Modified Homotopy Perturbation Method for nonlinear system of second-order BVPs

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Abstract: The purpose of this paper is to use a kind of analytical method called new homotopy perturbation method (NHPM), to solve nonlinear system of second order boundary value problems. The NHPM yields solutions in convergent series form with easily computable terms, and in some cases, yields exact solutions in one iteration. This method can be applied directly to the second order boundary systems needless of converting to the first order initial systems. To illustrate the application of the method, numerical results are derived using the calculated components of the NHPM. Comparisons with exact solutions show the efficiency and accuracy of NHPM in solving nonlinear system of second order boundary value problems.

Keywords: System of Second Order Boundary Value Problems, New Homotopy perturbation method.

1. Introduction

The homotopy perturbation method (HPM) was established by Ji-Huan He in 1999 [23]. The homotopy perturbation method (HPM) has been used to investigate a variety of mathematical and physical problems, since it is very effective, simple, and convenient to solve nonlinear problems. This technique is used for solving nonlinear wave equations [11], boundary value problems [9,10], the quadratic Riccati differential equation [7], partial differential algebraic equations [8], integral equations [12], systems of ordinary differential equations [13], stiff systems of ordinary differential equations [14], delay differential equation [15], systems of integro-differential equations [16], partial differential equations [17], and many others [18–20].

It is well known that a wide class of problems which arise in several branches of pure and applied sciences can be formulated as a system of boundary value problems. In this paper, we consider a class of nonlinear systems of second-order BVPs of the form:

$$\begin{aligned} u''+a_1(t)u'+a_2(t)u+a_3(t)v''+a_4(t)v'+a_5(t)v+G_1(t,u,v)&=f_1(t)\\ v''+b_1(t)v'+b_2(t)v+b_3(t)u''+b_4(t)u'+b_5(t)u+G_2(t,u,v)&=f_2(t) \end{aligned} \quad (1)$$

subject to the boundary conditions:

$$\begin{aligned} u(0)=u(1)=0\\ v(0)=v(1)=0 \end{aligned} \quad (2)$$

where $0\le t\le 1$, G_1 and G_2 are nonlinear functions of u and v. Also $a_i(t), b_i(t)$ for $i=1,2...5$ are given continuous functions and f_1 and f_2 are known.

However, many classic numerical methods used for second-order initial value problems cannot be applied to second-order boundary value problems (BVPs). For a nonlinear system of second-order BVPs, there are few valid methods to obtain numerical solutions [4-6]. M. Dehghan and A. Saadatmandi applied a numerical method based on Sinc-collocation method [1] and in [2] Chebyshev finite difference method has been used for this problem. Also a numerical method based on the cubic B-spline scaling functions is proposed in [22] to find the solutions of (1) and (2). In [3] HPM is applied for solving nonlinear system of second-order BVPs.

In the present paper, the system of BVPs will be solved by the NHPM which is introduced by Aminikhah and Hemmatnezhad [14]. This numerical scheme is based on the Taylor series expansion and is capable of finding the exact solution of many nonlinear differential equations.

NHPM has been successfully applied to stiff systems of ODEs [14], initial-type differential equations of heat transfer, nonlinear strongly differential equations, and stiff delay differential equations (DDEs) [21].

2. New Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(\mathbf{u})-f(t)=0,\qquad \mathbf{u}(0)=\mathbf{u}_0,\qquad t\in\Omega \quad (3)$$

where A is a general differential operator, $\mathbf{u}_0$ is an initial approximation of Eq. (3), and $f(t)$ is a known analytical function on the domain Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (3) can be, therefore, rewritten as follows:

$$L(\mathbf{u})+N(\mathbf{u})-f(t)=0$$

By the homotopy technique, we construct a homotopy $\mathbf{U}(t,p):\Omega\times[0,1]\to\Re$, which satisfies:

$$\mathrm{H}(\mathbf{U},\mathrm{p}) = (1-\mathrm{p})[\mathrm{L}\mathbf{U}(\mathrm{t})-\mathrm{L}\mathbf{u}_0(\mathrm{t})]+\mathrm{p}[\mathrm{A}\mathbf{U}(\mathrm{t})-\mathrm{f}(\mathrm{t})]=0,\quad \mathrm{p}\in[0,1],\ \mathrm{t}\in\Omega \quad (4)$$

or

$$\mathrm{H}(\mathbf{U},\mathrm{p}) = \mathrm{L}\mathbf{U}(\mathrm{t})-\mathrm{L}\mathbf{u}_0(\mathrm{t})+\mathrm{p}\mathrm{L}\mathbf{u}_0(\mathrm{t})+\mathrm{p}[\mathrm{N}\mathbf{U}(\mathrm{t})-\mathrm{f}(\mathrm{t})]=0,\quad \mathrm{p}\in[0,1],\ \mathrm{t}\in\Omega \quad (5)$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously from Eqs. (4) or (5) we will have

$$H(\mathbf{U},0) = L\mathbf{U}(t) - L\mathbf{u}_0(t) = 0, \; H(\mathbf{U},1) = A\mathbf{U}(t) - f(t) = 0 \qquad (6)$$

The changing process of p from zero to unity is just that of $\mathbf{U}(t,p)$ from $\mathbf{u}_0(t)$ to $\mathbf{u}(t)$. In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (4) or (5) can be written as a power series in p:

$$\mathbf{U} = \sum_{n=0}^{\infty} p^n \mathbf{U}_n = \mathbf{U}_0 + p\mathbf{U}_1 + p^2\mathbf{U}_2 + p^3\mathbf{U}_3 + \cdots \qquad (7)$$

Setting $p = 1$, results in the approximate solution of Eq. (3)

$$\mathbf{u}(t) = \lim_{p \to 1} \mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \mathbf{U}_3 + \cdots \qquad (8)$$

Applying the inverse operator $L^{-1} = \int_0^t (\cdot) dt$ to both sides of Eq. (5), we obtain

$$\mathbf{U}(t) = \mathbf{U}(0) + \int_0^t L\mathbf{u}_0(t)dt - p\int_0^t L\mathbf{u}_0(t)dt - p[\int_0^t (N\mathbf{U}(t) - f(t))dt] \qquad (9)$$

where $\mathbf{U}(0) = \mathbf{u}_0$.

Now, suppose that the initial approximations to the solutions, $L\mathbf{u}_0(t)$, have the form

$$L\mathbf{u}_0(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n P_n(t) \qquad (10)$$

where $\boldsymbol{\alpha}_n$ are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \ldots$ are specific functions.

Substituting (7) and (10) into (9) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : \mathbf{U}_0(t) = \mathbf{u}_0 + \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n \int_0^t P_n(t)dt \\ p^1 : \mathbf{U}_1(t) = -\sum_{n=0}^{\infty} \boldsymbol{\alpha}_n \int_0^t P_n(t)dt - \int_0^t (N\mathbf{U}_0(t) - f(t))dt \\ p^2 : \mathbf{U}_2(t) = -\int_0^t N\mathbf{U}_1(t)dt \\ p^j : \mathbf{U}_j(t) = -\int_0^t N\mathbf{U}_{j-1}(t)dt \end{cases} \qquad (11)$$

Now, if these equations are solved in such a way that $\mathbf{U}_1(t) = 0$, then Eq. (11) results in

$\mathbf{U}_1(t) = \mathbf{U}_2(t) = \mathbf{U}_3(t) = \cdots = 0$.

and therefore the exact solution can be obtained by using

$$\mathbf{U}(t) = \mathbf{U}_0(t) = \mathbf{u}_0 + \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n \int_0^t P_n(t)dt \qquad (12)$$

It is worth noting that, if $\mathbf{U}(t)$ is analytic at $t = t_0$, then their Taylor series

$$\mathbf{U}(t) = \sum_{n=0}^{\infty} \mathbf{a}_n (t - t_0)^n$$

can be used in Eq. (11), where $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots$ are known coefficients and $\boldsymbol{\alpha}_n$ are unknown ones, which must be computed.

