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# A Note on the Cyclic Groups

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***Abstract*:** In this paper, some homological functors for cyclic groups and linear groups of finite order are considered.

***Keywords*:** Cyclic groups, multiplier, linear groups.

## 1. Introduction

The Schur multiplier of a group $G$, $M(G)$, is named after Issai Schur. According to Hopf's Formula, $M(G) \cong (F' \cap R)/[F,R]$ where $R \rightarrowtail F \twoheadrightarrow G$ is presentation of the group $G$. The nonabelian tensor square has been discovered by Dennis [1] in 1976. The nonabelian tensor square, $G \otimes G$, of the group $G$ is a group generated by the symbols $g \otimes h$ and defined by the relations $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g$, $g'$, $h$, $h' \in G$, where $G$ acts on itself by conjugation, i.e. ${}^{g}g' = gg'g^{-1}$.

In 1987, Brown and Loday [2] introduced the concept of nonabelian tensor square in their paper. In their second paper [3], they provided a list of open problems on this topic, and this paper became one of the most original papers that is referred to by many mathematicians [4–6].

A group $G$ is capable if there exists a group $H$ such that $G \cong H/Z(H)$. Ellis [7] proved that a group $G$ is capable if and only if its exterior center $Z^{\wedge}(G)$ is trivial. In 1979, Beyl *et al.* [8] established a necessary and sufficient condition for a group to be capable, that is, a group is capable if and only if the epicenter $Z^{*}(G)$ of the group is trivial. In this thesis, the Schur multiplier, nonabelian tensor square and capability of groups of orders $p^2q$ and $p^3q$, special linear groups, projective special linear groups, symplectic groups and projective symplectic groups will be determined, where $p$ and $q$ are distinct primes and for groups of order $p^3q$, $p < q$.

## 2. Definitions and Basic Theorems

In this section some definitions and preparatory theorems that are necessary in the following chapters are stated.

*Definition 1:* [9] Let $G$ be a group and $X$ a subset of $G$ . Let $\{H_i | i \in I\}$ be the family of all subgroups of $G$ which contains $X$. Then $\bigcap_{i \in I} H_i$ is called the subgroup of $G$ generated by the set $X$, and is denoted by $< X >$.

*Definition 2:* [9] Let $G$ be a group. The subgroup of $G$ generated by the set $\{xyx^{-1}y^{-1} | x, y \in G\}$ is called the commutator subgroup of $G$ and denoted by $G'$.

*Definition 3:* [9] A group $G$ is said to be solvable if $G^{(n)} = \{1\}$ for some $n$, where $G^{(n)} = (G^{(n-1)})'$ is called the $n$-th derived subgroup of $G$.

*Definition 4:* [10] A finite $p$-group $G$ is called extra-special if $G'$ and $Z(G)$ coincide and have order $p$.

*Definition 5:* [11] A normal subgroup $N$ of $G$ is called a normal Hall subgroup of $G$ if the order of $N$ is coprime with its index in $G$.

*Definition 6:* [11] A group $G$ is termed metacyclic if there exists a normal subgroup $N$ of $G$ such that both $N$ and $G/N$ are cyclic.

A metacyclic group can be presented in the form

$$G = < a, b | a^m = 1, b^s = a^t, bab^{-1} = a^r >$$

where the positive integers $m$, $r$, $s$ and $t$ satisfy

$$r^s \equiv 1 \pmod{m}, \qquad m \big| t(r-1).$$

Some definitions on the linear groups are presented in the following:

*Definition 7:* [12] The general linear group $GL_n(F_q)$ of degree $n$ is the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication, that is, $GL_n(F_q) = \{A_{n \times n}; |A| \neq 0\}$ where $F_q$ is a field with $q$ elements.

*Definition 8:* [12] The special linear group, $SL_n(F_q)$, is the group of all matrices with determinant 1, that is, $SL_n(F_q) = \{A \in GL_n(F_q); |A| = 1\}$.

