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Products of Invertible L-Topological Spaces

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***Abstract*:** In this paper we study the invertibility of product spaces and establish that the invertibility of a coordinate space is sufficient for the invertibility of the product space, but it is not necessary. Also we study the invertibility of product spaces of completely invertible L-topological spaces.

***Keywords*:** L-topology, invertible, product space.

1. Introduction

The concept of topological invertibility was introduced by P. H. Doyle and J. G. Hocking [1] in 1961. Later, several authors [2], [3], [4] investigated the properties of invertible topological spaces and established the effect of invertibility on topological properties.

In [5], Mathew extended the concept of invertibility to fuzzy topological spaces and obtained some basic properties of such spaces. In [6], Seenivasan and Balasubramanian discussed some properties of the invertible fuzzy topological space. Later, Mathew and Jose studied the local to global nature of separation axioms and axioms of countability of invertible fuzzy topological spaces in [7]. In [8], Rao continued this investigation with more separation axioms. In [9], Jose and Mathew examined the effect of invertibility on compactness and connectedness of fuzzy topological space and in [10], they studied the quotient space and product space of invertible fuzzy topological spaces. They also extended the concept of invertibility to L-topological spaces and obtained certain properties of the same in [11].

In this paper the author investigates the product space of a family of invertible L-topological spaces and prove that the product space is invertible if at least one of the coordinate spaces is invertible. The product space of a family of completely invertible L-topological spaces is also investigated and it is observed that the product space of two L-topological spaces may be completely invertible, even if none of the coordinate spaces is so. Further, the product space of a family of L-topological spaces is type 2 completely invertible if and only if each coordinate space is type 2 completely invertible.

2. Preliminaries

In this section, we include certain definitions and known results needed for the subsequent development of the study. L-subset with a constant degree of membership α is denoted by $\underline{\alpha}$. The identity map on X is denoted by e. An L-subset v of X is said to be proper if $v \neq \underline{0}, \underline{1}$, where 0 (resp. 1) stands for the smallest (resp. largest) element of complete lattice L.

The family of all the finite subsets of a set X is denoted by $[X]^{<\omega}$.

Definition 1 ([12]): Let X be a non-empty ordinary set and $(L,\ ')$ be a Hutton algebra, that is a complete, completely distributive lattice L equipped with an order-reversing involution $'$. For any $u \in L^X$, using the order-reversing involution $'$, we define an operation $'$ on L^X by $u'(x) = (u(x))'\ \forall x \in X$, calling $' : L^X \to L^X$ the pseudo-complement operation on L^X, u' the pseudo-complement set of $u \in L^X$.

Theorem 1 ([12]): Let X be a non-empty ordinary set and $(L,\ ')$ be a Hutton algebra. Then the pseudo-complement operation $' : L^X \to L^X$ is an order reversing involution.

Definition 2 ([12]): Let X be a non-empty ordinary set, $(L,\ ')$ be a Hutton algebra and $\delta \subset L^X$. Then δ is called an L-topology on X, and (X, δ) is called an L-topological space, or L-ts for short, if δ satisfies the following three conditions:

(i) $\underline{0}, \underline{1} \in \delta$;

(ii) if $u, v \in \delta$, then $u \wedge v \in \delta$;

(iii) if $\mathcal{A} \subset \delta$, then $\vee \mathcal{A} \in \delta$.

Every element in δ is called an open L-subset in L^X. An L-ts (X, δ) where $\delta = \{\underline{0}, \underline{1}\}$ is said to be trivial.

Definition 3 ([12]): Let X, Y be two L-topological spaces, $\theta : X \to Y$ an ordinary mapping. Then for any L-subset v of X, $\theta(v)$ is an L-subset in Y defined by

$$\theta(v)(y) = \sup\{v(x) : x \in X,\ \theta(x) = y\};\ \forall v \in L^X,\ \forall y \in Y.$$

For an L-subset w in Y, $\theta^{-1}(w)$ is an L-subset of X defined by

$$\theta^{-1}(w)(x) = w(\theta(x)),\ \ \forall w \in L^Y,\ \forall x \in X.$$

Definition 4 ([12]): Let (X, δ), (Y, μ) be L-ts.

(i) $\theta : (X, \delta) \to (Y, \mu)$ is called an L-continuous mapping from (X, δ) to (Y, μ) if $\forall v \in \mu$, $\theta^{-1}(v) \in \delta$.

(ii) $\theta : (X, \delta) \to (Y, \mu)$ is said to be open if $\forall u \in \delta$, $\theta(u) \in \mu$.

(iii) $\theta : (X, \delta) \to (Y, \mu)$ is called an L-homeomorphism if it is bijective, continuous and open.

Theorem 2 ([11]): Let (X, δ), (Y, μ) be L-ts, and let $\theta : (X, \delta) \to (Y, \mu)$ be an L-homeomorphism. Then for every $u \in L^X$, $\theta(u') = (\theta(u))'$.

Theorem 3 ([12]): Let (X, δ), (Y, μ) be L-ts, and let $\theta^{-1} : (Y, \mu) \to (X, \delta)$ be an L-homeomorphism. Then for every $v \in L^Y$, $\theta^{-1}(v') = (\theta^{-1}(v))'$.

