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On the Convergence of the Cesaro Mean of the Fourier-Laplace Serieas

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Abstract: Objectives of this paper are to study convergence and summability of the Fourier-Laplace series in the spaces of singular distributions. The Cezaro means employed for summation of these series and estimations of the spectral function in the Sobolev spaces for this study. In this we employ embedding thoren in connection with the fractional powers of the Laplace operator on a sphere. It is established exact relations between singularity and order of the means which are sufficient for the convergence and summability. Findings of the paper can be used in the solutions of the problems of engineering and mathematical physics such as heat transfer and/or wave propagations on in the surfaces by the series methods in eigenfunction expansions associated with the differential operators defined on the surfaces.

Keywords: Convergence, summabilty, the Fourier-Laplace series, the Cesaro means, distributions.

1. Introduction

Let x and y are belong to the N-dimensional sphere centered at origin with unit radius S^N. We denote by $\gamma = \gamma(x,y)$ spherical distance between these two point, which is equal the angle between vectors x and y. That is why $0 \leq \gamma(x,y) \leq \pi$. Using this metrics define a ball $B(x,r)$ on the surface S^N with the radius equal r centered at $x \in S^N$ as $B(x,r) = \{y \in S^N : \gamma(x,y) < r\}$.

Differentiation on the surface S^N of the functions can be defined by projections to the local Cartesian coordinates in R^N or with the spherical coordinated by the separating spherical parts. Moreover, in this way one can define also differential operators. For example, let Δ_s denotes Laplace operator on S^N. This operator can be obtained from the Laplace operator Δ in R^{N+1} by representing it in spherical coordinates and separating the angular term in it:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_s$$

where

$$\begin{aligned}\Delta_s =& \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial \xi_1}\left(\sin^{N-1}\xi_1\frac{\partial}{\partial \xi_1}\right) \\ &+ \frac{1}{\sin^2\xi_1 \sin^{N-2}\xi_2}\frac{\partial}{\partial \xi_2}\left(\sin^{N-2}\xi_2\frac{\partial}{\partial \xi_2}\right) + \cdots \\ &+ \frac{1}{\sin^2\xi_1 \sin^2\xi_2 .... \sin^2\xi_{N-1}}\frac{\partial^2}{\partial \zeta^2}.\end{aligned}$$

Operator Δ_s we consider in the Hilbert space $L_2(S^N)$ with the domain of definition $C^\infty(S^N)$. It is a non-negative, symmetric operator and its closure $\overline{-\Delta_s}$ is a self-adjoin operator in $L_2(S^N)$. Eigenfunctions $\{Y_j^k\}_{j=1}^{a_k}$ of this operator are homogeneous harmonic polynomials, and mutually orthogonal, where $a_k = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{(N-2)!k!}$. Actually, this system of functions is a complete orthonormal system in the space $L_2(S^N)$. Thus, any function $f \in L_2(S^N)$ has unique representation by this orthonormal system:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{1}$$

where $f_{k,j}$ the Fourie coefficients of a function $f(x)$ and series Eq. (1) always converges in $L_2(S^N)$ norm.

Series Eq. (1) is called the Fourier-Laplace series of a function $f(x)$ on a sphere S^N. We denote by $S_n f(x)$ nth partial sum of the series Eq. (1). It can be represented as follows [1].

$$S_n f(x) = \langle f, \Theta(x,y,n)\rangle,$$

where $\langle f, \Theta(x,y,n)\rangle$ means inner product in the space $L_2(S^N)$ and $\Theta(x,y,n)$ is called a spectral function of the operator $\overline{-\Delta_s}$ and defined as follows,

$$\Theta(x,y,n) = \sum_{k=0}^{n}\sum_{j=1}^{a_k} Y_j^k(y) Y_j^k(x). \tag{2}$$

2. Main results and proofs.

Denote $D(S^N)$ a topological space of the infinite differentiable on S^N functions with topology defined with the seminorms:

$$P_m(\psi) = \|\psi\|_{C^m(S^N)},$$

$\psi \in D(S^N)$, m is non negative integer and $D'(S^N)$ is the space of distributions on $D(S^N)$ with the respect to this topology.