*Definition 9:* [12] The projective general linear group, $PGL_n(F_q)$ and the projective special linear group $PSL_n(F_q)$ are the quotients of $GL_n(F_q)$ and $SL_n(F_q)$ by their centers, respectively.

*Definition 10:* [12] A symplectic matrix is a $2n \times 2n$ matrix $M$ (whose entries are typically either real or complex) satisfying the condition $M^T \Omega M = \Omega$ where $M^T$ denotes the transpose of $M$ and $\Omega$ is a fixed nonsingular, skew-symmetric matrix. Typically $\Omega$ is chosen to be the block matrix

$$\begin{bmatrix} O & -I_n \\ I_n & O \end{bmatrix}.$$

*Definition 11:* [12] The symplectic group of degree $2n$ over a field $F_q$, denoted by $Sp_{2n}(F_q)$, is the group of $2n$ by $2n$ symplectic matrices with entries in $F_q$, and with the group operation that of matrix multiplication.

*Definition 12:* [12] The projective symplectic group $PSp_{2n}(F_q)$ is the group obtained from the symplectic group $Sp_{2n}(F_q)$ on factoring by the scalar matrices contained in that group.

The following theorem gives elementary results on group theory that will be used in the subsequent chapters.

*Theorem 1:* [9] Let $G$ be a group and $H,K \leq G$.
$(i)$ Suppose $H \trianglelefteq G$. Then $G/H$ is abelian if and only if $G' \subseteq H$.
$(ii)$ Then $G^{ab}$ is abelian.
$(iii)$ If $H,K \trianglelefteq G$, $K \leq H$, then

$$[H,G] \leq K \Leftrightarrow H/K \leq Z(G/K).$$

$(iv)$ If $G/Z(G)$ is cyclic, then $G$ is abelian.
$(v)$ $G$ is abelian if and only if $G' = \{1\}$.
$(vi)$ If $G$ is a nontrivial group and $exp(G^{ab}) \geq |G'|$, then $Z(G) \neq 1$, where $exp(G^{ab})$ is the least common multiple of the orders of all elements of the group $G^{ab}$.

In the following theorem, direct product and semidirect product of two groups are written in their presentation forms.
*Theorem 2:* [9]
$(i)$ If $G =< S_G|R_G >$ and $H =< S_H|R_H >$, where $S_G$ and $S_H$ are (disjoint) generating sets and $R_G$ and $R_H$ are defining relations. Then

$$G \times H =< S_G \cup S_H|R_G \cup R_H \cup R_P >$$

where $R_P$ is a set of relations specifying that each element of $S_G$ commutes with each element of $S_H$.
$(ii)$ Let $C_m =< a|a^m = e >$ and $C_n =< b|b^n = e >$ . Then the semidirect product of $C_m$ and $C_n$ is given by a single relation $aba^{-1} = b^k$ where $(k,n) = 1$.

The existence of a complement of a group $G$ is stated in the next theorem.
*Theorem 3:* (Schur-Zassenhaus):[10] Let $N$ be normal subgroup of $G$. Assume that $|N| = n$ and $[G : N] = m$ are relatively prime. Then $G$ contains subgroups of order $m$ and any two of them are conjugate in $G$.
This theorem asserts that the complement of $G$ exists and the two are conjugate. Also it shows that if $G/N$ is cyclic, then the complement of $N$ is too.

Since all the groups considered in this thesis are finite, it can be proved in some cases that each Sylow subgroup of a group $G$ are cyclic by the use of the following theorem.
*Theorem 4:* (Zassenhaus-Burnside-Holder): [10] Suppose $G$ is a finite group such that each Sylow subgroups of $G$ are cyclic, then $G =< a,b|a^m = b^n, b^{-1}ab = a^r >$, where $m$ is odd, $m\big|r^n - 1$, $0 \leq r \leq m-1$ and $(m,n(r-1)) = 1$. Conversely, if a group $G$ has this structure, then each Sylow subgroup of $G$ are cyclic. In this group $G'$ and $G^{ab}$ are cyclic .

