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Intersection Normal Graphs of Finite Groups

S. M. S. Omer [1,*], H. M. Mohammed Salih[2]

[1]Department of Mathematics, Faculty of Science, University of Benghazi, AlKufra, Libya
[2]Department of Mathematics, Faculty of Science, Soran University, Kawa St.Soran, Erbil-Iraq
[*]Corresponding author email: sanaa.omer@uob.edu.ly

***Abstract*:** In this paper, a new type of graph on a finite group G, namely the intersection normal graph is defined and studied. The graph is denoted by $\Gamma_G^{in}(N)$, where its vertices are normal subgroups of G, in which two distinct vertices N_1 and N_2 are adjacent if $N_1 \cap N_2 \subseteq N$ for a fixed normal subgroup N of G. In this paper, the intersection graph is shown as a simple connected with a diameter less than or equal to two. Several graph properties are considered. However the graph structure of $\Gamma_G^{in}(\{e\})$, is given for some finite groups such as the dihedral, quaternion and cyclic groups. We achieve this with the aid of the computer algebra system GAP and the YAGs package.

***Keywords*:** Normal subgroup, planer graph, cut-vertex.

1. Introduction

A group theory can be considered as the study of symmetry. A group is basically the collection of symmetries of some object preserving some of its structure; therefore many mathematicians could associated the group theory with graph theory such as in [1–3]. It has been proved that graphs can be interesting tools for the study of groups. Groups linked with graphs have been arguably the most famous and productive area of algebraic graph theory. In the following context, some basics and related works are provided.

A graph is connected if there is a path connecting any two distinct vertices. The distance between two distinct vertices u and v is the length of the shortest path connecting them and denoted by $d(u, v)$ (if such a path does not exist, define $d(u, v) = \infty$). The diameter of a graph G, denoted by $diam(G)$, is defined by the supremum of the distances between vertices. The girth of a graph, denoted by $g(G)$ is the length of the shortest cycle in the graph G. A graph with no cycles has infinite girth. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the graph G and denoted by $rad(G)$. The r-partite graph is one whose vertex can be partitioned into r subsets so that an edge has both ends in no subset. A complete r-partite graph is an r-partite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite graph with part sizes m and n is denoted by $K_{n,m}$. A graph is called complete if each pair of vertices is joined by an edge. We use K_n to denote the complete graph with n vertices.

Two graphs G and H are isomorphic, denoted by $G \cong H$, if there is a bijection $\phi : G \to H$ of vertices such that the vertices x and y are adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are adjacent in H. A connected graph that can be drawn without any edges crossing is called planar. A vertex v of a connected graph G is called a cut vertex of G, if $G \setminus v$ (delete v from G) results in a disconnected graph. Removing a cut vertex from a graph breaks it into two or more graphs [4, 5]. A non empty subset S of a group H is called a subgroup if S is a group, denoted by $S \leq H$. A subgroup S of H is called normal if $hsh^{-1} \in S$ for all $h \in H$ and $s \in S$ [6].

Theorem 1 ([6]): Let N be a minimal normal subgroup of G. For all normal subgroups M of G, either $N \leq M$ or $N \cap M = \{e\}$.

Theorem 2 ((Kuratowski's Theorem) [4]): A graph is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Theorem 3 ([7]): Let G be a finite solvable group. If G has a unique non trivial normal subgroup, then

1. G is a cyclic p-group of order p^2.
2. G is a semidirect product $G = P \rtimes Q$, where P is an elementary abelian p-group and Q is a cyclic group of order q, with p and q being distinct primes. Moreover, the action of Q on P is irreducible.

2. Main Results

In this section, a new graph namely the intersection normal graph is introduced. Besides, some related results are obtained.

Definition 1: Let G be a finite group and $N(G)$ be the set of all normal subgroups of G and N in $N(G)$. The intersection normal graph, denoted by $\Gamma_G^{in}(N)$, is an undirected graph whose vertex set is $N(G)$ and two distinct vertices N_1 and N_2 are adjacent if $N_1 \cap N_2 \subseteq N$.

In the following, the notation $|N(G)|$ denotes the cardinality of $N(G)$. The following two examples indicate the concept of the intersection graph.

Example 1: Consider the group of integers modulo 4, that is $\mathbb{Z}_4$. The intersection normal graphs for normal subgroups $\{0\}, \{0, 2\}$ and $\mathbb{Z}_4$ are $\Gamma_G^{in}(\{0\}) = K_{1,2}$ and $\Gamma_G^{in}(\{0, 2\}) = \Gamma_G^{in}(\mathbb{Z}_4) = K_3$.

