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Controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$

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***Abstract*:** In this paper, we will introduce a new concept, that of controlled K-operator frame for the space $End_{\mathcal{A}}^{*}(\mathcal{H})$ of all adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ where $\mathcal{A}$ is a C^*-algebra. Next, we give a characterization of controlled K-operator frame in $End_{\mathcal{A}}^{*}(\mathcal{H})$. Also we establish some results. The presented results are new and of interest for people working in this area. Some illustrative examples are provided to advocate the usability of our results.

***Keywords*:** Operator Frame, Controlled operator frame, K-operator frame, Controlled K-operator frame, C^*-algebra, Hilbert $\mathcal{A}$-modules.

1. Introduction

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [6] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [4] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [8]. Frames have been used in signal processing, image processing, data compression and sampling theory.

In 2012, L. Gavruta [9] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K.

Controlled frames in Hilbert spaces have been introduced by P. Balazs [2] to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Rahimi [12] defined the concept of controlled K-frames in Hilbert spaces and showed that controlled K-frames are equivalent to K-frames due to which the controlled operator C can be used as preconditions in applications. Controlled frames in C^*-modules were introduced by Rashidi and Rahimi [10], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space.

K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ has been study by Rossafi and Kabbaj [16].

Motivated by the above literature, we introduce the notion of controlled K-operator frame for Hilbert C^*-modules.

2. Preliminaries

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-modules. Our references for C^*-algebras as [3, 5]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$.

Definition 1: [13] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x,x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x,x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax+y,z \rangle_{\mathcal{A}} = a\langle x,z \rangle_{\mathcal{A}} + \langle y,z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x,y,z \in \mathcal{H}$.

(iii) $\langle x,y \rangle_{\mathcal{A}} = \langle y,x \rangle_{\mathcal{A}}^{*}$ for all $x,y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x,x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in the C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x,x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, A map $T : \mathcal{H} \rightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \rightarrow \mathcal{H}$ such that $\langle Tx,y \rangle_{\mathcal{A}} = \langle x,T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^{*}(\mathcal{H})$.

The following lemmas will be used to prove our mains results.

Lemma 1: [1]. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{K})$. Then the following statements are equivalent:

(i) T is surjective.

(ii) T^* is bounded below with respect to norm, i.e.: there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathcal{K}$.

(iii) T^* is bounded below with respect to the inner product, i.e.: there is $m' > 0$ such that,

$$\langle T^*x, T^*x \rangle_{\mathcal{A}} \geq m' \langle x,x \rangle_{\mathcal{A}}, x \in \mathcal{K}$$

.

Lemma 2: [13]. Let $\mathcal{H}$ be an Hilbert $\mathcal{A}$-module. If $T \in End_{\mathcal{A}}^{*}(\mathcal{H})$, then

$$\langle Tx,Tx \rangle_{\mathcal{A}} \leq \|T\|^2 \langle x,x \rangle_{\mathcal{A}}, \qquad x \in \mathcal{H}.$$

For the following theorem, $R(T)$ denote the range of the operator T.

Theorem 1: [7] Let E, F and G be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$. Let $T \in End_{\mathcal{A}}^{*}(E,F)$ and $T' \in End_{\mathcal{A}}^{*}(G,F)$ with $\overline{(R(T^*))}$ is orthogonally complemented. Then the following statements are equivalent:

(1) $T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$.

(2) There exists $\mu > 0$ such that $\|(T')^*x\| \leq \mu\|T^*x\|$ for all $x \in F$.

(3) There exists $D \in End^*(G,E)$ such that $T' = TD$, that is the equation $TX = T'$ has a solution.

(4) $R(T') \subseteq R(T)$.

3. Controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$

We begin this section with the following definitions.

Definition 2: [14] Let $\mathcal{H}$ be a Hilbrt $\mathcal{A}$-module over a unital C^*-algebra, $C \in GL^+(\mathcal{H})$ and $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$. A family $\{x_i\}_{i\in I}$ in $\mathcal{H}$ is said to be C-controlled K-frame if there exist two constants $0 < A \leq B < \infty$ such that

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle Cx_i,x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}},$$

for all $x \in \mathcal{H}$.