In the following theorem, the classification of groups of order $pq$ is presented.
*Theorem 5:* [9] Let $p$ and $q$ be primes such that $p > q$. If $q \nmid p-1$, then every group of order $pq$ is isomorphic to the cyclic group $\mathbb{Z}_{pq}$. If $q\big|p-1$, then there are (up to isomorphism) exactly two distinct groups of order $pq$: the cyclic group $\mathbb{Z}_{pq}$ and a nonabelian group $K$ generated by elements $c$ and $d$ such that $|c| = p$, $|d| = q$, $dc = c^s d$, where $p \nmid s-1$ and $p\big|s^q - 1$.

For groups of order $p^2q$, the following theorem is stated:
*Theorem 6:* [13] Let $G$ be a group of order $p^2q$ where $p$ and $q$ are distinct primes. Then exactly one of the following holds:
$(i)$ If $p > q$, then $G$ has a normal Sylow $p$-subgroup .
$(ii)$ If $q > p$, then $G$ has a normal Sylow $q$-subgroup.
$(iii)$ If $p = 2$, $q = 3$, then $G \cong A_4$ and $G$ has a normal Sylow 2-subgroup.

In the following theorem the classification of groups of order $p^3$ is stated:
*Theorem 7:* [10] Let $G$ be a group of order $p^3$, where $p$ is an odd prime. Then exactly one of the following holds:
(2.7.1) $G \cong \mathbb{Z}_{p^3}$.
(2.7.2) $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.
(2.7.3) $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
(2.7.4) $G \cong< a,b|a^p = b^p = 1, [a,b]^a = [a,b] = [a,b]^b >$. In this case $exp(G) = p$.
(2.7.5) $G \cong< a,b|a^{p^2} = b^p = 1, a^b = a^{p+1} >$. In this case $exp(G) = p^2$.

The following theorem is a well-known fact concerning the commutator subgroup and center of groups of order $p^3$.
*Theorem 8:* Let $G$ be a nonabelian group of order $p^3$, where $p$ is a prime. Then
$(i)$ $Z(G) \cong \mathbb{Z}_p$.
$(ii)$ $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
$(iii)$ $G' \cong \mathbb{Z}_p$

In 1899, Western [14] obtained the classification of groups of order $p^3q$. Western proved that there are 27 types of groups of order $p^3q$, where $p < q$. In the next theorem, the classification of groups of order $p^3q$ where $p < q$ is stated. These classifications are the most important that will be used in the subsequent chapters to determine the Schur multiplier, nonabelian tensor square and capability of groups of order $p^3q$.
*Theorem 9:* [14] Let $G$ be a nonabelian group of order $p^3q$, where $p$ and $q$ are distinct primes and $p < q$. Then exactly one of the following holds:
(2.9.1) $< a,b,c|a^4 = b^2 = c^q = 1, bab = a^{-1}, ac = ca, bc = cb >$.
(2.9.2)$< a,b,c|a^4 = b^4 = c^q = 1, b^2 = a^2, b^{-1}ab = a^{-1}, ac = ca, bc = cb >$.
(2.9.3) $< a,b|a^8 = b^q = 1, a^{-1}ba = b^{-1} >$.
(2.9.4) $< a,b,c|a^4 = b^2 = c^q = 1, ab = ba, ac = ca, bcb = c^{-1} >$.
(2.9.5) $< a,b,c|a^4 = b^2 = c^q = 1, ab = ba, a^{-1}ca = c^{-1}, bc = cb >$.
(2.9.6) $< a,b,c,d|a^2 = b^2 = c^2 = d^q = 1, ab = ba, ac = ca, bc = cb, ad = da, bc = cb, cdc = d^{-1} >$.
(2.9.7) $< a,b,c|a^4 = b^2 = c^q = 1, bab = a^{-1}, ac = ca, bcb = c^{-1} >$.
(2.9.8) $< a,b,c|a^4 = b^2 = c^q = 1, bab = a^{-1}, a^{-1}ca = c^{-1}, bc = cb >$, $q \equiv 1 \pmod 2$.
(2.9.9)$< a,b,c|a^4 = b^4 = c^q = 1, b^2 = a^2, b^{-1}ab = a^{-1}, ac = ca, b^{-1}cb = c^{-1} >$.
(2.9.10) $< a,b|a^8 = b^q = 1, a^{-1}ba = c^m$, where $m$ is any primitive root of $m^4 \equiv 1 \pmod q$ and $q \equiv 1 \pmod 4$.
(2.9.11) $< a,b,c|a^4 = b^2 = c^q = 1, ab = ba, a^{-1}ca = c^m, bc = cb >$, where $m$ is any primitive root of $m^4 \equiv 1 \pmod q$ and $q \equiv 1 \pmod 4$.
(2.9.12) $< a,b|a^8 = b^q = 1, a^{-1}ba = b^m$, where $m$ is any primitive root of $m^8 \equiv 1 \pmod q$ and $q \equiv 1 \pmod 8$.
(2.9.13) $< a,b,c,d|a^2 = b^2 = c^2 = d^q = 1, ab = ba, ac =$

