A Study of the Feasibility of Citizens' Equal Access to Urban Services by Gini Coefficient

Hossein Jabbari Khamnei[1], Zeinab Khajavi [2]

[1]Department of Statistics, University of Tabriz
[2]Department of Industrial Engineer, University of Alghadir

h_jabbari@tabrizu.ac.ir, z.khajavi88@gmail.com

Abstract: Finding the best location is the first and most fundamental step in planning for offering a service or product to costumers. To locate appropriate location, scholars have suggested a variety of models based on circumstances and conditions. Equality is of importance in locating models in the utility sector and in making equal accessibility to urban utilities for members of the society; it has an effective role alongside other criteria such as efficiency and effectiveness. Finding a proper location for retail chain supermarkets with equality as the main concern, would contribute to some factors such as reduction of costs, feasibility of access, and equality in distribution. This research was conducted to find the best location for retail chain supermarkets so that each costumer has an equal distance to the nearest store. In this research, minimization of the Gini coefficient of the Lorenz curve based on service distance has been applied. Based on Gini coefficient of the Lorenz curve function, an algorithm has been designed to find an optimal location for a retail chain supermarket in a bounded flat environment with defined population centers and demand.

Keywords: Lorenz curve, Gini coefficient, Location, Retail chain stores.

1. Introduction

Typical location objectives are concerned with minimizing cost or maximizing efficiency of service. For example, the p-median objective minimizes the average service distance; the p-center objective minimizes the maximum service distance; the p-max-cover objective maximizes the number of customers served in a given distance. For a review of these problems, see [5, 6, 9].

With the rapid growth of public, non-profit services, it has become apparent that equity consideration may be as important as efficiency and effectiveness. Many location models employ equity objectives. In the context of facility location, the objective is to equalize the distances between demand points and a facility that serves them. Maimon [17] considered the problem of minimizing the variance of the distances in a network setting. Drezner et al. [12] analyzed the minimization of the range of distances that customers travel in the plane. This is equivalent to locating a line that minimizes the maximum vertical distance to the demand points. Drezner and Drezner [8] analyzed minimizing both objectives, the variance of the distances and the range of distances in the plane. Models that assign equitable loads to facilities were investigated for discrete planar problems [7], continuous planar demand [2], and the network environment [3]. Eiselt and Laporte [13] list 19 equity measures used in location models, prominent among them is the Gini coefficient of the Lorenz curve based on service distances. Eiselt and Laporte [13] say "Among the most popular measures of equality is the Gini index." Indeed, when economists measure inequity in income distribution, they measure the Gini coefficient [1, 4, 14, 22, 9] rather than the variance, range, or inter-quartile range. These measure dispersion and not necessarily inequity. Note that all of these objectives attain their minimum value when all distances are equal. For a review of equity models in location, see [17, 9].

In this research, minimization of the Gini coefficient of the Lorenz curve based on service distance has been applied. Based on the Gini coefficient of the Lorenz curve function, an algorithm has been designed to find an optimal location for a chain retail supermarket in a bounded flat environment with defined population centers and demand.