For any $f \in D'(S^N)$, we define a partial sums of the Cezaro means of the partial sum of the series Eq. (1)

$$\begin{aligned}S_n^\alpha f(x) =& \langle f, \Theta(x,y,n)\rangle \\ =& \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^\alpha \sum_{j=1}^{a_k} Y_j^k(x)\langle f, Y_j^k(y)\rangle,\end{aligned} \tag{3}$$

where $\langle f, Y_j^k(y)\rangle$ means action of the distribution f on a basic function $Y_j^k(y)$ and A_n^α are the Cesaro constants [2].

From the definitions it follows that if $f \in D(S^N)$, then

$$S_n f(x) \to f(x) \tag{4}$$

in the topology of the space $D(S^N)$ and if $f \in D'(S^N)$ then Eq. (4) is valid in the topology of $D'(S^N)$.

Let ℓ is a real number. By $W_2^\ell(S^N)$ denote the Sobolev spaces which can be defined as follows: it is a subspace of the distributions $f \in D'(S^N)$ with the finite norm [3]

$$\|f\|_\ell = \sqrt{\sum_{k=0}^{\infty}(k^2+(N-1)k)^\ell \sum_{j=1}^{a_k} |\langle f, Y_j^k\rangle|^2} \tag{5}$$

Note, that the space $W_2^\ell(S^N)$ is a Hilbert space and the space $D(S^N)$ is dense in $W_2^\ell(S^N)$ for $\ell > 0$. For any $\ell > 0$ the norm (5) for the distributions can be expressed as follows [4]

$$\|f\|_{-\ell} = \sup_{u \in W_2^\ell(S^N)} \frac{|\langle f, u\rangle|}{\|u\|_\ell}.$$

Theorem 1: Let $f \in W_2^{-\ell}(S^N), \ell > 0$ and let $f = 0$ in the domain $V \subset S^N$. If $\alpha \geq \ell + (N-1)/2$, then uniformly in any compact set $K \subset V$

$$\lim_{n\to\infty} S_n^\alpha f(x) = 0.$$

The statement of the theorem follows from the lemmas below:

Lemma 1: Let a domain $V \subset S^N$ and $K \subset V$ is a compact. Then uniformly with the respect x on K the following estimation is valid

$$\|\Theta^\alpha(x,y,n)\|_{L_2(S^N\setminus V)} \leq c \cdot n^{\frac{N-1}{2}-\alpha}. \tag{6}$$

Proof: There exists a constant $r_0 > 0$ such that for all $y \in S^N \setminus V$ and $x \in K$ we have $\gamma(x,y) \geq r_0$. Taking into account this we estimate a norm as follows

$$\|\cdot\|_{L_2(S^N\setminus V)} = \|\cdot\|_{L_2(r_0 \leq \gamma \leq \pi - \frac{1}{n})} + \|\cdot\|_{L_2(\pi - \frac{1}{n} \leq \gamma \leq \pi)}$$

By using the asymptotic of the spectral function obtain

$$\|\Theta^\alpha(x,y,n)\|_{L_2(S^N\setminus V)} \tag{7}$$

$$\begin{aligned} \leq & c \cdot n^{\frac{N-1}{2}-\alpha}\sqrt{\int_{r_0\leq\gamma\leq\pi-\frac{1}{n}} (sin\gamma)^{-N+1} d\sigma(y)} \\ &+ c \cdot n^{\frac{N-3}{2}-\alpha}\sqrt{\int_{r_0\leq\gamma\leq\pi-\frac{1}{n}} (sin\gamma)^{-N-1} d\sigma(y)} \\ &+ c \cdot n^{-1}\sqrt{\int_{r_0\leq\gamma\leq\pi-\frac{1}{n}} d\sigma(y)} \\ &+ c \cdot n^{N-1-\alpha}\sqrt{\int_{\pi-\frac{1}{n}\leq\gamma\leq\pi} d\sigma(y)}. \end{aligned} \tag{8}$$

