Journal of Computer Science & Computational Mathematics, *Volume 11, Issue 2, June 2021*
DOI: 10.20967/jcscm.2021.02.003

Application of the $\exp(-\phi(\xi))$-Expansion Method for Exact Solutions of the Non-Homogeneous Radhadkrishnan-Kundu-Lakshmanan Equation

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***Abstract*:** The aim of this paper, we introduce the non-homogeneous Radhadkrishnan-Kundu-Lakshmanan (RKL) equation. The RKL equation is nonlinear partial differential equation and non-homogeneous which has external force term at right hand side. We are interested in finding the exact traveling wave solutions by applying the $\exp(-\phi(\xi))$-expansion method for the RKL equation. These solutions are obtained, including kinds trigonometric function, hyperbolic function, exponential function and rational function solutions. The solutions show that the simplicity and efficiency of the used approaches that can be applied for nonlinear equations as well as linear ones. For the aid of symbolic computation, we use Maple program, to determine the solutions which are validated with all exact solutions. Some solutions have been represented by graphical. The graphical solutions of the RKL equation are also shown with real part, imaginary part and complex modulus.

***Keywords*:** the $\exp(-\phi(\xi))$-expansion method, the non-homogeneous Radhadkrishnan-Kundu-Lakshmanan equation, trigonometric function solutions, hyperbolic function solutions, exponential function solutions

1. Introduction

In various branches of chemistry, biology, physics and engineering, the partial differential equations play as a basic tool for solving different kinds of problems which arises in these areas. In the past, the higher order nonlinear Schrödinger equations are essential module for nonlinear optics which explained the propagation especially short pulses in optical fibers and have a wide applications in ultrafast signal-routing, telecommunication system etc. The parameters involving in higher order Schrödinger equations are utilized to explain the pulse propagation in optical fibers. Optical solitons occur due the balance of group velocity dispersion and nonlinear effect. Now, the governing model for the propagation of solitons through an optical fiber is given by the following generalized form of the non-homogeneous Radhadkrishnan-Kundu-Lakshmanan equation [1]:

$$iu_t + au_{xx} + bF(|u|^2)u = i\alpha\left(F(|u|^2)u\right)_x - i\beta u_{xxx} + G(x,t), \qquad (1)$$

where $i=\sqrt{-1}$ and a, b, α, $\beta \in \mathbb{R}$. The dependent variable $u(x,t)$ represents the complex valued wave function with the independent variables being x and t that represent space and time. The function F is some types of nonlinearity that will be considered and $G(x,t)$ is an external force term. Under external force field, the particle acceleration or deceleration process during this time interval results in a distortion of the distribution function in the velocity space [2]. The four laws of nonlinearity are given by

1. Kerr law : $F(s) = s$,
2. Power law :$F(s) = s^m$,
3. Parabolic law : $F(s) = s + k_1 s^2$,
4. Dual-power law : $F(s) = s^m + k_2 s^{2m}$,

where the index m dictates the power law or the dual-power law nonlinearity, the constants k_1 and k_2 bind the two laws of nonlinearity in parabolic law and dual-power law. Thus, if $k_1 = 0$, parabolic law falls back to Kerr law and if $k_2 = 0$, the dual-power law collapses to the power law nonlinearity. Also, if $m = 1$, power law reduces to Kerr law and dual-power law reduces to parabolic law.

Many influential methods have been developed for a solitary waves solution of a nonlinear partial equation such as the homogeneous balance method [3], the extended $\tanh$-function method [4], the Jacobi elliptic function expansion method [5], the expo method [6], the homotopy perturbation method [7], the F-expansion method [8], the simplest equation method [9], the extended simple equation method [10], the (G'/G)-expansion method [11], the Hirotas bilinear method [12], the exp function method [13], the $\tan(\phi(\xi)/2)$-expansion method [14], the general projective Riccati equation method [15], the modified simple equation method [16], the extended direct algebraic method [17], the auxiliary method, the Ansatz method [18], the Kudryashov method [19], the modified trial equation method [20], the first integral method [21], the fractional (G'/G)-expansion method [22], the fractional expo-function method [23], the fractional sub-equation method [24], the fractional functional variable method [25], the fractional modified trial equation method

[26], and so on. The study about solutions, structures, interaction and further properties of methods give much more attention and different meaningful results are fruitfully obtained [27].