2. The Lorenz Curve and the Gini Coefficient

To measure equitable (or inequitable) distribution of a good across a population, a graphical representation was proposed by Lorenz [16]. This graph (an example shown here in Fig.1) is known as the Lorenz curve. Traditionally, the abscissa measures the cumulative percentage of the population, and the ordinate measures the cumulative percentage of the good, such as income or wealth. Since 0% of the population holds 0% of the income, and 100% of the population hold 100% of the income, the Lorenz curve has extremes at (0,0) and (1,1) with the most equitable distribution (x% of the population has x% of the good) being the straight 45º line that connects these two extremes. As disparity or inequity occurs, x% of the population holds less than x% of the good and the Lorenz curve drops below the straight "equity" line. The larger the disparity, the larger the drop. Thus, the area between the Lorenz curve and the straight "equity" line is a measure of equality or inequity in the distribution of a good. The Gini coefficient [15] captures the magnitude of this inequity and is widely used by economists in the domain of income or wealth distributions [1, 4, 14, 22]. The Gini coefficient (G) is the ratio of the area between the Lorenz curve and the straight "equity" line to the entire area below the equity line with $0 \leq G \leq 1$. If $G = 0$, then there is no disparity, all members of the population have the same share of the good. That is, the Lorenz curve is the straight "equity" line and the area of disparity does not exist. If G = 1, then one member of the population has all of the good, with the area of disparity being equal to the entire area under the straight "equity" line. Therefore, the most equitable distribution is the one that results from G being closest to 0, or the distribution that minimizes G. One may think of wealth as a pie and each members' share being of equal size would result in G = 0. As one member of the population takes a larger slice, at least one other member must take less and disparity occurs ($G > 0$). If one member holds the whole pie to him or herself, $G = 1$. In the context of location analysis, the good that is being "distributed" is the distance between the customers and the closest service providing facility, which we term, *service distance*. Unlike income or wealth, which has a fixed aggregate value for a population, service distance is theoretically unbounded. One can always locate a facility farther and farther from the demand area. The concept of dividing the pie cannot be said directly about service distance because the total distance is not fixed. However, an analogy can be made. Assume a service facility is located on a line equidistant from a two-member population connected by this line. As a service facility moves on that line, it is moving toward one member and away from the other, hence resulting in inequality; one member is being made better off at the expense of another. For a review, see [9].

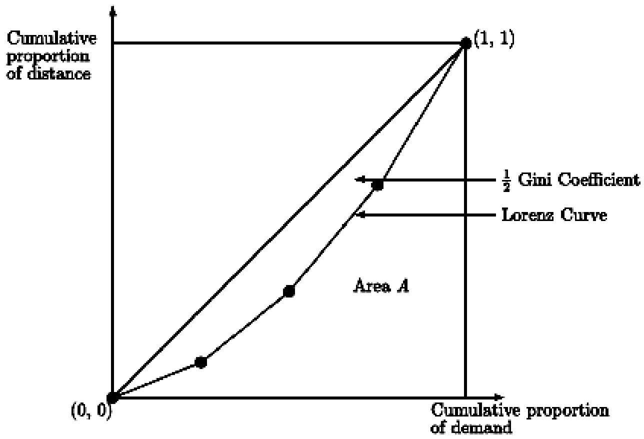

Figure 1. The Lorenz curve and the Gini coefficient.

One of our key insights that has been overlooked in previous research applying the Gini coefficient as a measure of equity in service distance distribution is that G is only a meaningful measure of equity when the potential service locations are within the convex hull of the demand area [9].

In this section, we assume that demand is generated at demand points. Let

n	be the number of demand points,
p	be the number of facilities,
$X = (x_i, y_i)$	For $i = 1,2,\ldots,n$ be the locations of the demand point i,
d_{ij}	be the distance between demand points i and j,
$X = (x, y)$	be the location of the new facility,
$d_i(X)$	be the distance between demand point i and the new facility.

The resulting Gini coefficient is [13, 21]:

$$G(X) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left|d_i(X) - d_j(X)\right|}{2n\sum_{i=1}^{n} d_i(X)} \qquad (1)$$

Its objective is minimizing the numerator of the Gini coefficient, i.e., minimizing

$$H(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|d_i(X) - d_j(X)\right| \qquad (2)$$

3. Analysis of discrete demand

We show how to convert the mean difference objective $H(X)$ (2) to an ordered median objective and the Gini coefficient objective $G(X)$ (1) to a ratio between an ordered median objective and the sum of all distances. The ordered median objective is described in the next section.