$ca,bc=cb,ad=da,d^{-1}bd=c,d^{-1}cd=bc>$.
(2.9.14) $<a,b,c|a^4=b^4=c^3=1,a^2=b^2,b^{-1}ab=a^{-1},c^{-1}ac=b,c^{-1}bc=ab>$.
(2.9.15) $<a,b,c|a^4=b^4=c^3=1,bab=a^{-1},c^{-1}a^2b=b,c^{-1}bc=a^2b,a^{-1}ca=c^2a^2b>$.
(2.9.16) $<a,b,c,d|a^2=b^2=c^2=d^7=1,ab=ba,ac=ca,bc=cb,d^{-1}ad=b,d^{-1}bd=c,d^{-1}cd=ab>$.
(2.9.17) $<a,b,c|a^{p^2}=b^p=c^q=1,b^{-1}ab=a^{p+1},ac=ca,bc=cb>$.
(2.9.18) $<a,b,c,d|a^p=b^p=c^p=d^q=1,ab=ba,ac=ca,c^{-1}bc=ab,ad=da,bd=db,cd=dc>$.
(2.9.19)$<a,b|a^{p^3}=b^q=1,a^{-1}ba=b^ma$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$ and $q\equiv 1 \pmod p$.
(2.9.20) $<a,b,c|a^{p^2}=b^p=c^q=1,ab=ba,ac=ca,b^{-1}cb=c^m>$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$ and $q\equiv 1 \pmod p$.
(2.9.21) $<a,b,c|a^{p^2}=b^p=c^q=1,ab=ba,a^{-1}ca=c^m,bc=cb>$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$ and $q\equiv 1 \pmod p$.
(2.9.22) $<a,b,c,d|a^p=b^p=c^p=d^q=1,ab=ba,ac=ca,bc=cb,ad=da,bd=db,c^{-1}dc=d^m>$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$ and $q\equiv 1 \pmod p$.
(2.9.23) $<a,b,c|a^{p^2}=b^p=c^q=1,b^{-1}ab=a^{p+1},ac=ca,b^{-1}cb=c^n>$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$, $q\equiv 1 \pmod p$ and $n=m,m^2,...,m^{p-1}$.
(2.9.24) $<a,b,c,d|a^p=b^p=c^p=d^q=1,ab=ba,ac=ca,ad=da,bd=db,c^{-1}bc=ab,c^{-1}dc=d^m>$, where $m$ is any primitive root of $m^p\equiv 1 \pmod q$ and $q\equiv 1 \pmod p$.
(2.9.25) $<a,b|a^{p^3}=b^q=1,a^{-1}ba=b^m$, where $m$ is any primitive root of $m^{p^2}\equiv 1 \pmod q$ and $q\equiv 1 \pmod{p^2}$.
(2.9.26) $<a,b,c|a^{p^2}=b^p=c^q=1,ab=ba,a^{-1}ca=c^a,bc=cb>$, where $m$ is any primitive root of $m^{p^2}\equiv 1 \pmod q$ and $q\equiv 1 \pmod{p^2}$.
(2.9.27) $<a,b|a^{p^3}=b^q=1,a^{-1}ba=b^m$, where $m$ is any primitive root of $m^{p^3}\equiv 1 \pmod q$ and $q\equiv 1 \pmod{p^3}$.
In this research, these groups are referred to as groups of type (2.9.1)-(2.9.27).