In this paper, the non-homogeneous Radhakrishnan-Kundu-Lakshmanan (RKL) equation [28] with Kerr law nonlinearity, in dimensionless form, for the propagation of solitons through an optical fiber is introduced

$$iu_t + au_{xx} + b|u|^2u - i\alpha\left(|u|^2u\right)_x + i\beta u_{xxx} = \delta e^{i(kx-ct)}, \tag{2}$$

where δ is real constant. The first term on the left side represents the temporal evolution of the nonlinear wave, while the coefficient a is the group-velocity dispersion (GVD) and b represents the coefficient of nonlinearity. For the perturbation terms, the coefficient of α represents the self-steepening term for short pulses and the coefficient of third order dispersion term is represented by β. We are interested in the $\exp(-\phi(\xi))$-expansion method for finding new exact solutions of the dimensionless form of the non-homogeneous with time-dependent GVD, nonlinearity and intermodulation distortion (IMD).

2. Description of the $\exp(-\phi(\xi))$-expansion method

Consider a nonlinear partial differential equations (NPDEs) of the type:

$$P(u, u_t, u_x, u_tu_t, u_tu_x, \ldots) = 0, \tag{3}$$

where u is an unknown function. P is a polynomial of u and its partial derivatives, which the highest order derivatives and the nonlinear terms are involved.
This makes the solution procedure extremely simple by using the traveling wave variable as

$$u(x,t) = U(\xi), \quad \xi = \tau x + \lambda t, \tag{4}$$

where τ and λ are nonzero arbitrary constants. We can rewrite Eq. (3) in the following nonlinear ordinary differential equations (NODEs):

$$Q(U, U', U'', U''', \ldots) = 0, \tag{5}$$

where the prime denotes the derivation with respect to ξ. We should integrate Eq. (5) term by term as soon as possible.

Step 1: Suppose a solution of Eq. (5) can be expressed as follows:

$$U(\xi) = \sum_{i=-n}^{n} a_i (e^{-\phi(\xi)})^i, \tag{6}$$

where a_i, $(i = 0, \pm1, \pm2, ..., \pm n)$ are constants to be determined later. The function $\phi(\xi)$ is a general solution of equation:

$$\phi'(\xi) = pe^{-\phi(\xi)} + qe^{\phi(\xi)} + r, \tag{7}$$

where p, q and r are constants. We can know that Eq. (7) has different solutions as follows:

Type 1: When $p = 1$, we obtain

$$\phi_1(\xi) = \ln\left(\frac{-\sqrt{r^2-4q}\tanh\left(\frac{1}{2}\sqrt{r^2-4q}(\xi+\xi_0)\right) - r}{2q}\right),$$

where $q \neq 0$ and $r^2 - 4q > 0$,

$$\phi_2(\xi) = \ln\left(\frac{\sqrt{r^2-4q}\tan\left(\frac{1}{2}\sqrt{r^2-4q}(\xi+\xi_0)\right) - r}{2q}\right),$$

where $q \neq 0$ and $r^2 - 4q < 0$,

$$\phi_3(\xi) = -\ln\left(\frac{r}{e^{r(\xi+\xi_0)} - 1}\right), \quad q = 0, r \neq 0, r^2 - 4q > 0,$$

$$\phi_4(\xi) = -\ln\left(-\frac{2r(\xi+\xi_0)+4)}{r^2(\xi+\xi_0)}\right), q \neq 0, r \neq 0, r^2 - 4q = 0.$$

Type 2: When $r = 0$, we obtain

$$\phi_5(\xi) = \ln\left(\sqrt{\frac{p}{q}}\tan\left(\sqrt{pq}(\xi+\xi_0)\right)\right), \quad p > 0, q > 0,$$

$$\phi_6(\xi) = \ln\left(-\sqrt{\frac{p}{q}}\cot\left(\sqrt{pq}(\xi+\xi_0)\right)\right), \quad p > 0, q > 0,$$

$$\phi_7(\xi) = \text{sgn}(p)\ln\left(-\sqrt{\frac{-p}{q}}\tanh\left(\sqrt{-pq}(\xi+\xi_0)\right)\right), \quad pq < 0,$$

$$\phi_8(\xi) = \text{sgn}(p)\ln\left(-\sqrt{\frac{-p}{q}}\coth\left(\sqrt{-pq}(\xi+\xi_0)\right)\right), \quad pq < 0.$$

Type 3: When $q = 0$ and $r = 0$, we obtain

$$\phi_9(\xi) = \ln\left(p(\xi+\xi_0)\right),$$

where ξ_0 is an integrating constant.