3.1 The ordered median formulation

A new approach for formulating and solving many location problems is the ordered median formulation [20]. A vector of real valued weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ is given. The objective is to find locations for p facilities that minimize the weighted sum of distances to the closest facility. The shortest distance is multiplied by the weight λ_1, the second shortest distance by λ_2, and so on, the largest distance is multiplied by λ_n. Let the sorted vector of distances noted by $d_i(X)$ for $i = 1,2,\ldots,n$. The demand point $k = (i)$ is the ith shortest distance in the vector $\{d_k(X)\}$ for $k = 1,2,\ldots,n$. This means that the shortest distance in the vector $\{d_k(X)\}$ for $k = 1,2,\ldots,n$ is $d_{(1)}(X)$, the second shortest distance is $d_{(2)}(X)$, and so on. The largest distance is $d_{(n)}(X)$. Formula (2) can be rewritten as [19, 9]

$$H(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|d_i(X) - d_j(X)\right| = \sum_{j=1}^{n}(4i - 2 - 2n)d_i(X)$$

Consequently

$$G(X) = \frac{\sum_{i=1}^{n}\left[\frac{2i-1}{n} - 1\right]d_i(X)}{\sum_{i=1}^{n} d_i(X)} \qquad (3)$$

4. Locating one facility in the plane with discrete demand

We solve the problem of locating one facility in a feasible region, which is a union of convex polygons such as the convex hull of the demand points. Without such a restriction, the optimal location minimizes the Gini coefficient. Distances are Euclidean.

We consider the minimization of objective:

Gini coefficient: $G(X)$ is defined in Eq. (1) and converted to a ratio of an ordered one-median objective and the sum of distances by Eq. (3).

In this article, we solve the objective using the "big triangle small triangle" (BTST) global optimization technique [11]. The mean difference objective can be solved by the general approach, using BTST, suggested in [10] for the solution of ordered one-median problems. The Gini coefficient objective requires the establishment of a lower bound for the objective function in a triangle. We first briefly describe the BTST algorithm. The details of the algorithm are discussed in [11].☐

5. The BTST approach

A feasible region consisting of a finite number of convex polygons is given and let $\varepsilon > 0$ be a given relative accuracy.

Phase 1: Each convex polygon is triangulated using the Delaunay triangulation. The vertices of the triangles are the demand points and the vertices of the convex polygon. The union of the triangulations is the initial set of triangles.

Phase 2: Calculate a lower bound, *LB*, and an upper bound, *UB*, for each triangle. Let the lowest *UB* be $\overline{UB}$. Discard all triangles for which $LB = \overline{UB}(1-\varepsilon)$.

Phase 3: Choose the triangle with the lowest *UB* and divide it into four small triangles by connecting the center so it fits sides. Calculate *LB* and *UB* for each triangle, and update the $\overline{UB}$ if necessary. The large triangle and all triangles for which $LB = \overline{UB}(1-\varepsilon)$ are discarded.

Stopping criterion: The branch and bound is terminated when there are no triangles left. The solution $\overline{UB}$ is within a relative accuracy of ε from the optimum.

6. Lower bounds in a triangle

Two rigorous lower bounds and one heuristic lower bound are proposed in [10] for the solution of any ordered one-median problem. These bounds are briefly summarized below. For complete details, the reader is referred to [10]. One rigorous lower bound (which we term here LB_D) is based on the shortest and largest distances between a demand point and all of the points in a triangle.

$$LB_D = \sum_{i=1}^{n} \left[\max\{\lambda_i, 0\}\delta_{(i)} + \min\{\lambda_i, 0\}\Delta_{(i)} \right] \qquad (4)$$

Where δ_i and Δ_i are the shortest and largest distances between demand point i and the triangle, respectively. $\delta_{(i)}$ and $\Delta_{(i)}$ are the distances in the sorted vectors $\{\delta_i\}$ and $\{\Delta_i\}$.

Another rigorous lower bound, which we term here LB_L, is based on the objective function satisfying the Lipschitz condition.

$$LB_L = \overline{F} - \left(\sum_{i=1}^{n} |\lambda_i| \right) r \qquad (5)$$

where $\overline{F}$ is the smallest value of the objective function on the three vertices of the triangle and r is the1-center solution in the triangle. r is half of the largest side of the triangle if the triangle is obtuse, and is the radius of the circumscribing circle of the triangle otherwise.