In the following two theorems, the commutator subgroup of some linear groups are given.

*Theorem 10:* [15] Let $q>3$. Then
$(i)$ $(SL_2(F_q))'=SL_2(F_q)$.
$(ii)$ $(PSL_2(F_q))'=PSL_2(F_q)$.
$(iiii)$ $(GL_2(F_q))'=SL_2(F_q)$.
$(iv)$ $(PGL_2(F_q))'=PSL_2(F_q)$.

*Theorem 11:* [15]
$(i)$ $SL_n(F_q)$ and $PSL_n(F_q)$ are perfect groups, except when $(n,q)=(2,2),(2,3)$.
$(ii)$ $(GL_n(F_q))'=SL_n(F_q)$.
$(iii)$ $(PGL_n(F_q))'=PSL_n(F_q)$.
$(iv)$ If $(n,q)\neq(2,2),(2,3),(4,2)$, then $Sp_{2n}(F_q)$ and $PSp_{2n}(F_q)$ are perfect groups, where

$$PSp_2(F_2)\cong PSL_2(F_2)\cong S_3,PSp_2(F_2)\cong S_6.$$

## 3. The Schur Multiplier

In this section, some definitions are stated to define the Schur multiplier of a group $G$. Basic results on the Schur multiplier will also be presented in this section.

*Definition 13:* [10] The homology $H(\mathbf{C})$ of the complex $\mathbf{C}$ is the sequence of $R$- modules $H_n(\mathbf{C})$=Ker $\partial_n$/Im $\partial_{n+1}$ which are usually referred to as the homology groups of $\mathbf{C}$. Thus $\mathbf{C}$ is exact if and only if all the homology groups vanish.
*Definition 14:* [10] A complex $\mathbf{C}$ is called positive if $C_n=0$ for $n<0$.
*Definition 15:* [10] Let $M$ be a right $R$-module. By a right R-resolution of $M$ is meant a positive right $R$-complex $\mathbf{C}$ and an epimorphism $\varepsilon: C_0\twoheadrightarrow M$ such that

$$...\to C_2\to C_1\to C_0\to M\to 0$$

is exact. The resolution is said to be free (projective) if $\mathbf{C}$ is free (projective).
*Definition 16:* [10] The complex of $\mathbb{Z}$-modules

$$...\to M\otimes_{\mathbb{Z}G}P_{n+1}\to M\otimes_{\mathbb{Z}G}P_n\to M\otimes_{\mathbb{Z}G}P_{n-1}\to ...$$

is denoted by $M\otimes_{\mathbb{Z}G}\mathbf{P}$ and the $n$-th homology group of $G$ with coefficients in $M$ to be the abelian group $H_n(G,M)=H_n(M\otimes_{\mathbb{Z}G}\mathbf{P})$.
The definition of the Schur multiplier is stated in the following:
*Definition 17:* [10] Let $G$ be any group and $\mathbb{Z}$ as a trivial $G$-module. Then $H_2(G,\mathbb{Z})=H_2(G)$ is known as the Schur multiplier of $G$.
In this research, the Schur multiplier of a group $G$ is denoted as $M(G)$.
*Definition 18:* [10] A group $G^*$ is said to be a covering group of $G$ if $G^*$ has a subgroup $A$ such that
$(i)$ $A\subseteq Z(G^*)\cap[G^*,G^*]$,
$(ii)$ $A\cong M(G)$,
$(iii)$ $G\cong G^*/A$.
*Theorem 12:* [10] (Hopf's Formula). If $R\rightarrowtail F\twoheadrightarrow G$ is a presentation of a group $G$, then $M(G)\cong(F'\cap R)/[F,R]$. In particular, this factor does not depend on the presentation.
*Theorem 13:* [11]
$(i)$ $M(G)$ is a finite group whose elements have order dividing the order of $G$. $(ii)$ $M(G)=1$ if $G$ is cyclic.
*Theorem 14:* [11] If for all $p\big||G|$ the Sylow $p$-subgroups of $G$ are cyclic, then $M(G)=1$.
By the use of the following two theorems, the Schur multiplier of a group $G$ can be computed whenever $G$ is isomorphic to a direct product (or a semidirect product) of two groups.
*Theorem 15:* [11] If $G_1$ and $G_2$ are finite groups, then