Step 2: The value of the number n is a positive integer and given by

$$n = \frac{p - rs}{q + s - 1}. \tag{8}$$

By balancing, where the parameter p is the highest linear term in Eq. (5)

$$D\left(D_\xi^{(p)}U(\xi)\right), \quad p = 1, 2, 3, \ldots,$$

and three parameters q, r and s are in the highest nonlinear term in Eq. (5) as:

$$D\left(U^q\left(D_\xi^{(r)}U(\xi)\right)^s\right), \; q = 1, 2, \ldots, r = 1, 2, \ldots, s = 0, 1, 2, \ldots.$$

Step 3: Substitute Eq. (6) into Eq. (5) along with Eq. (7). After collecting all the terms with the same order $(e^{-\phi(\xi)})^j$ together. It obtains the polynomial equation in $(e^{-\phi(\xi)})^j$, $(j = 0, 1, 2, ...)$. Equating each coefficient of the resulting polynomial to be zero, then an over-determined set of algebraic equations for a_i, $(i = 0, 1, 2, ..., n)$ is obtained. Consequently, we obtain the exact solutions of Eq. (3).

3. Exact solutions by the $\exp(-\phi(\xi))$-expansion method

In this section, we show step by step of the $\exp(-\phi(\xi))$-expansion method to construct the exact solutions of the non-homogeneous RKL equation as following :

$$iu_t + au_{xx} + b|u|^2u - i\alpha\left(|u|^2u\right)_x + i\beta u_{xxx} = \delta e^{i(kx-ct)}, \tag{9}$$

where k, c, a, b, α, β and δ are real constants and u denoted a solution of Eq. (9). First we take the traveling wave transformation as

$$u(x,t) = e^{i\theta}U(\xi), \ \ \theta = kx - ct \ \text{ and } \ \xi = \gamma x + vt, \tag{10}$$

where k, c, γ and v are non-zero arbitrary constants. We derive all partial derivatives as:

$$\begin{aligned}
u_t &= ve^{i\theta}U' - ice^{i\theta}U = e^{i\theta}\left(vU' - icU\right),\\
u_x &= \gamma e^{i\theta}U' + ike^{i\theta}U = e^{i\theta}\left(U' + ikU\right),\\
u_{xx} &= \gamma^2 e^{i\theta}U'' + 2i\gamma ke^{i\theta}U' - k^2e^{i\theta}U,\\
u_{xxx} &= \gamma^3e^{i\theta}U''' + 3i\gamma^2ke^{i\theta}U'' - 3\gamma k^2e^{i\theta}U' - ik^3e^{i\theta}U\\
&= e^{i\theta}\left(\gamma^3U''' + 3i\gamma^2kU'' - 3\gamma k^2U' - ik^3U\right),\\
\left(|u|^2u\right)_x &= \left(e^{i\theta}U^3\right)_x = 3\gamma e^{i\theta}U^2U' + ike^{i\theta}U^3\\
&= e^{i\theta}\left(3\gamma U^2U' + ikU^3\right).
\end{aligned}$$

Substituting all derivatives, Eq. (9) can be reduced to the integrable nonlinear ordinary differential equation

$$e^{i\theta}\Big[\left(c - ak^2 + \beta k^3\right)U + \left(a\gamma^2 - 3\beta\gamma^2k\right)U'' + (b+\alpha k)U^3 - 3i\alpha U^2U' + i\beta\gamma^3U''' + i\left(v + 2a\gamma k - 3\beta\gamma k^2\right)U' - \delta\Big] = 0,$$

we finally have

$$\begin{aligned}
&i\left[\left(v + 2\gamma ak - 3\beta\gamma k^2\right)U' - 3\alpha U^2U' + \beta\gamma^3U'''\right]\\
&+ \left(c - ak^2 + \beta k^3\right)U + \left(a\gamma^2 - 3\beta\gamma^2k\right)U''\\
&+ (b+\alpha k)U^3(\xi) = \delta
\end{aligned} \tag{11}$$

Separating into real and imaginary parts, we obtain

$$\text{Re:} \ \left(c - ak^2 + \beta k^3\right)U + \left(a\gamma^2 - 3\beta\gamma^2k\right)U'' + (b+\alpha k)U^3 = \delta, \tag{12}$$

$$\text{Im}: \ \left(v + 2\gamma ak - 3\beta\gamma k^2\right)U' - 3\alpha U^2U' + \beta\gamma^3U''' = 0. \tag{13}$$

Integrating Eq. (13), we get

$$\text{Im:} \ \left(v + 2\gamma ak - 3\beta\gamma k^2\right)U - \alpha U^3 + \beta\gamma^3U'' = c_1. \tag{14}$$