We explain this method by considering several examples.

3. Test Problems

3.1 Example 1.

Consider the system of second-order boundary value problems [3].

$$\begin{cases} u''(t) + tu(t) + tv(t) = 2 \\ v''(t) + 2tv(t) + 2tu(t) = -2 \qquad 0 \le t \le 1 \end{cases} \qquad (13)$$

subject to the boundary conditions

$u(0) = u(1) = 0, \quad v(0) = v(1) = 0$

The exact solutions of this problem are

$$u(t) = t^2 - t, \quad v(t) = t - t^2 \qquad (14)$$

For solving system (14) by NHPM, we construct the following homotopy:

$$\begin{cases} (1-p)[LU(t) - Lu_0(t)] + p[LU(t) + tU(t) + tV(t) - 2] = 0 \\ (1-p)[LV(t) - Lv_0(t)] + p[LV(t) + 2tV(t) + 2tU(t) + 2] = 0 \end{cases} \qquad (15)$$

or

$$\begin{cases} LU(t) - Lu_0(t) + pLu_0(t) + p[tU(t) + tV(t) - 2] = 0 \\ LV(t) - Lv_0(t) + pLv_0(t) + p[2tV(t) + 2tU(t) + 2] = 0 \end{cases} \qquad (16)$$

where $L = \frac{d^2}{dt^2}$ and $p \in [0,1]$ is an embedding parameter.

Assume that

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t), \quad Lv_0(t) = \sum_{n=0}^{\infty} \beta_n P_n(t), \qquad P_n(t) = t^n \qquad (17)$$

and from the initial conditions

$U(0) = 0, V(0) = 0$

and for $U'(0)$ and $V'(0)$ we let $U'(0) = \lambda_1, V'(0) = \lambda_2$ where λ_1 and λ_2 are unknown constants which should be determined.

Substituting (17) into (16) and applying the inverse operator $L^{-1} = \int_0^t \int_0^t (\cdot) dtdt$ to Eq. (16), we have

$$\begin{cases} U(t) = U(0) + U'(0)t + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - p\int_0^t \int_0^t (tU(t) + tV(t) - 2)dtdt, \\ V(t) = V(0) + V'(0)t + \sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - p\int_0^t \int_0^t (2tV(t) + 2tU(t) + 2)dtdt, \end{cases}$$

or

$$\begin{cases} U(t) = \lambda_1 t + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - p\int_0^t \int_0^t (tU(t) + tV(t) - 2)dtdt, \\ V(t) = \lambda_2 t + \sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - p\int_0^t \int_0^t (2tV(t) + 2tU(t) + 2)dtdt, \end{cases} \qquad (18)$$

Suppose the solutions of system (13) to be in the following form

$$U=\sum_{n=0}^{\infty}p^nU_n=U_0+pU_1+p^2U_2+p^3U_3+\cdots \quad (19)$$
$$V=\sum_{n=0}^{\infty}p^nV_n=V_0+pV_1+p^2V_2+p^3V_3+\cdots$$

where in U_i and V_i for j=0,1,2,3,... are functions which should be determined.

Substituting (19) into (18) and equating the coefficients of p with the same powers leads to

$$p^0:\begin{cases}U_0(t)=\lambda_1 t+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2},\\ V_0(t)=\lambda_2 t+\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}\end{cases}$$
$$p^1:\begin{cases}U_1(t)=-\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-\int_0^t\int_0^t(tU_0(t)+tV_0(t)-2)dtdt,\\ V_1(t)=-\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-\int_0^t\int_0^t(2tV_0(t)+2tU_0(t)+2)dtdt,\end{cases}$$
$$p^j:\begin{cases}U_j(t)=-\int_0^t\int_0^t(tU_{j-1}(t)+tV_{j-1}(t))dtdt,\\ V_j(t)=-\int_0^t\int_0^t(2tV_{j-1}(t)+2tU_{j-1}(t))dtdt,\end{cases}\quad j=2,3,\ldots$$