The sequence $\{x_i\}_{i\in I}$ is called a C-controlled Bessel sequence with bound B, if there exists $B > 0$ such that,

$$\sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle Cx_i,x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x \in \mathcal{H}$$

where the sum in the above inequalities converges in norm.

If A = B, we call this C-controlled K-frame a tight C-controlled K-frame, and if A = B = 1 it is called a Parseval C-controlled K-frame.

Definition 3: [15] A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist two positives constants $A, B > 0$ such that

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_ix,T_ix\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x \in \mathcal{H}. \quad (1)$$

The numbers A and B are called lower and upper bound of the operator frame, respectively.

Definition 4: [16] Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$. A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist two positives constants $A, B > 0$ such that

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_ix,T_ix\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x \in \mathcal{H}. \quad (2)$$

The numbers A and B are called lower and upper bound of the operator frame, respectively.

Let $GL^+(\mathcal{H})$ be the set for all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

Definition 5: [11] Let $C, C' \in GL^+(\mathcal{H})$, a family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a (C,C')-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x \in \mathcal{H}. \quad (3)$$

The numbers A and B are called lower and upper bounds of the (C,C')-controlled operator frame , respectively.

If $A = B = \lambda$, the (C,C')-controlled operator frame is called λ-tight.

If $A = B = 1$, it is called a normalized tight (C,C')-controlled operator frame or a Parseval (C,C')-controlled operator frame .

If only upper inequality of Eq. (3) hold, then $\{T_i\}_{i\in i}$ is called a (C,C')-controlled operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Definition 6: Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a (C,C')-controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist two positives constants $A, B > 0$ such that

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x \in \mathcal{H}. \quad (4)$$

The numbers A and B are called lower and upper bounds of the (C,C')-controlled K-operator frame , respectively.

If $A = B = \lambda$, the (C,C')-controlled K-operator frame is called λ-tight.

If $A = B = 1$, it is called a normalized tight (C,C')-controlled K-operator frame or a Parseval (C,C')-controlled operator frame .

If only upper inequality of Eq. (4) hold, then $\{T_i\}_{i\in i}$ is called an (C,C')-controlled K-operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Example 1: Let $C \in GL^+(\mathcal{H})$, $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{x_i\}_{i\in I}$ be a C-controlled K-frame for $\mathcal{H}$.

Let $(\Gamma_i)_{i\in I} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ such that :

$$\Gamma_i(x) = \langle x,x_i\rangle e_i, \qquad for\ all \quad i \in I \quad and \quad x \in \mathcal{H},$$

where $\langle e_i,e_j\rangle_{\mathcal{A}} = \delta_{ij}1_{\mathcal{A}}$.

Since $\{x_i\}_{i\in I}$ is a C-controlled K-frame for $\mathcal{H}$, then there exist $A, B > 0$ such that :

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle Cx_i,x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}},$$

for all $x \in \mathcal{H}$.

Hence,

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle e_i,e_i\rangle_{\mathcal{A}}\langle Cx_i,x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}} \qquad x \in \mathcal{H}.$$

So,

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle\langle x,x_i\rangle_{\mathcal{A}}e_i, \langle x,Cx_i\rangle_{\mathcal{A}}e_i\rangle_{\mathcal{A}} \\ \leq B\langle x,x\rangle_{\mathcal{A}} \qquad x\in\mathcal{H}.$$

Since C is a selfadjoint operator then,

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle\langle x,x_i\rangle_{\mathcal{A}}e_i, \langle Cx,x_i\rangle_{\mathcal{A}}e_i\rangle_{\mathcal{A}} \\ \leq B\langle x,x\rangle_{\mathcal{A}} \qquad x\in\mathcal{H}.$$

Therefore,

$$A\langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle\Gamma_i x, \Gamma_i Cx\rangle_{\mathcal{A}} \\ \leq B\langle x,x\rangle_{\mathcal{A}} \qquad x\in\mathcal{H}.$$

Since C is a surjective operator, from lemma 1, there exists $m>0$, such that

$$m\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \langle C^{\frac{1}{2}}K^*x, C^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}}.$$

Then,

$$Am\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle\Gamma_i x,\Gamma_i Cx\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}.$$

Then $(\Gamma_i)_{i\in I}$ is a $(Id_{\mathcal{H}},C)$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proposition 1: Every $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ is a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proof 1: For any $K\in End_{\mathcal{A}}^*(\mathcal{H})$, we have,

$$\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \|K\|^2\langle x,x\rangle_{\mathcal{A}}.$$

Let $\{T_i\}_{i\in I}$ be a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B.