By balancing the order between U'' and U^3 in Eq. (12) and Eq. (14) expressed as $U'' : D\left(\frac{d^pU(\xi)}{d\xi^p}\right)$ is $p = 2$, and the degree of U^3 in Eq. (12) expressed as $U^3 : D\left[U^q\left(\frac{d^rU(\xi)}{d\xi^r}\right)^s\right]$ is $q = 3$ and $s = 0$, we can obtain $n = 1$. Now, the solution of the $\exp(-\phi(\xi))$-expansion method for Eq. (12) and Eq. (14) can be written by

$$U = a_{-1}e^{\phi} + a_0 + a_1e^{-\phi}, \tag{15}$$

where the function ϕ with respect to ξ and we have the ordinary differential equation

$$\phi' = pe^{-\phi} + qe^{\phi} + r. \tag{16}$$

Taking derivatives and Eq. (16), it obtains

$$\begin{aligned}
U' &= a_{-1}\phi'e^{\phi} - a_1\phi'e^{-\phi}\\
&= a_{-1}p + a_{-1}qe^{2\phi} - a_1q + \frac{a_{-1}r}{e^{-\phi}} - \frac{a_1p}{e^{2\phi}} - \frac{a_1r}{e^{\phi}},
\end{aligned} \tag{17}$$

$$\begin{aligned}
U'' &= 2qa_{-1}\phi'e^{2\phi} + ra_{-1}\phi'e^{\phi} + 2a_1p\phi'e^{-2\phi} + a_1r\phi'e^{-\phi}\\
&= \frac{3a_{-1}qr}{e^{-2\phi}} + \frac{2a_{-1}pq}{e^{-\phi}} + a_{-1}pr + \frac{3a_1pr}{e^{2\phi}} + a_1qr + \frac{a_1r^2}{e^{\phi}}\\
&\quad + \frac{2a_1pq}{e^{\phi}} + \frac{2a_{-1}q^2}{e^{-3\phi}} + \frac{a_{-1}r^2}{e^{-\phi}} + \frac{2a_1p^2}{e^{3\phi}}.
\end{aligned} \tag{18}$$

Substituting U and U'' into Eq. (12) and collecting all the terms with same order of $e^{j\phi}$, $j = -3, -2, -1, 0, 1, 2, 3$ and equating each coefficient to zero, then the system of seven algebraic equations become

$$\begin{aligned}
e^{-3\phi} &: 2aa_1\gamma^2p^2 - 6a_1\gamma^2kp^2\beta + a_1^3k\alpha + a_1^3b = 0,\\
e^{-2\phi} &: 3a_0a_1^2b + 3a_0a_1^2\alpha k - 9a_1pr\beta\gamma^2k + 3a_1pra\gamma^2 = 0,\\
e^{-\phi} &: a_1\beta k^3 - a_1ak^2 + 3a_{-1}a_1^2b + 3a_0^2a_1b + 3a_0^2a_1\alpha k\\
&\quad + 2a_1pqa\gamma^2 - 6a_1pq\beta\gamma^2k + a_1r^2a\gamma^2 + 3a_{-1}a_1^2\alpha k\\
&\quad - 3a_1r^2\beta\gamma^2k + a_1c = 0,\\
e^{0} &: 6a_{-1}a_0a_1b + a_0c + a_0^3b - a_0ak^2 + a_0^3\alpha k + a_0\beta k^3\\
&\quad + 6a_{-1}a_0a_1\alpha k + a_{-1}pra\gamma^2 - 3a_{-1}pr\beta\gamma^2k - \delta\\
&\quad + a_1qra\gamma^2 - 3a_1qr\beta\gamma^2k = 0,\\
e^{\phi} &: a_{-1}\beta k^3 - a_{-1}ak^2 + 3a_{-1}^2a_1b + 3a_{-1}a_0^2b + a_{-1}c\\
&\quad + 3a_{-1}^2a_1\alpha k + 3a_{-1}a_0^2\alpha k + 2a_{-1}pqa\gamma^2\\
&\quad + a_{-1}r^2a\gamma^2 - 6a_{-1}pq\beta\gamma^2k - 3a_{-1}r^2\beta\gamma^2k = 0,\\
e^{2\phi} &: 3a_{-1}^2a_0b + 3a_{-1}^2a_0\alpha k - 9a_{-1}qr\beta\gamma^2k\\
&\quad + 3a_{-1}qra\gamma^2 = 0,\\
e^{3\phi} &: 2aa_{-1}g^2q^2 - 6a_{-1}\gamma^2kq^2\beta + a_{-1}^3k\alpha + a_{-1}^3b = 0.
\end{aligned} \tag{19}$$