Inequality Eq. (6) follows from Eq. (7) with appropriate estimations of the integrals in Eq. (7). Lemma 1 is proved. □

Lemma 2: Let $f \in W_2^{-\ell}(S^N), \ell > 0$ and $V \subset S^N$ is a domain, $K \subset V$ a compact and let $f = 0$ in $V \subset S^N$. Then uniformly by x on K we have

$$|S_n^\alpha f(x)| \leq c \cdot n^{\frac{N-1}{2}-\alpha+\ell}\|f\|_{-\ell}. \tag{9}$$

Proof: Introduce a symbol $\|\cdot\|_{\ell, S^N\setminus V}$ which denotes a norm in the space $W_2^\ell(S^N \setminus V)$. Taking into consideration that support of f is outside of the domain $V \subset S^N$, it follows

$$|\langle f, u\rangle| \leq \|f\|_{-\ell}\|u\|_{\ell, S^N\setminus V} \tag{10}$$

where $u \in C^\infty(S^N)$. The operator $\overline{-\Delta}^{\frac{\ell}{2}}$ is continuous from L_2 in W_2^ℓ which means for any $u \in L_2$

$$\|(\overline{-\Delta}^{\frac{\ell}{2}})u\|_\ell \leq \|u\|_{L_2}. \tag{11}$$

Then from Eq. (10) and Eq. (11) get

$$\begin{aligned} |\langle f, \Theta^\alpha(x,y,n)\rangle| \leq & \|f\|_{-\ell} \cdot \|\Theta^\alpha(x,y,n)\|_{\ell, S^N\setminus V} \\ \leq & c \cdot n^\ell \cdot \|f\|_{-\ell} \cdot \|(\overline{-\Delta}^{\frac{\ell}{2}})\Theta^\alpha(x,y,n)\|_{\ell, S^N\setminus V} \\ \leq & c \cdot n^\ell \cdot \|f\|_{-\ell} \cdot \|\Theta^\alpha(x,y,n)\|_{L_2(S^N\setminus V)}. \end{aligned} \tag{12}$$

Then from Lemma 1 and Eq. (12) obtain Eq. (9). Lemma 2 is proved. □

3. Literature review

The uniform convergence and localization for the arithmetic means of the Fourier–Laplace series in the Nikolsk'ii functional spaces studied in [2]. These problems also studied in [5] and in [6] in other functional spaces. The problems of almost-everywhere convergence of expansions in eigenfunctions of the Laplace operator on the sphere studied in [7]. These problems for the Riesz means studied in the papers [8] and [1]. In particular in [8] the localization conditions ofthe Fourier-Laplace series of distribution studied. The problems for the multiple Fourier trigonometric series of distributions studied in [9]. The uniform convergence of Fourier series on the closed domains studied in [3]. Statements related to the embedding properties, properties of the distributions, interpolation theorems, precise definitions of the function spaces and thier relations with the differential operators can be found in [4].

4. Conclusions

This paper is a study of a certain range of problems of localization of spectral expansions of distributions with compact supports and negative smoothness. The exact relationships between the orders of the averages, the "smoothness" of the distribution and the degree of summation are established, under which the principle of localization of spectral expansions is valid for Fourier-Laplace series on the sphere.

Findings of the paper can serve as a further development of the spectral theory of elliptic differential operators, and can also be used in solving evolutionary equations of mathematical physics and in substantiating the method for solving problems in natural sciences.

The results obtained can be useful in understanding the problems arising in the numerical solution of some mathematical models of natural science problems, which are solved using the method of separation of variables. The use of the results of the dissertation can make it possible to determine the correct approaches to finding a solution to these problems, for example, in determining the necessary average indicator for the summability of the corresponding spectral expansions, depending on the degree of singularity of the initial or boundary modes of the mathematical model of the problem being solved.

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