightarrow\frac{d}{dx}\ln(1-p)=\ln(x+1).$$

Applying the integral on both sides, we have

$$\int\mathrm{d}\ln(1-p)=\int\ln(x+1)\mathrm{d}x$$

$$\Rightarrow\ln(1-p)=((x+1)\ln(x+1)-x+c_1)$$

$$1-p=\exp((x+1)\ln(x+1)-x+c_1)$$

$$\Rightarrow p=1-\exp((x+1)\ln(x+1)-x+c_1).$$

Alternatively, we have

$$p=\frac{d^{\oplus}y}{dx}=1-\exp\frac{d}{dx}\ln(1-y),$$

Therefore

$$1-\exp\frac{d}{dx}\ln(1-y)$$
$$=\ 1-\exp((x+1)\ln(x+1)-x+c_1)$$
$$\frac{d}{dx}\ln(1-y)=(x+1)\ln(x+1)-x+c_1,$$

Applying integration on both sides, we have

$$\int d\ln(1-y)=\int((x+1)\ln(x+1)-x+c_1)\mathrm{d}x.$$

Using integration by parts for the right side, we have

$$u=\ln(x+1)\Rightarrow du=\frac{1}{x+1}dx$$
$$dv=(x+1)dx\Rightarrow v=\frac{(x+1)^2}{2},$$

Therefore

$$\ln(1-y)=(\frac{(x+1)^2}{2}\ln(x+1)$$
$$-\int\frac{(x+1)^2}{2}.\frac{1}{x+1}\mathrm{d}x)-\frac{x^2}{2}+c_1x+c_2$$
$$\ln(1-y)=\frac{1}{2}(x+1)^2\ln(x+1)$$
$$-\frac{1}{4}(x+1)^2-\frac{1}{2}x^2+c_1x+c_2.$$

We finally obtain

$$y=1-$$
$$\exp(\frac{1}{2}(x+1)^2\ln(x+1)-\frac{1}{4}(x+1)^2-\frac{1}{2}x^2+c_1x+c_2),$$

which is the solution to the equation (17).

TYPE 2 When the independent variable is missing, we have

$$\frac{d^{(2)\oplus}y}{dx^2}=f(y,\frac{d^{\oplus}y}{dx}).$$

Again, we set $p=\frac{d^{\oplus}y}{dx}$ and obtain

$$\frac{d^{\oplus}p}{dx}=f(y,p).$$

We attempt to treat y as a new independent variable. Then

$$\frac{d^{\oplus}y}{dx}=p,$$

$$\frac{d^{(2)\oplus}y}{dx^2}=\frac{d^{\oplus}p}{dy}.\frac{d^{\oplus}y}{dx}=p.\frac{d^{\oplus}p}{dy}=f(y,p),$$

we solve this and then integrate $y=\frac{d^{\oplus}p}{dx}$ to obtain y.

Example 7 As an example, let us solve the following equation, using the classical method:

$$-\ln(-y''e^{-y}+y'^2e^{-y})+y(y-\ln(-y'))=0. \qquad (20)$$

This can be done by choosing $g(u)=e^{-u}$ and $g^{-1}(u)=-\ln u$. By means of the formula concerning first and second order pseudo-derivatives, we obtain

$$\frac{d^{(2)\oplus}y}{dx^2}+y\frac{d^{\oplus}y}{dx}=0.$$

Hence, our pseudo equation reduces to

$$p\frac{d^{\oplus}p}{dy}+y.p=0\Rightarrow\frac{d^{\oplus}p}{dy}+y=0.$$

Alternatively, we have

$$\frac{d^{\oplus}p}{dy}=g^{-1}(\frac{d}{dy}g(p))=-\ln(\frac{d}{dy}e^{-p}).$$

Therefore, we have

$$-\ln(\frac{d}{dy}e^{-p})=-y$$

$$\Rightarrow\frac{d}{dy}e^{-p}=e^{y}\Rightarrow de^{-p}=e^{y}dy,$$

Applying the integral on both sides, we obtain

$$\int \mathrm{d}e^{-p}=e^{y}\mathrm{d}y$$

$$\Rightarrow e^{-p}=e^{y}+c_1\Rightarrow -p=\ln(e^{y}+c_1).$$

We have $p=\frac{d^{\oplus}y}{dx}$, therefore,

$$\ln(\frac{d}{dx}e^{-y})=\ln(e^{y}+c_1)$$

$$\Rightarrow\frac{d}{dx}e^{-y}=e^{y}+c_1\Rightarrow -y'e^{-y}=e^{y}+c_1$$

$$y'=-\frac{c_1+e^{y}}{e^{-y}}=-\frac{c_1+e^{y}}{\frac{1}{e^{y}}}=-e^{y}(c_1+e^{y})$$

$$\Rightarrow\frac{dy}{dx}=-e^{y}(c_1+e^{y})$$

$$\frac{dy}{-e^{y}(c_1+e^{y})}=dx,$$

by integration on both sides, we have

$$\int\frac{\mathrm{d}y}{-e^{y}(c_1+e^{y})}=\int\mathrm{d}x$$

$$\frac{-1}{c_1}\int\left(\frac{1}{e^y}+\frac{-1}{c_1+e^y}\right)\mathrm{d}y=\int \mathrm{d}x$$

$$\frac{-1}{c_1}\left(\int e^{-y}\,\mathrm{d}y-\int\frac{\mathrm{d}y}{c_1+e^y}\right)=\int \mathrm{d}x,$$

by variation of variable of $u=c_1+e^y$, for $\int\frac{\mathrm{d}y}{c_1+e^y}$, we have

$$\int\frac{\mathrm{d}u}{(u-c_1)u}=\int\left(\frac{1}{(u-c_1)}+\frac{-1}{u}\right)\mathrm{d}u$$

$$=\frac{1}{c_1}(\ln(u-c_1)-\ln u)$$

$$=\frac{1}{c_1}(\ln(e^y)-\ln(c_1+e^y))$$

$$=\frac{1}{c_1}(y-\ln(c_1+e^y)).$$

Then, we finally have

$$\frac{-1}{c_1}\left(-e^{-y}-\frac{1}{c_1}(y-\ln(c_1+e^y))\right)=x+c_2.$$

Therefore, we obtain the solution to the equation (20).

4. Conclusion

In this paper, we have introduced a new method for solving nonlinear differential equations. Using pseudo-analysis, the reduction of a pseudo-differential equation, as it was shown, allows us to obtain solutions of some nonlinear differential equations using the classical method, which were not achieved by the classical tools. Some further developments related to more general pseudo-operations with applications on nonlinear differential equations were obtained in [5, 6].

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