Similarly, substituting U and U'' into Eq. (14), it obtains the system of seven algebraic equations

$$
\begin{aligned}
&e^{-3\phi}: 2a_1\gamma^3p^2\beta - a_1^3\alpha = 0,\\
&e^{-2\phi}: 3a_1\gamma^3pr\beta - 3a_0a_1^2\alpha = 0,\\
&e^{-\phi}: 2\beta\gamma^3a_1pq - 3a_{-1}a_1^2\alpha - 3a_0^2a_1\alpha + a_1v - 3a_1\beta\gamma k^2\\
&\qquad + 2a_1\gamma ak + \beta\gamma^3a_1r^2 = 0,\\
&e^{0}: \beta\gamma^3a_1qr - 3a_0\beta\gamma k^2 + 2a_0\gamma ak - 6\alpha a_{-1}a_0a_1\\
&\qquad + a_0v - \alpha a_0^3 + \beta\gamma^3a_{-1}pr - c_1 = 0,\\
&e^{\phi}: 2\beta\gamma^3a_{-1}pq - 3\alpha a_{-1}^2a_1 - 3\alpha a_{-1}a_0^2 - 3a_{-1}\beta\gamma k^2\\
&\qquad + 2a_{-1}\gamma ak + \beta\gamma^3a_{-1}r^2 + a_{-1}v = 0,\\
&e^{2\phi}: 3a_{-1}\gamma^3qr\beta - 3a_0a_{-1}^2\alpha = 0,\\
&e^{3\phi}: 2a_{-1}\gamma^3q^2\beta - a_{-1}^3\alpha = 0.
\end{aligned}
\tag{20}
$$

Combining Eq. (19) and Eq. (20), solving the system of all algebraic equations with the aid the symbolic mathematical software Maple 17, yields one case of the coefficients a_{-1}, a_0, a_1, k, c_1, δ, c and v as follows

$$
\begin{aligned}
&a_{-1} = \frac{qa_1}{p},\ \ a_0 = \frac{ra_1}{2p},\ \ a_1 = p\gamma\sqrt{\frac{2\beta\gamma}{\alpha}},\ \ k = \frac{b\gamma\beta + a\alpha}{\alpha\beta(3-\gamma)},\\
&c_1 = -2\gamma^4q\beta a_1,\ \ \delta = \frac{2r\gamma^3qa_1(a\alpha + 3b\beta)}{\alpha(3-\gamma)},\\
&v = \frac{\gamma}{2\beta\alpha^2(\gamma-3)^2}(72\gamma^2\beta^2pq\alpha^2 - 6\gamma^3r^2\beta^2\alpha^2 + 4\gamma a^2\alpha^2\\
&\qquad + \gamma^4r^2\beta^2\alpha^2 - 48\gamma^3\beta^2pq\alpha^2 + 4\gamma^2\beta ab\alpha + 6\gamma^2\beta^2b^2\\
&\qquad - 6a^2\alpha^2 + 9\gamma^2r^2\beta^2\alpha^2 + 8\gamma^4\beta^2pq\alpha^2),\\
&c = \frac{1}{2\beta^2\alpha^3(\gamma-3)^3}(24\gamma^5\beta^3bpq\alpha^2 - 48\gamma^4\beta^2\alpha^3apq\\
&\qquad + 2\gamma\alpha^3a^3 - 144\gamma^4\beta^3bpq\alpha^2 + 72\gamma^3\beta^2\alpha^3apq\\
&\qquad + 216\gamma^3\beta^3bpq\alpha^2 - 4\alpha^3a^3 + 27\gamma^3r^2\beta^3b\alpha^2\\
&\qquad + 2\gamma^3\beta^3b^3 + 2\gamma^3\beta^2ab^2\alpha + 8\gamma^5\beta^2\alpha^3apq - 6\gamma\beta a^2b\alpha^2\\
&\qquad + 4\gamma^2\beta a^2b\alpha^2 + 3\gamma^5r^2\beta^3b\alpha^2 - 6\gamma^4r^2\beta^2\alpha^3a\\
&\qquad - 18\gamma^4r^2\beta^3b\alpha^2 + 9\gamma^3r^2\beta^2a\alpha^3 + \gamma^5r^2\beta^2\alpha^3a),
\end{aligned}
$$

where b, γ, β, a, α, q, p and r are arbitrary constants, which $\gamma \neq 3$. Substituting these results into Eq. (15) and combining with all solutions from **Type 1-3**, one can obtain abundant exact solutions, including trigonometric function, hyperbolic function, exponential function and rational function solutions.