Example 2: Consider the Klein four group $V_4 = \{e, a, b, c\}$ where $a^2 = b^2 = c^2 = e$ and $ab = ba, ac = ca, bc = ca$. The intersection normal graph $\Gamma_G^{in}(\{e\})$, is given in Figure 1.

Proposition 1: A finite group G is simple if and only if $\Gamma_G^{in}(N) = K_2$.

Proof: Let G be a simple group. Then it has exactly two normal subgroups $\{e\}$ and G. So they are adjacent. Thus

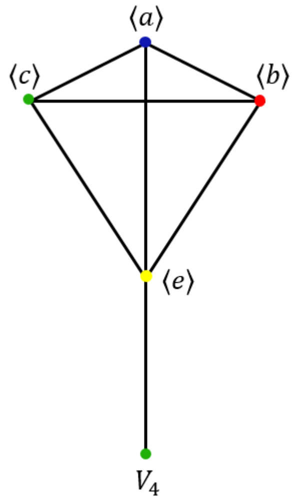

Figure 1. The intersection normal graph of $\{e\}$

$\Gamma_G^{in}(N) = K_2$. Conversely, let $\Gamma_G^{in}(N) = K_2$. This means that G has only two normal subgroups. Therefore, G is simple group. □

The following result tells us the intersection normal graph of the whole group always gives the complete graph. Furthermore, the intersection normal graphs have linear property over intersection.

Lemma 1: Let $N_1, N_2, ..., N_l$ be normal subgroups of a finite group G. Then

1. $\Gamma_G^{in}(G) = K_{|\mathcal{N}(G)|}$.
2. $\Gamma_G^{in}(\bigcap_{i\in\Lambda} N_i) = \bigcap_{i\in\Lambda} \Gamma_G^{in}(N_i)$.

Proof: The proof is straightforward. □

Lemma 2: Let N be a minimal normal subgroup of G and L be a non trivial normal subroup of G. Then $deg_{\Gamma_G^{in}(L)}(\{e\}) = deg_{\Gamma_G^{in}(L)}(N) = |\mathcal{N}(G)|$.

Proof: Based on Theorem 1, thus $N \cap L = N$ or $N \cap L = \{e\}$. In both cases we obtain $N \subseteq L$ or $\{e\} \subset L$. Hence $deg_{\Gamma_G^{in}(L)}(N) = |\mathcal{N}(G)|$. □

Proposition 2: If G has normal subgroups N_i such that $\{e\} = N_0 \subset N_1 \subset N_2 \subset ... \subset N_r \subset N_{r+1} = G$, then

1. $\Gamma_G^{in}(G)$ and $\Gamma_G^{in}(N_r)$ are identical.
2. $\Gamma_G^{in}(\{e\}) = K_{1,r+1}$
3. $\Gamma_G^{in}(N_i)$ is a subgraph of $\Gamma_G^{in}(N_l)$ where $l > i$.

Proof:

1. It is clear that they have the same number of vertices. Let $e = N_iN_j$ be an edge in $\Gamma_G^{in}(N_r)$, that is $N_i \cap N_j \subseteq N_r \subseteq G$. This implies that $e = N_iN_j$ is an edge in $\Gamma_G^{in}(G)$. On the other hand, let $e = N_iN_j$ be an edge in $\Gamma_G^{in}(G)$, that is $N_i \cap N_j \subseteq G$. If either N_i or N_j is N_r, then we are done. If neither N_i nor N_j is N_r, then $N_i \cap N_j = N_{\bar{r}} \subseteq N_r$ where $\bar{r} = min\{i,j\}$. Thus e is an edge of $\Gamma_G^{in}(G)$.
2. Since $N_i \cap N_j = N_l \not\subset \{e\}$ for $i,j \in \{1,2,...,r+1\}$ where $l = min\{i,j\}$ and $N_i \cap \{e\} = \{e\}$, then the result follows.
3. the proof is clear.

□

As a direct consequence of Proposition 2, the following results are obtained.

Corollary 1: If G has normal subgroups $N_1, ..., N_r$ such that $\{e\} \lhd N_1 \lhd ... \lhd N_r \lhd G$.