Now, if we set the Taylor series of $U_1(t)$ and $V_1(t)$ at $t=0$ equal to zero, leads to

$$\alpha_0=2,\alpha_1=0,\alpha_2=-\lambda_1-\lambda_2,\alpha_3=0,\alpha_4=0,\alpha_5=\frac{1}{4}\lambda_1+\frac{1}{4}\lambda_2,$$
$$\alpha_6=\alpha_7=0,\alpha_8=-\frac{1}{56}\lambda_1-\frac{1}{56}\lambda_2,\ldots$$
$$\beta_0=-2,\beta_1=0,\beta_2=-2\lambda_2-2\lambda_1,\beta_3=0,\beta_4=0,\beta_5=\frac{1}{2}\lambda_2+\frac{1}{2}\lambda_1$$
$$\beta_6=\beta_7=0,\beta_8=-\frac{1}{28}\lambda_2-\frac{1}{28}\lambda_1,\cdots$$

Substituting this numbers in $U_1(t)$ and $V_1(t)$, we find:

$$U_1(t)=\lambda_1 t+t^2+\frac{1}{12}(-\lambda_1-\lambda_2)t^4+\frac{1}{42}\left(\frac{1}{4}\lambda_1+\frac{1}{4}\lambda_2\right)t^7+\frac{1}{90}\left(-\frac{1}{56}\lambda_1-\frac{1}{56}\lambda_2\right)t^{10}+\cdots$$
$$V_1(t)=\lambda_2 t-t^2+\frac{1}{12}(-2\lambda_1-2\lambda_2)t^4+\frac{1}{42}\left(\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2\right)t^7+\frac{1}{90}\left(-\frac{1}{28}\lambda_1-\frac{1}{28}\lambda_2\right)t^{10}+\cdots$$

with unknown coefficients λ_1 and λ_2. For determining λ_1 and λ_2 we use the boundary condition at $t=1$. We have:

$$\begin{cases}U_1(1)=0\\V_1(1)=0\end{cases}\Rightarrow\begin{cases}\frac{4649}{5040}\lambda_1+1-\frac{391}{5040}\lambda_2=0\\ \frac{2129}{2520}\lambda_1-1-\frac{391}{2520}\lambda_2=0\end{cases}$$

By solving this system we have

$\lambda_1=-1,\quad \lambda_2=1$

Therefore, the approximate solutions of the system of differential equation (14) can be expressed as

$$u(t)=U_0(t)=-t+t^2,$$
$$v(t)=V_0(t)=t-t^2,$$

which are the exact solutions. In standard HPM, $U_2(t)$ and $V_2(t)$ gives the exact solutions.

3.2 Example 2.

Consider the following differential equations [3]

$$\begin{cases}u''(t)+(2t-1)u'(t)+\cos(\pi t)v'(t)=f_1(t)\\ v''(t)+tu(t)=f_2(t)\qquad 0\le t\le 1\end{cases}\quad (20)$$

with the boundary conditions:

$u(0)=u(1)=0,\quad v(0)=v(1)=0$,

where $f_1(t)=-\pi^2\sin(\pi t)+(2t-1)\pi\cos(\pi t)+(2t-1)\cos(\pi t)$ and $f_2(t)=2+t\sin(\pi t)$. The exact solutions are $u(t)=\sin(\pi t)$ and $v(t)=t^2-t$.

For solving system (20) by NHPM, we construct the following homotopy:

$$\begin{cases}(1-p)[LU(t)-Lu_0(t)]+\\ \quad p[LU(t)+(2t-1)U'(t)+\cos(\pi t)V'(t)-f_1(t)]=0\\ (1-p)[LV(t)-Lv_0(t)]+p[LV(t)+tU(t)-f_2(t)]=0\end{cases}$$
(21)

or

$$\begin{cases}LU(t)-Lu_0(t)+pLu_0(t)+\\ \quad p[(2t-1)U'(t)+\cos(\pi t)V'(t)-f_1(t)]=0\\ LV(t)-Lv_0(t)+pLv_0(t)+p[tU(t)-f_2(t)]=0\end{cases}\quad (22)$$

where $L=\frac{d^2}{dt^2}$ and $p\in[0,1]$ is an embedding parameter.