Type 1: When $p = 1$,

$$
\begin{aligned}
u_1 = &-e^{i\theta}\gamma\sqrt{\frac{\beta\gamma}{2\alpha}}\left(\sqrt{r^2-4q}\tanh\left(\frac{\sqrt{r^2-4q}}{2}(\xi+\xi_0)\right)\right.\\
&\left. + \frac{2\sqrt{2}q}{\sqrt{r^2-4q}\tanh\left(\frac{\sqrt{r^2-4q}}{2}(\xi+\xi_0)\right)+r}\right),
\end{aligned}
\tag{21}
$$

where $q \neq 0$ and $r^2 - 4q > 0$.

$$
\begin{aligned}
u_2 = &e^{i\theta}\gamma\sqrt{\frac{\beta\gamma}{2\alpha}}\left(\sqrt{4q-r^2}\tan\left(\frac{\sqrt{4q-r^2}}{2}(\xi+\xi_0)\right)\right.\\
&\left. + \frac{2\sqrt{2}q}{\sqrt{4q-r^2}\tan\left(\frac{\sqrt{4q-r^2}}{2}(\xi+\xi_0)\right)-r}\right),
\end{aligned}
\tag{22}
$$

where $q \neq 0$ and $r^2 - 4q < 0$.

$$
u_3 = e^{i\theta}r\gamma\left(\sqrt{\frac{\beta\gamma}{2\alpha}} + \frac{\sqrt{\frac{2\gamma\beta}{\alpha}}}{e^{r(\xi+\xi_0)}-1}\right),
\tag{23}
$$

where $q = 0$, $r \neq 0$ and $r^2 - 4q > 0$.

$$
\begin{aligned}
u_4 = &e^{i\theta}\gamma\sqrt{\frac{\gamma\beta}{2\alpha}}\left(r - \frac{r(\xi+\xi_0)+2}{\xi+\xi_0}\right.\\
&\left. - \frac{r^2(\xi+\xi_0)}{r(\xi+\xi_0)+2}\right),
\end{aligned}
\tag{24}
$$

where $q \neq 0$, $r \neq 0$ and $r^2 - 4q = 0$.

Type 2: When $r = 0$,

$$
\begin{aligned}
u_5 = &e^{i\theta}\gamma\sqrt{\frac{2\beta\gamma pq}{\alpha}}\left(\tan(\sqrt{pq}(\xi+\xi_0))\right.\\
&\left. + \frac{1}{\tan(\sqrt{pq}(\xi+\xi_0))}\right),
\end{aligned}
\tag{25}
$$

where $p > 0$ and $q > 0$.

$$
\begin{aligned}
u_6 = &-e^{i\theta}\gamma\sqrt{\frac{2\beta\gamma pq}{\alpha}}\left(\cot(\sqrt{pq}(\xi+\xi_0))\right.\\
&\left. + \frac{1}{\cot(\sqrt{pq}(\xi+\xi_0))}\right),
\end{aligned}
\tag{26}
$$

where $p > 0$ and $q > 0$.

$$
\begin{aligned}
u_7 = &-\gamma e^{i\theta}\mathrm{sgn}(p)\sqrt{-\frac{2\beta\gamma pq}{\alpha}}\left(\tanh(\sqrt{-pq}(\xi+\xi_0))\right.\\
&\left. + \frac{1}{\tanh(\sqrt{-pq}(\xi+\xi_0))}\right),
\end{aligned}
\tag{27}
$$

where $pq < 0$.

$$
\begin{aligned}
u_8 = &-\gamma e^{i\theta}\mathrm{sgn}(p)\sqrt{-\frac{2\beta\gamma pq}{\alpha}}\left(\coth(\sqrt{-pq}(\xi+\xi_0))\right.\\
&\left. + \frac{1}{\coth(\sqrt{-pq}(\xi+\xi_0))}\right),
\end{aligned}
\tag{28}
$$

where $pq < 0$.

Type 3: When $q = 0$ and $r = 0$, we obtain

$$
u_9 = e^{i\theta}\sqrt{\frac{2\beta\gamma}{\alpha}}\left(\frac{\gamma}{\xi+\xi_0}\right),
\tag{29}
$$

where $\theta = kx - ct$ and $\xi = \gamma x + vt$.