1. If it has length 3, then $\Gamma_G^{in}(\{e\}) \subseteq \Gamma_G^{in}(N_1) \subseteq \Gamma_G^{in}(G)$ or $(K_{1,2} \subseteq K_3 \subseteq K_3)$.
2. If it has length 4, then $\Gamma_G^{in}(\{e\}) \subseteq \Gamma_G^{in}(N_1) \subseteq \Gamma_G^{in}(N_2) \subseteq \Gamma_G^{in}(G)$
   or $(K_{1,3} \subseteq K_4 \setminus \{one\,edge\} \subseteq K_4 \subseteq K_4)$.

Proposition 3: If G is a finite solvable group and $\Gamma_G^{in}(\{e\}) = K_{1,2}$, then

1. G is a cyclic p-group of order p^2.
2. G is a semidirect product $G = P \rtimes Q$, where P is an elementary abelian p-group and Q is a cyclic group of order q, with p and q being distinct primes. Moreover, the action of Q on P is irreducible

Proof: Since $\Gamma_G^{in}(\{e\}) = K_{1,2}$, then G has a unique non trivial normal subgroup. The proof of rests follow from Theorem 3. □

Note that the converse of 1. in Proposition 3 is true and it can be seen in Proposition 12. The following example shows that the converse of 2. in Proposition 3 is not true.

Example 3: Any group of order 20 has a normal 5-Sylow subgroup and is a semidirect product. But $\Gamma_G^{in}(\{e\}) = K_{1,2}$.

Proposition 4: If G is a finite non solvable group and $\Gamma_G^{in}(\{e\}) = K_{1,2}$, then G can be described as in Theorem 3.

Proof: The proof is similar as Proposition 3. □

Example 4: Consider the general linear group $G = GL(2,3)$ which has 5 normal subgroups such as $\{e\}, C_2, Q_8, SL(2,3)$ and G. The corresponding intersection normal graphs are $K_{1,4} \subseteq H \subseteq K_4 - e \subseteq K_4 \subseteq K_4$ where H is given in Figure 2.

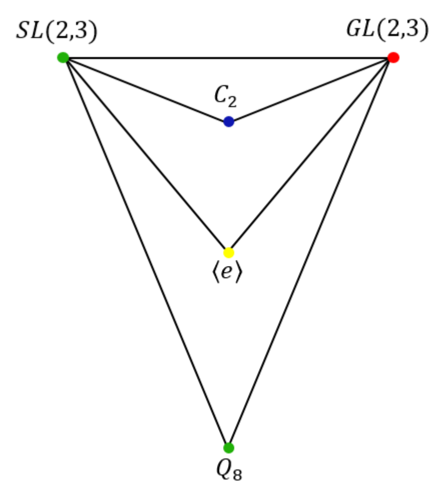

Figure 2. The intersection normal graph of H

Lemma 3: Let $\Gamma_G^{in}(N)$ be the intersection graph. Then

$$d(H_i,H_j)=\begin{cases}1 & if\, H_i\cap H_j\subseteq N\\ 2 & if\, H_i\cap H_j\not\subseteq N\end{cases}$$

Proof: The proof is clear. □

Proposition 5: Let G be a finite group and N be a normal subgroup of G. Then $\Gamma_G^{in}(N)$ is connected graph with diameter at most 2 and radius 1.

Proof: The proof of the connectedness is straightforward and the rest of the proof follows from Lemma 3. □

Proposition 6: If $\Gamma_G^{in}(N)$ contains a cycle, then $\Gamma_G^{in}(N)$ has girth 3.

Proof: Since $\Gamma_G^{in}(N)$ contains a cycle, then there are normal subgroups N_1 and N_2 of G such that $N_1\cap N_2=\{e\}$.These normal subgroups together with $\{e\}$ give K_3. Based on Proposition 2, $\Gamma_G^{in}(\{e\})$ is a subgraph of $\Gamma_G^{in}(N)$. Hence $\Gamma_G^{in}(N)$ has girth 3. □

Proposition 7: Let G be a finite group with at least two minimal normal subgroups. Then $\Gamma_G^{in}(\{e\})$ is not a tree graph.

Proof: Since G has at least two minimal normal subgroups N_1 and N_2, thus the normal subgroups with trivial normal subgroup gives the cycle K_3 in $\Gamma_G^{in}(\{e\})$. Therefore $\Gamma_G^{in}(\{e\})$ is not a tree. □

Theorem 4: Let G be a finite group with non trivial normal proper subgroups $N_1,...,N_r$. If $|E(\Gamma_G^{in}(N_i))|=|E(\Gamma_G^{in}(N_j))|$ and $|N_i|=|N_j|$ for some i,j. Then $\Gamma_G^{in}(N_i)$ and $\Gamma_G^{in}(N_j)$ are isomorphic.