Then,

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}.$$

Hence,

$$A\|K\|^{-2}\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}} \\ \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}.$$

Therefore, $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds $A\|K\|^{-2}$ and B.

Proposition 2: Let $\{T_i\}_{i\in I}$ be a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. If K is surjective then $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proof 2: Suppose that K is surjetive, from lemma 1 there exists $0<m$ such that

$$\langle K^*x,K^*x\rangle_{\mathcal{A}} \geq m\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H} \tag{5}$$

Let $\{T_i\}_{i\in I}$ be a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B. Hence,

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}. \tag{6}$$

Using 5 and 6, we have

$$Am\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proposition 3: Let $C,C^{'}\in GL^+(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be a K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. Assume that C and C' commute with T_i and K. Then $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proof 3: Let $\{T_i\}_{i\in I}$ be a K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Then there exist $A,B>0$ such that

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_ix,T_ix\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, x\in\mathcal{H}. \tag{7}$$

Since,

$$\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}} = \sum_{i\in I}\langle T_i(CC')^{\frac{1}{2}}x, T_i(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\ \leq B\langle (CC')^{\frac{1}{2}}x,(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\ \leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}},$$

then,

$$\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}} \leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}}. \tag{8}$$

Moreover,

$$\sum_{i\in I}\langle T_i(CC')^{\frac{1}{2}}x, T_i(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} \geq A\langle K^*(CC')^{\frac{1}{2}}x, K^*(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\ \geq A\langle (CC')^{\frac{1}{2}}K^*x,(CC')^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}}.$$

Since $(CC')^{\frac{1}{2}}$ is a surjecive operator, then there exists $m>0$ such that,

$$\langle (CC')^{\frac{1}{2}}K^*x,(CC')^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}} \geq m\langle K^*x,K^*x\rangle_{\mathcal{A}}. \tag{9}$$

From 8 and 9, we have,

$$Am\langle K^*x,K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}} \leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}}.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled Bessel K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. We assume that C and C' commute with T_i and T_i^*.

We define the operator $T_{(C,C')}:\mathcal{H}\to l^2(\mathcal{H})$ by

$$T_{(C,C')}x=\{T_i(CC')^{\frac{1}{2}}x\}_{i\in I}.$$

There adjoint operator is defined by $T_{(C,C')}^*: l^2(\mathcal{H})\to\mathcal{H}$ given by,

$$T_{(C,C')}^*(\{a_i\}_{i\in I})=\sum_{i\in I}(CC')^{\frac{1}{2}}T_i^*a_i$$

is called the synthesis operator.

If C and $C^{'}$ commute between them, and commute with the operators $T_i^*T_i$ for each $i\in I$. We define the $(C,C^{'})$-controlled Bessel K-operator frame by:

$$\begin{aligned}S_{(C,C')}:&\mathcal{H}\longrightarrow\mathcal{H}\\ &x\longrightarrow S_{(C,C')}x=T_{(C,C')}T_{(C,C')}^*x.\end{aligned}$$

$T_{(C,C')}$ and $T_{(C,C')}^*$ are called the synthesis and analysis operator of $(C,C')-$controlled Bessel K-operator frame $\{T_i\}_{i\in I}$ respectively.

It's clear to see that $S_{(C,C')}$ is positive, bounded and selfadjoint.

Theorem 2: Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled Bessel K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. The following statements are equivalent:

(1) $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame.

(2) There is $A>0$ such that $S_{(C,C')}\geq AKK^*$.