Assume that

$$Lu_0(t)=\sum_{n=0}^{\infty}\alpha_nP_n(t),$$
$$Lv_0(t)=\sum_{n=0}^{\infty}\beta_nP_n(t),\qquad P_n(t)=t^n \quad (23)$$

and from the initial conditions

$U(0)=0,V(0)=0$

and for $U'(0)$ and $V'(0)$ we let $U'(0)=\lambda_1, V'(0)=\lambda_2$ where λ_1 and λ_2 are unknown constants which should be determined.

Substituting (23) into (22) and applying the inverse operator $L^{-1}=\int_0^t\int_0^t(\cdot)dtdt$ to Eq. (22), we have

$$\begin{cases}U(t)=U(0)+U'(0)t+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-\\ \quad p\int_0^t\int_0^t((2t-1)U'(t)+\cos(\pi t)V'(t)-f_1(t))dtdt,\\ V(t)=V(0)+V'(0)t+\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-p\int_0^t\int_0^t(tU(t)-f_2(t))dtdt,\end{cases}$$

or

$$\begin{cases}U(t)=\lambda_1 t+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-\\ \quad p\int_0^t\int_0^t((2t-1)U'(t)+\cos(\pi t)V'(t)-f_1(t))dtdt,\\ V(t)=\lambda_2 t+\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-p\int_0^t\int_0^t(tU(t)-f_2(t))dtdt,\end{cases}\quad (24)$$

Suppose the solutions of system (20) to be in the following form

$$U=\sum_{n=0}^{\infty}p^nU_n=U_0+pU_1+p^2U_2+p^3U_3+\cdots \quad (25)$$
$$V=\sum_{n=0}^{\infty}p^nV_n=V_0+pV_1+p^2V_2+p^3V_3+\cdots$$

where in U_i and V_i for j=0,1,2,3,... are functions which should be determined.
Substituting (25) into (24) and equating the coefficients of p with the same powers leads to

$$p^0:\begin{cases}U_0(t)=\lambda_1 t+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2},\\ V_0(t)=\lambda_2 t+\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}\end{cases}$$

$$p^1:\begin{cases}U_1(t)=-\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-\int_0^t\int_0^t\left((2t-1)U_0'(t)+\cos(\pi t)V_0'(t)-f_1(t)\right)dtdt,\\ V_1(t)=-\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-\int_0^t\int_0^t\left(tU_0(t)-f_2(t)\right)dtdt,\end{cases}$$

$$p^j:\begin{cases}U_j(t)=-\int_0^t\int_0^t\left((2t-1)U_{j-1}'(t)+\cos(\pi t)V_{j-1}'(t)-f_1(t)\right)dtdt,\\ V_j(t)=-\int_0^t\int_0^t\left(tU_{j-1}(t)-f_2(t)\right)dtdt,\end{cases}\qquad j=2,3,\ldots$$

Now, if we set the Taylor series of $U_1(t)$ and $V_1(t)$ at $t=0$ equal to zero, leads to

$$\alpha_0=-1+\lambda_1-\pi-\lambda_2,\ \alpha_1=-\pi^3-1-\lambda_1+\pi-\lambda_2,$$

$$\alpha_2=\frac{3}{2}-\frac{5}{2}\lambda_1+\frac{5}{2}\pi+\frac{3}{2}\lambda_2+\frac{1}{2}\pi^2\lambda_2+\frac{1}{2}\pi^2,$$

$$\alpha_3=\frac{1}{6}\pi^5+\frac{1}{2}\lambda_1-\frac{1}{2}\pi+\frac{3}{2}+\frac{3}{2}\lambda_2+\frac{1}{6}\pi^2\lambda_2+\frac{1}{6}\pi^2,\ldots$$

$$\beta_0=2,\ \beta_1=0,\ \beta_2=-\lambda_1+\pi,\ \beta_3=-\frac{1}{2}\lambda_1+\frac{1}{2}\pi+\frac{1}{2}+\frac{1}{2}\lambda_2,$$

$$\beta_4=\frac{1}{6}+\frac{1}{6}\lambda_1-\frac{1}{6}\pi+\frac{1}{6}\lambda_2,\ldots$$