The heuristic lower bound, which we term here LB_H, is based on an estimate of the Lipschitz constant in a triangle. A grid of 21 points in the triangle is established, and the Lipschitz constant estimated rather than using $\sum_{i=1}^{n} |\lambda_i|$ for it. For more details, see [10]. These lower bounds can be used for the minimization of the mean difference objective. To obtain a lower bound for the minimization of the Gini coefficient, which is a ratio of two functions, we observe that $\sum_{i=1}^{n} d_i(X)$ is a convex function, which obtains its maximum at one of the three vertices of the triangle. Therefore, any lower bound on the numerator, which is an ordered one-median objective, divided by the maximum value of $\sum_{i=1}^{n} d_i(X)$ is a lower bound for the ratio. We term these lower bounds LB_D, LB_L, and LB_H, respectively.

7. Case Study

To analyze the citizens' equal access to retail chain stores in the third district of Tabriz we used the Gini coefficient Lorenz curve as a case study. The reason for choosing this part of Tabriz (Laleh) is that it is in the intersection of the Shahrivar Seventeen from north to 22 Bahman St. and from south to Boulevard Azadi Road; it is considered a high population density area; and groups with low-income and high income reside in this area. If the distance traveled to reach these retail stores for those two groups differ substantially, then the position of these stores will be considered unfair.

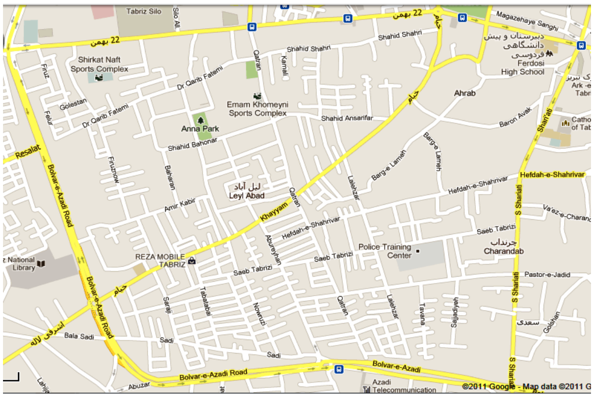

Figure 2. Area selected for case study

After designing the model and algorithm to solve it, we reviewed and implemented them on the third district of Tabriz. The first step in selecting a suitable location to place a store is triangulation of the locations that need such stores by the help of Gini coefficient Lorenz curve. We used the incremental algorithm Delaunay triangulation for this purpose. See figure 3.

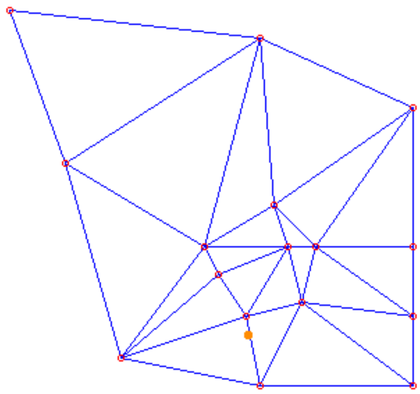

Figure 3. Delaunay triangulation and the ideal location for another chain store

The problem was solved three times, using the three lower bounds discussed in Section 3. We used $\varepsilon = 10^{-4}$ for LB_D

and LB_L. An accuracy of $\varepsilon = 10^{-8}$ was applied in the *BTST* using the heuristic lower bound LB_H.

8. Results

As can be seen, node triangles are the demand specified in the range of the case study. A site selected according to minimizing the Gini coefficient of the Lorenz curve area of the city of Tabriz is displayed in Table 1. The optimal location based on minimizing the Gini coefficient shows. Also, the lower bound results heuristic of the objective function was more efficient than others were and the minimum value of the Gini coefficient in this bound is achieved.