$$M(G_1\times G_2)=M(G_1)\times M(G_2)\times(G_1\otimes G_2).$$

*Theorem 16:* [11] Let $N$ be a normal Hall subgroup of $G$, i.e. $(|G|,|G/N|)=1$ and $T$ be a complement of $N$ in $G$. Then $M(G)\cong M(T)\times M(N)^T$.
In this research, a group of order $p^2q$ and $p^3q$ can be presented as a direct product or semidirect product of two

groups, and in some cases one of them is $S_n, A_n, D_n, Q_n$ or metacyclic group.

In the following two theorems, the Schur multiplier of these groups are presented.

*Theorem 17:* [11]

$$M(S_n) = \begin{cases} 1 & ; n \leq 3, \\ \mathbb{Z}_2 & ; n > 4. \end{cases}$$

where $S_n$ is symmetric group of order $n!$.

*Theorem 18:* [11] Let $G$ be a finite metacyclic group, i.e. $G = < a, b | a^m = e, b^s = a^t, bab^{-1} = a^r >$, where the positive integers $m$, $r$, $s$ and t satisfy $r^s \equiv 1 \pmod{m}$ and $m \big| t(r-1)$ and $t \big| m$. Then $M(G) \cong \mathbb{Z}_n$, where $n = \frac{(r-1,m)(1+r+r^2+\ldots+r^{s-1},t)}{m}$.

The next theorem states three equivalent conditions that can be used for computing the Schur multiplier of some types of groups given in Theorem 2.9.

*Theorem 19:* [11] Let $Z$ be a central subgroup of a finite group $G$. Then the following conditions are equivalent:
$(i)$ $M(G) \cong M(G/Z)/(G' \cap Z)$,
$(ii)$ $Z \subseteq Z^*(G)$,
$(iii)$ the natural map $M(G) \to M(G/Z)$ is injective.

In the following theorem, the Schur multiplier and covering group of a finite perfect group is stated.

*Theorem 20:* [11] If $G$ is a finite perfect group and $G \cong F/R$, where $F$ is a free group. Then
$(i)$ $F'/[F,R]$ is a covering group of $G$ and the central extension

$$1 \to (F' \cap R)/[F,R] \to F'/[F,R] \to G \to 1$$

is universal.
$(ii)$ If $1 \to A \to G^* \to G \to 1$ is a universal central extension, then $A \cong M(G)$ and $G^*$ is a covering group of $G$.

Steinberg [16] obtained an universal central extension for $PSL_n(F_q)$ and $PSp_{2n}(F_q)$ in the following theorem.

*Theorem 21:* [16] If $q$ is finite, $|F_q| > 4$ and $SL_2(F_9)$ is excluded, then the natural extension
$(i)$ $1 \to Z(SL_n(F_q)) \to SL_n(F_q) \to PSL_n(F_q) \to 1$ is universal.
$(ii)$ $1 \to \{\pm I\} \to Sp_{2n}(F_q) \to PSp_{2n}(F_q) \to 1$ is universal.

Huppert and Hannebauer [15, 17] obtained the following well-known facts concerning the Schur multiplier in the following theorem.

*Theorem 22:* [15, 17]
$(i)$ $M(SL_2(F_q)) = 1$.
$(ii)$ $M(GL_2(F_q)) = 1$.
$(iii)$ If $q \neq 2^n$, then $M(PSL_2(F_q)) = \mathbb{Z}_2$.
$(iv)$ If $q \neq 2^n$, then $M(PGL_2(F_q)) = \mathbb{Z}_2$.

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