(3) $K=S_{(C,C')}^{\frac{1}{2}}Q$, for some $Q\in End_{\mathcal{A}}^{*}(\mathcal{H})$.

Proof 4: (1) $\Longrightarrow$ (2)

Assume that $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and B, with frame operator $S_{(C,C')}$, then

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}}\leq\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Therefore,

$$A\langle KK^*x,x\rangle_{\mathcal{A}}\leq\langle\sum_{i\in I}C'T_i^*T_iCx,x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Hence,

$$S_{(C,C')}\geq AKK^*.$$

(2) $\Longrightarrow$ (3)

Let $A>0$ such that

$$S_{(C,C')}\geq AKK^*.$$

This give,

$$S_{(C,C')}^{\frac{1}{2}}S_{(C,C')}^{\frac{1}{2}*}\geq AKK^*.$$

From theorem 1, we have,

$$K=S_{(C,C')}^{\frac{1}{2}}Q$$

with $Q\in End_{\mathcal{A}}^{*}(\mathcal{H})$.

(3) $\Longrightarrow$ (1)

Suppose that ,

$$K=S_{(C,C')}^{\frac{1}{2}}Q$$

for some $Q\in End_{\mathcal{A}}^{*}(\mathcal{H})$.

From theorem 1 there exists $A>0$ such that,

$$AKK^*\leq S_{(C,C')}^{\frac{1}{2}}S_{(C,C')}^{\frac{1}{2}*}.$$

Hence,

$$AKK^*\leq S_{(C,C')}.$$

Therefore, $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Theorem 3: Let $K,Q\in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. Suppose that Q commute with C , C' and K. Then $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled Q^*K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Proof 5: Suppose that $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame with frame bounds A and B. Then,

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}}\leq\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Hence,

$$\begin{aligned}A\langle K^*Qx,K^*Qx\rangle_{\mathcal{A}}&\leq\sum_{i\in I}\langle T_iCQx,T_iC^{'}Qx\rangle_{\mathcal{A}}\\&\leq B\langle Qx,Qx\rangle_{\mathcal{A}},x\in\mathcal{H}.\end{aligned}$$

So,

$$\begin{aligned}A\langle(Q^*K)^*x,(Q^*K)^*x\rangle_{\mathcal{A}}&\leq\sum_{i\in I}\langle T_iQCx,T_iQC^{'}x\rangle_{\mathcal{A}}\\&\leq B\|Q\|^2\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.\end{aligned}$$

Therefore $\{T_iQ\}_{i\in I}$ is a (C,C')-controlled Q^*K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and $B\|Q\|^2$.

Theorem 4: Let $K\in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be a (C,C')-controlled tight K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bound A_1. Then $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bound A_2 if and only if $K_r^{-1}=\frac{A_1}{A_2}K^*$.

Proof 6: Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled tight K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bound A_1.

Assume that $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bound A_2. Then,

$$\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}=A_2\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Since $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, then we have,

$$A_1\langle K^*x,K^*x\rangle_{\mathcal{A}}=\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}.$$

Hence,

$$A_1\langle K^*x,K^*x\rangle_{\mathcal{A}}=A_2\langle x,x\rangle_{\mathcal{A}}.$$

So,

$$\langle KK^*x,x\rangle_{\mathcal{A}}=\langle\frac{A_2}{A_1}x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Then,

$$KK^*=\frac{A_2}{A_1}Id_{\mathcal{A}}.$$

Therefore,

$$K_r^{-1}=\frac{A_1}{A_2}K^*.$$

For the converse, assume that

$$K_r^{-1}=\frac{A_1}{A_2}K^*.$$

Then,

$$KK^*=\frac{A_2}{A_1}Id_{\mathcal{A}}.$$

This give that,

$$\langle KK^*x,x\rangle=\langle\frac{A_2}{A_1}x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Since $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bound A_1, the we have,

$$\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}=A_2\langle x,x\rangle_{\mathcal{A}},x\in\mathcal{H}.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Corollary 1: Let $K\in End_{\mathcal{A}}^*(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be a sequence for $End_{\mathcal{A}}^*(\mathcal{H})$. Then those statements are true,