Substituting this numbers in $U_1(t)$ and $V_1(t)$, we find:

$$U_1(t)=\lambda_1 t+\frac{1}{2}(-1-\pi+\lambda_1-\lambda_2)t^2+\frac{1}{6}\left(-\pi^3-1+\pi-\lambda_1-\lambda_2\right)t^3+$$
$$\frac{1}{12}\left(\frac{1}{2}\pi^2+\frac{3}{2}+\frac{5}{2}\pi-\frac{5}{2}\lambda_1+\frac{3}{2}\lambda_2+\frac{1}{2}\pi^2\lambda_2\right)t^4+\cdots$$

$$V_1(t)=\lambda_2 t+t^2+\frac{1}{12}(-\lambda_1+\pi)t^4+\frac{1}{20}\left(-\frac{1}{2}\lambda_1+\frac{1}{2}\pi+\frac{1}{2}+\frac{1}{2}\lambda_2\right)t^5+\cdots$$

with unknown coefficients λ_1 and λ_2. For determining λ_1 and λ_2 we use the boundary condition at $t=1$ and we have:

$$\begin{cases}U_1(1)=0\\ V_1(1)=0\end{cases}$$

By solving this system we find

$$\lambda_1=\pi,\quad \lambda_2=-1$$

Therefore, the approximate solutions of the system of differential equation (20) can be expressed as

$$u(t)=U_0(t)=\pi t-\frac{1}{6}(\pi t)^3+\frac{1}{120}(\pi t)^5-\frac{1}{5040}(\pi t)^7+\cdots$$
$$=\sin(\pi t)$$

$$v(t)=V_0(t)=t^2-t,$$

which are compatible to the exact solutions.

If we use $\sum_{n=0}^{N}\alpha_n t^n$ and $\sum_{n=0}^{N}\beta_n t^n$ instead of infinite series, we get approximate values for λ_1 and λ_2, hence we achieve approximate $u(t)$ and $v(t)$.

Table 1 shows the comparison between exact solution and the series solution obtained by the standard HPM [3] by n-term approximation and NHPM by N+1 terms of summations $\sum_{n=0}^{N}\alpha_n t^n$ and $\sum_{n=0}^{N}\beta_n t^n$.

Table 1. The comparison between the exact solution, HPM, and NHPM

n	9	11	13
E_{U_n} (HPM)	1.8×10^{-8}	1.4×10^{-11}	1.8×10^{-14}
E_{V_n} (HPM)	6.1×10^{-9}	1.4×10^{-11}	3.8×10^{-14}
N	15	20	25
E_{U_0} (NHPM)	1.8×10^{-8}	8.8×10^{-12}	2.5×10^{-17}
E_{V_0} (NHPM)	9.0×10^{-10}	4.1×10^{-13}	1.1×10^{-18}

Where

$$E_{U_n}=\max\{|U_n(t)-u(t)|,0\le t\le 1\}$$
$$E_{V_n}=\max\{|V_n(t)-v(t)|,0\le t\le 1\}$$

3.3 Example 3.

In this example, consider the non-linear system [3]

$$\begin{cases}u''(t)-tv'(t)+u(t)=f_1(t)\\ v''(t)+tu'(t)+u(t)v(t)=f_2(t)\end{cases}\qquad 0\le t\le 1 \tag{26}$$

with the boundary conditions $u(0)=u(1)=0,$ and $v(0)=v(1)=0$ where

$f_1(t)=t^3-2t^2+6t$ and $f_2(t)=t^5-t^4+2t^3+t^2-t+2$.