4. Graphical solutions of the RKL equation

In previous section, all solutions are complex valued functions. Real part, imaginary part and complex modulus of the solutions are depicted with the parameters $\xi_0 = 5$, $\gamma = 2$, $\alpha = 3$, $\beta = 2$, $a = 2$ and $b = -4$. We select tanh-hyperbolic function of $u_1(x,t)$ as

$$u_1 = -e^{i\theta} 2\sqrt{\frac{2}{3}}\left(\sqrt{r^2-4q}\tanh\left(\frac{\sqrt{r^2-4q}}{2}(\xi+5)\right)\right.$$
$$\left. + \frac{2\sqrt{2}q}{\sqrt{r^2-4q}\tanh\left(\frac{\sqrt{r^2-4q}}{2}(\xi+5)\right)+r}\right),$$

where $\xi = 2x + (8r^2 + \frac{140}{3} + 64q)t$ and $\theta = -\frac{5}{3}x - \left(192q + 24r^2 + \frac{400}{27}\right)t$. Let $q = 2$ and $r = 3$, then the solution is shown in figures as the following

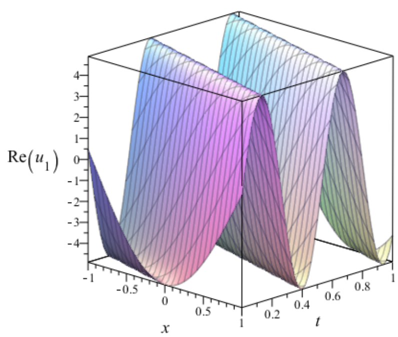

Figure 1. Graph of the periodic wave solution for $u_1(x,t)$ obtained from Eq. (21) of real part with $-1 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$.

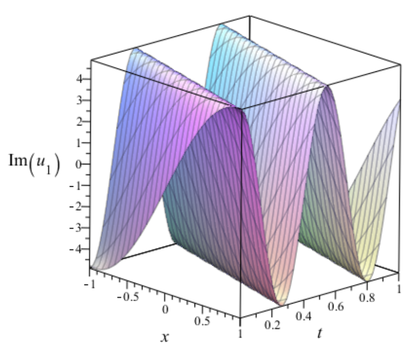

Figure 2. Graph of the periodic wave solution for $u_1(x,t)$ obtained from Eq. (21) of imaginary part with $-1 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$.

We select the solution $u_2(x,t)$ in the form of tan-trigonometric function is depicted for real part, imaginary part and complex modulus of the solution as the following figures.

$$u_2 = e^{i\theta} 2\sqrt{\frac{2}{3}}\left(\sqrt{4q-r^2}\tan\left(\frac{\sqrt{4q-r^2}}{2}(\xi+5)\right)\right.$$
$$\left. + \frac{2\sqrt{2}q}{\sqrt{4q-r^2}\tan\left(\frac{\sqrt{4q-r^2}}{2}(\xi+5)\right)-r}\right),$$

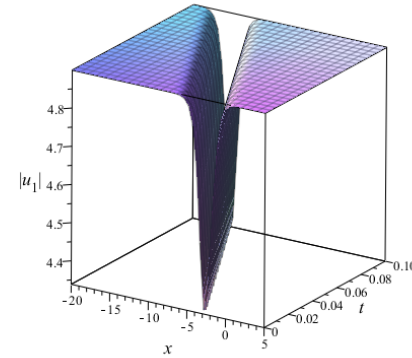

Figure 3. Graph of the dark soliton solution for $u_1(x,t)$ obtained from Eq. (21) of complex modulus with $-20 \leqslant x \leqslant 5$ and $0 \leqslant t \leqslant 0.1$.

where $\xi = 2x + (8r^2 + \frac{140}{3} + 64q)t$ and $\theta = -\frac{5}{3}x - \left(192q + 24r^2 + \frac{400}{27}\right)t$. When given parameters $q = 3$ and $r = 2$, the graph is shown the following as

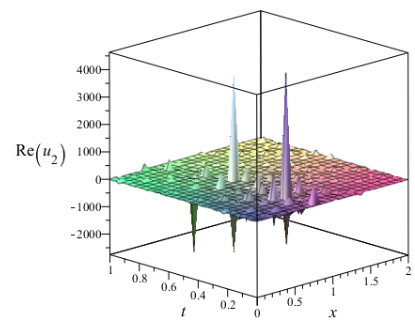

Figure 4. Graph of the wave solution for $u_2(x,t)$ obtained from Eq. (22) of real part with $0 \leqslant x \leqslant 2$ and $0 \leqslant t \leqslant 1$.