Table 1. Location selected with the objective of minimizing the Gini coefficient

Lower bound	LB_L	LB_D	LB_H
Gini coefficient Value	0.674032	0.9262	0.1573
Optimal point	(164, 28, 302, 86)	(90, 70)	(185, 89, 99)
Iterations	21	0	22

9. Conclusions

The Gini coefficient of the Lorenz curve is a commonly used measure of inequity in income and wealth distribution. When applied to location models, equity is measured by the distances customers must travel to be served. It is viewed as unfair and inequitable if some customers must travel longer distances than others do to be served. To achieve equity, we intend to locate facilities such that distances from customers to the service providing facility are equitable. To achieve this goal, the best location for new facilities is defined as the one that minimizes the Gini coefficient of the distances [9].

Based on the results, Gini coefficient of the Lorenz curve has reached its lowest level in the first section, and the ideal location for another chain store with the goal of creating equal distance to be traveled by the people would be an area within the boundary of Azadi square, Khayam Street and Aboreyhan square.

References

[1] Atkinson AB. On the measurement of inequality. Journal of Economic Theory 1970; 2:244–63.

[2] Baron O, Berman O, Krass D, Wang Q. The equitable location problem on the plane. European Journal of Operational Research 2007; 183:578–90.

[3] Berman O, Drezner Z, Tamir A, Wesolowsky GO. Optimal location with equitable loads. Annals of Operations Research 2009; 167:307–26.

[4] Cowell FA. Measuring inequality. Oxford: Philip Allan; 1977.

[5] Current J, Daskin M, Schilling D. Discrete network location problems. In: Drezner Z, Hamacher H, editors. Facility location: applications and theory. Berlin: Springer; 2002. p. 81–118.

[6] Daskin MS. Network and discrete location: models, algorithms, and applications. New York: Wiley, Interscience; 1995.

[7] Drezner T, Drezner Z. Equity models in planar location. Computational Management Science 2007;4:1–16.

[8] Drezner T, Drezner Z. Multiple facilities location in the plane using the gravity model. Geographical Analysis 2006; 38:391–406.

[9] Drezner T, Drezner Z, Guyse J. Equitable service by a facility: Minimizing the Gini coefficient. Computers & Operations Research, 2009; 36:3240-3246.

[10] Drezner Z, Nickel S. Solving the ordered one-median problem in the plane. European Journal of Operational Research 2009; 195:46–61.

[11] Drezner Z, Suzuki A. The big triangle small triangle method for the solution of non-convex facility location problems. Operations Research 2004; 52:128–35.

[12] Drezner Z, Thisse J-F, Wesolowsky GO. The minimax-min Location Problem. Journal of Regional Science 1986; 26:87–101.

[13] Eiselt HA, Laporte G. Objectives in location problems. In: Drezner Z, editor. Facility location: a survey of applications and methods. NY: Springer; 1995. p. 151–80.

[14] Gastwirth JL. The estimation of the Lorenz curve and Gini index. The Review of Economics and Statistics 1972; 54:306–16.

[15] Gini C. Measurement of inequality and incomes. The Economic Journal 1921; 31:124–6.

[16] Lorenz MO. Methods of measuring the concentration of wealth. Publications of the American Statistical Association 1905; 9:209–19.

[17] Maimon O. The variance equity measure in locational decision theory. Annals of Operations Research 1986; 6:147–60.

[18] Marsh M, Schilling D. Equity measurement in facility location analysis: a review and framework. European Journal of Operational Research 1994; 74:1–17.

[19] Mesa JA, Puerto J, Tamir A. Improved algorithm for several network location problems with equity measures. Discrete Applied mathematics 2003; 130: 437–48.

[20] Nickel S, Puerto J. Location theory: a unified approach. Berlin: Springer; 2005.

[21] Ogryczak W. Inequality measures and equitable approaches to location problems. European Journal of Operational Research 2000; 122:374–91.

[22] Sen A. On economic inequality. Oxford, United Kingdom: Clarendon Press; 1973.