(1) If $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$, then $\{T_i(K^n)^*\}_{i\in I}$ is a $(C,C')-$controlled tight K^{n+1}-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

(2) If $\{T_i\}_{i\in I}$ is a $(C,C')-$controlled tight operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ then $\{T_iK^*\}_{i\in I}$ is a $(C,C')-$controlled tight K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Theorem 5: Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with best frame bounds A and B. If $Q:\mathcal{H}\rightarrow\mathcal{H}$ is an adjointable and invertible operator such that Q^{-1} commutes with K^*, then $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with best frame bounds M and N satisfying the inequalities,

$$A\|Q^{-1}\|^{-2}\leq M\leq A\|Q\|^2\,and\,A\|Q^{-1}\|^{-2}\leq N\leq B\|Q\|^2. \tag{10}$$

Proof 7: Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with best frame bounds A and B. Then,

$$\sum_{i\in I}\langle T_iCQx,T_iC^{'}Qx\rangle_{\mathcal{A}}\leq B\langle Qx,Qx\rangle_{\mathcal{A}}\leq B\|Q\|^2\langle x,x\rangle_{\mathcal{A}}.$$

Also we have,

$$\begin{aligned}A\langle K^*x,K^*x\rangle_{\mathcal{A}}&=A\langle K^*Q^{-1}Qx,K^*Q^{-1}Qx\rangle_{\mathcal{A}}\\&=A\langle Q^{-1}K^*Qx,Q^{-1}K^*Qx\rangle_{\mathcal{A}}\\&\leq\|Q^{-1}\|^2\sum_{i\in I}\langle T_iCQx,T_iC^{'}Qx\rangle_{\mathcal{A}}\\&=\|Q^{-1}\|^2\sum_{i\in I}\langle T_iQCx,T_iQC^{'}x\rangle_{\mathcal{A}}.\end{aligned}$$

Hence,

$$\begin{aligned}A\|Q^{-1}\|^{-2}\langle K^*x,K^*x\rangle_{\mathcal{A}}&\leq\sum_{i\in I}\langle T_iQCx,T_iQC^{'}x\rangle_{\mathcal{A}}\\&\leq B\|Q\|^2\langle x,x\rangle_{\mathcal{A}}.\end{aligned}$$

Therefore, $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds $A\|Q^{-1}\|^{-2}$ and $B\|Q\|^2$.

Now let M and N be the best bounds of the $(C,C')-$controlled K-operator frame $\{T_iQ\}_{i\in I}$. Then,

$$A\|Q^{-1}\|^{-2}\leq M\quad and\quad N\leq B\|Q\|^2. \tag{11}$$

Also, $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame bounds M and N, and

$$\begin{aligned}\langle K^*x,K^*x\rangle_{\mathcal{A}}=\langle QQ^{-1}K^*x,QQ^{-1}K^*x\rangle_{\mathcal{A}}\leq\|Q\|^2\\\langle K^*Q^{-1}x,K^*Q^{-1}x\rangle_{\mathcal{A}},x\in\mathcal{H}.\end{aligned}$$

Hence

$$\begin{aligned}M\|Q\|^{-2}\langle K^*x,K^*x\rangle_{\mathcal{A}}&\leq M\langle K^*Q^{-1}x,K^*Q^{-1}x\rangle_{\mathcal{A}}\\&\leq\sum_{i\in\mathbb{J}}\langle T_iQCQ^{-1}x,T_iQC'Q^{-1}x\rangle_{\mathcal{A}}\\&\leq\sum_{i\in\mathbb{J}}\langle T_iQQ^{-1}Cx,T_iQQ^{-1}C'x\rangle_{\mathcal{A}}\\&=\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}\\&\leq N\|Q^{-1}\|^2\langle x,x\rangle_{\mathcal{A}}.\end{aligned}$$

Since A and B are the best bounds of $(C,C')-$controlled K-operator frame $\{T_i\}_{i\in I}$, we have

$$C\|Q\|^{-2}\leq A\quad and\quad B\leq D\|Q^{-1}\|^2. \tag{12}$$

Therfore the inequality 10 follows from 12 and 11.

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