The exact solutions are $u(t)=t^3-t$ and $v(t)=t^2-t$.

For solving system (26) by NHPM, we construct the following homotopy:

$$\begin{cases}(1-p)\left[LU(t)-Lu_0(t)\right]+\\ \quad p\left[LU(t)-tV'(t)+U(t)-f_1(t)\right]=0\\ (1-p)\left[LV(t)-Lv_0(t)\right]+p\left[LV(t)+tU'(t)+U(t)V(t)-f_2(t)\right]=0\end{cases}$$

or

$$\begin{cases}LU(t)-Lu_0(t)+pLu_0(t)+\\ \quad p\left[-tV'(t)+U(t)-f_1(t)\right]=0\\ LV(t)-Lv_0(t)+pLv_0(t)+p\left[tU'(t)+U(t)V(t)-f_2(t)\right]=0\end{cases} \tag{27}$$

where $L=\frac{d^2}{dt^2}$. Assume from the initial conditions that $U(0)=0, V(0)=0$ and let $U'(0)=\lambda_1, V'(0)=\lambda_2$ where λ_1 and λ_2 are unknown constants.

Substituting (23) into (27) and applying the inverse operator $L^{-1}=\int_0^t\int_0^t(\cdot)dtdt$ to Eq. (27), we have

$$\begin{cases}U(t)=U(0)+U'(0)t+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\alpha_n}{n+2}t^{n+2}-\\ \quad p\int_0^t\int_0^t\left(-tV'(t)+U(t)-f_1(t)\right)dtdt,\\ V(t)=V(0)+V'(0)t+\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}-p\sum_{n=0}^{\infty}\frac{\beta_n}{n+2}t^{n+2}\\ \quad -p\int_0^t\int_0^t\left(tU'(t)+U(t)V(t)-f_2(t)\right)dtdt,\end{cases}$$

or

$$\begin{cases} U(t) = \lambda_1 t + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - \\ \qquad p\int_0^t \int_0^t \left(-tV'(t) + U(t) - f_1(t)\right) dt dt, \\ V(t) = \lambda_2 t + \sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - p\sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} \\ \qquad - p\int_0^t \int_0^t \left(tU'(t) + U(t)V(t) - f_2(t)\right) dt dt, \end{cases} \quad (28)$$

Suppose the solutions of system (26) to be in the following form

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots$$

$$V = \sum_{n=0}^{\infty} p^n V_n = V_0 + pV_1 + p^2 V_2 + p^3 V_3 + \cdots \quad (29)$$

where in U_i and V_i for j=0,1,2,3,... are functions which should be determined.

Substituting (29) into (28) and equating the coefficients of p with the same powers leads to

$$p^0: \begin{cases} U_0(t) = \lambda_1 t + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2}, \\ V_0(t) = \lambda_2 t + \sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} \end{cases}$$

$$p^1: \begin{cases} U_1(t) = -\sum_{n=0}^{\infty} \frac{\alpha_n}{n+2} t^{n+2} - \int_0^t \int_0^t \left(-tV_0'(t) + U_0(t) - f_1(t)\right) dt dt, \\ V_1(t) = -\sum_{n=0}^{\infty} \frac{\beta_n}{n+2} t^{n+2} - \int_0^t \int_0^t \left(tU_0'(t) + U_0(t)V_0(t) - f_2(t)\right) dt dt, \end{cases}$$

Now, if we set the Taylor series of $U_1(t)$ and $V_1(t)$ at $t = 0$ equal to zero, leads to

$$\alpha_0 = 0, \quad \alpha_1 = 6 + \lambda_2 - \lambda_1, \quad \alpha_2 = 0,$$

$$\alpha_3 = -\frac{1}{2} - \frac{1}{3}\lambda_1 - \frac{1}{6}\lambda_2, \quad \alpha_4 = \frac{1}{3} - \frac{1}{3}\lambda_1\lambda_2, \ldots$$

$$\beta_0 = 2, \quad \beta_1 = -1 - \lambda_1, \quad \beta_2 = 1 - \lambda_1\lambda_2,$$

$$\beta_3 = -\frac{1}{2}\lambda_1 - 1 - \frac{1}{2}\lambda_2,$$

$$\beta_4 = -1 + \frac{1}{6}\lambda_1 + \frac{1}{6}\lambda_1^2 - \lambda_2 - \frac{1}{6}\lambda_2^2 + \frac{1}{6}\lambda_1\lambda_2, \ldots$$