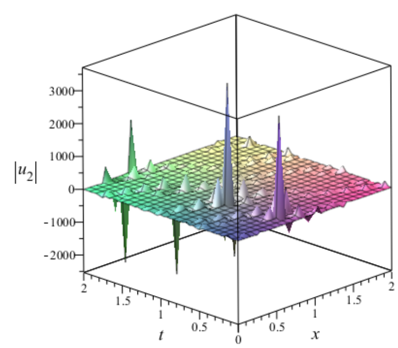

Figure 5. Graph of the wave solution for $u_2(x,t)$ obtained from Eq. (22) of imaginary part with $0 \leqslant x \leqslant 2$ and $0 \leqslant t \leqslant 2$.

Finally, $u_9(x,t)$ of the non-homogeneous RKL equation has been shown solution in real part, imaginary part and complex modulus.

$$u_9 = e^{-i\left(\frac{5}{3}x+\frac{400}{27}t\right)}\sqrt{\frac{2}{3}}\left(\frac{4}{2x+\frac{140}{3}t+5}\right).$$

When given parameters $p = 3$, the graphs of solution $u_9(x,t)$ are shown as follows

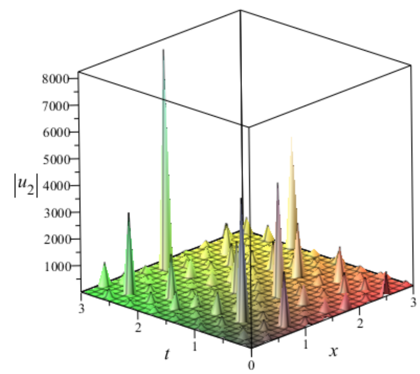

Figure 6. Graph of the wave solution for $u_2(x,t)$ obtained from Eq. (22) of complex modulus with $0 \leqslant x \leqslant 3$ and $0 \leqslant t \leqslant 3$.

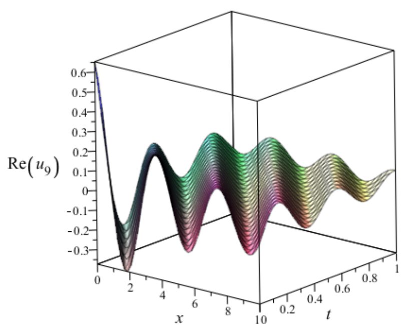

Figure 7. Graph of the wave solution for $u_9(x,t)$ obtained from Eq. (29) of real part with $0 \leqslant x \leqslant 10$ and $0 \leqslant t \leqslant 1$.

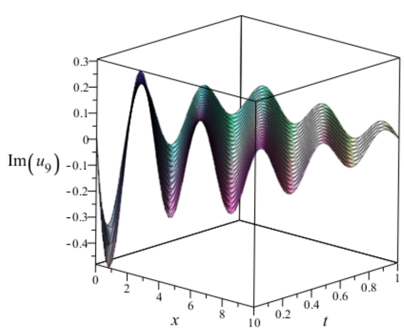

Figure 8. Graph of the wave solution for $u_9(x,t)$ obtained from Eq. (29) of imaginary part with $0 \leqslant x \leqslant 10$ and $0 \leqslant t \leqslant 1$.

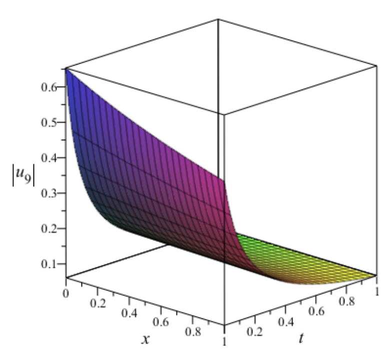

Figure 9. Graph of the wave solution for $u_9(x,t)$ obtained from Eq. (29) of complex modulus with $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$.

5. Conclusions

In summary, we has successfully implemented a powerful method to seek exact traveling wave solutions of the non-homogeneous Radhadkrishnan-Kundu-Lakshmanan equation by using the $\exp(-\phi(\xi))$-expansion method. The main advantage of this method over other methods are that it possesses all types of exact travelling wave solution, including kinds of trigonometric function solutions, kinds of hyperbolic function solutions, exponential function solutions and rational function solution. This method is extremely simple, easy to use and is very accurate for wide classes of problem. The aid of symbolic computation, Maple program can be used to obtain the solutions and validate all exact travelling solutions. Finally, we also show graphical solutions of the non-homogeneous Radhadkrishnan-Kundu-Lakshmanan equation.

Acknowledgment

This work was partially financial supported by the Graduate College and Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok.

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