Proof: Define $f: V(\Gamma_G^{in}(N_i))\to V(\Gamma_G^{in}(N_j))$ by $f(H_i)=H_j$, $f(H_j)=H_i$ and $f(H_l)=H_l$ where $l\neq i,j$. It is clear that f is bijective. Let H_mH_s be an edge in $\Gamma_G^{in}(N_i)$ that is $H_m\cap H_s\subseteq H_i$. If $m,s\neq i,j$ then $f(H_m\cap H_s)\subseteq f(H_i)$, that is $H_m\cap H_s\subseteq H_j$. Thus, H_mH_s is an edge in $\Gamma_G^{in}(N_j)$. If $m=i$ or $s=i$ ($m=j$ or $s=j$), then the result follows. □

The following examples show that the converse of Theorem 4 is not true.

Example 5: Let $G=C_{15}$ be a cyclic group. Then, $\Gamma_G^{in}(C_3)\cong\Gamma_G^{in}(C_5)$.

Example 6: The groups D_8 and Q_8 have the same number of normal subgroups. Thus, their intersection normal graphs are identical.

Proposition 8: Let G be a finite group with at least four minimal normal subgroups. Then $\Gamma_G^{in}(N)$ is not a planar graph.

Proof: Suppose that G is a finite group with four minimal normal subgroups N_i for $i=1,2,3,4$. It is clear that $N_i\cap N_j=\{e\}$ for $i\neq j$. These minimal normal subgroups together with trivial normal group produce K_5 in $\Gamma_G^{in}(\{e\})$. From Theorem 2, the result is obtained. □

Proposition 9: Let $\Gamma_G^{in}(\{e\})$ be the intersection graph, with at least two minimal normal subgroups of G then $\{e\}$ is a cut vertex.

Proof: If $\Gamma_G^{in}(\{e\})$ is a star graph, then the proof is clear. If $\Gamma_G^{in}(\{e\})$ is not star graph, then using Lemma 2, $deg_{\Gamma_G^{in}(\{e\})}(\{e\})=l$ where $l=|\mathcal{N}(G)|$ and $deg_{\Gamma_G^{in}(\{e\})}(G)=1$. Therefore, vertex G is isolated vertex in $\Gamma_G^{in}(\{e\})\setminus\{e\}$. Thus, the graph is disconnected and $\{e\}$ is a cut vertex. □

Proposition 10: Let $G=D_{2n}$ be the dihedral group. Then

1. $\Gamma_G^{in}(\{e\})=K_{1,m+1}$ if n is odd and $n=p^m$.
2. $\Gamma_G^{in}(\{e\})=K_{1,m+3}$ if n is even and $n=p^m$.
3. $\Gamma_G^{in}(\{e\})$ is not a tree graph if there exist distinct prime numbers p_i and p_j such that $n=p_1^{m_1}\dots p_r^{m_r}$.

Proof:

1. if n is odd, then it is clear that G has $m+2$ normal subgroups. The rest follows from Proposition 2.
2. if n is even, then G has $m+4$ normal subgroups. By Proposition 2, we have $K_{1,m+3}$.
3. since $n=p_1^{m_1}\dots p_r^{m_r}$ then without loss of generality we assume that $n=p_1^{m_1}p_2^{m_2}$. The proof of the rest follows from Proposition 7.

□

Proposition 11: Let $G=Q_{4n}$ be a quaternion group. Then

1. $\Gamma_G^{in}(\{e\})=K_{1,m+4}$ if $n=2^m$.
2. $\Gamma_G^{in}(\{e\})$ is not tree, otherwise

Proof: The proof is similar as Proposition 10. □

Proposition 12: If G is a cyclic group of order p^n where p is a prime number, then $\Gamma_G^{in}(\{e\})$ is a star graph ($\Gamma_G^{in}(\{e\})=K_{1,n}$).

Proof: If $n=1$, then G is a simple group. From Proposition 1, we have $\Gamma_G^{in}(\{e\})=K_{1,1}$ and the proof of $n>1$ follows from Proposition 2. □

Proposition 13: If G is a finite abelian group such that G is not cyclic, then $\Gamma_G^{in}(\{e\})$ is not a tree graph.

Proof: The proof is similar as in Proposition 10. □

3. Conclusion

In this paper, we introduced a new graph called the intersection normal graph whose vertices are normal subgroups. The graph is found for some famous groups such as dihedral groups, quaternion groups, cyclic groups and others. Besides, some properties of the intersection graph were determined.

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