Substituting this numbers in $U_1(t)$ and $V_1(t)$, we find $U_1(t)$ and $V_1(t)$ with unknown coefficients λ_1 and λ_2. For determining λ_1 and λ_2 we use the boundary condition at $t = 1$ and we have:

$$\begin{cases} U_1(1) = 0 \\ V_1(1) = 0 \end{cases}$$

By solving this system we find

$$\lambda_1 = -1, \quad \lambda_2 = -1$$

and

$$\lambda_1 = 35.574680, \lambda_2 = 124.90329$$

If we choose $\lambda_1 = -1$ and $\lambda_2 = -1$, then $U_1(t)$ and $V_1(t)$ are the exact solution of this example. The second values for λ s are not acceptable because $U_1(t)$ and $V_1(t)$ with these values do not satisfy in the differential equations.

Table 2 shows the comparison between exact solution and the series solution obtained by the standard HPM [3] by n-term approximation and NHPM by N+1 terms of summations $\sum_{n=0}^{N} \alpha_n t^n$ and $\sum_{n=0}^{N} \beta_n t^n$:

Table 2. The comparison between exact solution, HPM, and NHPM

n	7	9	11
E_{U_n} (HPM)	8.9×10^{-4}	3.4×10^{-5}	4.6×10^{-6}
E_{V_n} (HPM)	2.0×10^{-3}	2.9×10^{-4}	2.2×10^{-5}
N	2		
E_{U_0} (NHPM)	0.0		
E_{V_0} (NHPM)	0.0		

3.4 Example 4.

Consider the non-linear system

$$\begin{cases} u''(t) + tv(t) + tu^2(t) = f_1(t) \\ v''(t) + tu'(t) + v(t) = f_2(t) \end{cases} \qquad 0 \le t \le 1 \quad (30)$$

where

$u(0) = u(1) = 0, \quad v(0) = v(1) = 0$,

$f_1(t) = -\pi^2 \sin(\pi t) + t\sin(\pi t)^2 + t^4 - 3t^3 + 2t^2$ and $f_2(t) = \pi t\cos(\pi t) + t^3 - 3t^2 + 8t - 6$. The exact solutions are $u(t) = \sin(\pi t)$ and $v(t) = t^3 - 3t^2 + 2t$.

If we do similar to previous examples, we get $\lambda_1 = \pi$ and $\lambda_2 = 2$, and hence

$$U_1(t) = \pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \frac{(\pi t)^7}{7!} + \cdots = \sin(\pi t)$$

$$V_1(t) = 2t - 3t^2 + t^3$$

which are compatible with the exact solutions.

Table 3 shows the comparison between exact solution and the series solution obtained by NHPM by N+1 terms of summations $\sum_{n=0}^{N} \alpha_n t^n$ and $\sum_{n=0}^{N} \beta_n t^n$:

Table 3. The comparison between exact solution and NHPM

N	20	25
E_{U_0} (NHPM)	8.9×10^{-12}	2.6×10^{-17}
E_{V_0} (NHPM)	7.1×10^{-13}	2.1×10^{-18}

4. Conclusion

In this paper, we've proposed an efficient modification of the HPM which achieves the exact or approximate solution of the nonlinear systems of BVPs with less computational work compared to the standard HPM and solves the problem without any need to discretise the variables.

The new modification can usually provide the exact solution using just one single iteration and improves the performance of the standard HPM. This method doesn't need to transform the BVPs into higher order systems of integral equations.

Several examples were tested by applying the NHPM and the results have shown remarkable performance.

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