Definition 5 ([12]): Let (X,δ) be an L-ts, $\beta \subset \delta$. β is called a base of δ, if $\delta = \{\vee\mathcal{A} : \mathcal{A} \subset \beta\}$. $\mathcal{S} \subset \delta$ is called a subbase of δ, if the family $\{\wedge\mathcal{B} : \mathcal{B} \in [\mathcal{S}]^{<\omega} - \{\emptyset\}\}$ is a base of δ.
Definition 6 ([11]): An L-ts (X,δ) is said to be invertible with respect to a proper open L-subset v if there is an L-homeomorphism θ of (X,δ) such that $\theta(v') \leq v$. This L-homeomorphism θ is called an inverting map for v and v is said to be an inverting L-subset of (X,δ).

When we say (X,δ) is invertible with respect to v, it is understood that $v \in \delta$ and $v \neq \underline{0}, \underline{1}$ and there exists an inverting map θ of v. This v and θ together are called an inverting pair of (X,δ).
Definition 7 ([11]): A non-trivial L-ts (X,δ) is said to be completely invertible if for any $v \neq \underline{0}, \in \delta$, there is an L-homeomorphism θ of (X,δ) such that $\theta(v') \leq v$.
It should be noted that for a completely invertible L-ts every non empty open L-subset is an inverting L-subset.
Theorem 4 ([11]): Let (X,δ) be an invertible L-ts and $v \in \delta$. Then (v,e) is an inverting pair if and only if $v' \leq v$.
Theorem 5 ([11]): Let (X,δ) be an L-ts with β as a base. Then (X,δ) is completely invertible if and only if (X,δ) is invertible with respect to all members of β.
Definition 8 ([11]): An invertible L-ts (X,δ) is said to be type-1 if identity is an inverting map.
Definition 9 ([11]): An invertible L-ts (X,δ) is said to be type-2 if identity is an inverting map for all the inverting L-subsets.
Theorem 6: An L-ts (X,δ) is type 2 completely invertible if and only if $v' \leq v$ for every $v \neq \underline{0} \in \delta$.

3. Product spaces of invertible L-topologies

Definition 10 ([12]): Let $S = \{(X_t,\delta_t), t \in T\}$ be a family of L-ts, $\mathcal{A} = \{u_t : t \in T\}$ a family of L-subsets where $u_t \in X_t$ for every $t \in T$. Denote $X = \prod_{t\in T} X_t$. For every $t \in T$, suppose $p_t : X \to X_t$ is the ordinary projection, define the projection from L-space L^X to L^{X_t} as $p_t : L^X \to L^{X_t}$. Define the product topology of L-topologies $\{\delta_t : t \in T\}$ on X, denoted by $\prod_{t\in T} \delta_t$, as the L-topology δ on X generated by the subbase $\{p_t^{-1}(u_t) : u_t \in \delta_t, t \in T\}$, and call the L-ts (X,δ) the product space of L-ts $\{(X_t,\delta_t), t \in T\}$. For every $t \in T$, (X_t,δ_t) is called a coordinate space. Define the product of L-subsets $\mathcal{A} = \{u_t : t \in T\}$, denoted by $\prod\mathcal{A} = \prod_{t\in T} u_t = \bigwedge\{p_t^{-1}(u_t) : t \in T\}$.
Theorem 7 ([12]): Let $\{(X_t,\delta_t) :\ t \in T\}$ be a family of L-ts, (X,δ) be their product space. Then $\{\bigwedge_{t\in F} p_t^{-1}(u_t) : F \in [T]^{<\omega}\}$ is a base of the product topology δ.
Definition 11 ([12]): Let $\{(X_t,\delta_t) : t \in T\}$ be a family of L-ts. The family $\{p_t^{-1}(u_t) : u_t \in \delta_t, t \in T\}$ and the family $\{\bigwedge_{t\in F} p_t^{-1}(u_t) : F \in [T]^{<\omega}, \forall t \in F, u_t \in \delta_t\}$ are called the canonical subbase and the canonical base of the product topology $\prod_{t\in T} \delta_t$ respectively.
Theorem 8: The product topology (X,δ) of a family of L-topologies (X_t,δ_t), $t \in T$ is invertible if (X_t,δ_t) is invertible for some $t \in T$.
Proof 1: Let p_t be the projection of the product X into the t^{th} coordinate set X_t and $\mathcal{S}$ be the canonical subbase for (X,δ). Suppose (X_k,δ_k) is invertible for some $k \in T$. Let (u_k,θ_k) be an inverting pair of (X_k,δ_k). Now consider $v = p_k^{-1}(u_k)$, clearly $v \in \mathcal{S}$ so that $v \in \delta$. Define $\theta : X \to X$ by $\theta(x) = y$ where $y = (y_t)$ such that $y_t = x_t,\ t \neq k,\ y_k = \theta_k(x_k)$. Clearly θ is an L-homeomorphism of (X,δ). Let $x \in X$ and $\theta^{-1}(x) = z$. Then $\theta(v'(x)) = v'(z) = (p_k^{-1}(u_k))'(z) = (u_k)'(z_k) = (u_k)'(\theta_k^{-1}(x_k)) = \theta_k(u_k)'(x_k) \leq u_k(x_k) = p_k^{-1}(u)(x) = v(x)$ so that (v,θ) is an inverting pair of (X,δ).
Corollary 1: The product of a family of invertible L-topological spaces is invertible.
Proof 2: Follows from theorem 8
Theorem 9: The product L-topology (X,δ) of a family of L-ts' (X_t,δ_t), $t \in \delta$ is type 1 invertible if (X_t,δ_t) is type 1 invertible for some $t \in T$.
Proof 3: Let p_t be the projection of the product X into the t^{th} coordinate set X_t and $\mathcal{S}$ be the canonical subbase for (X,δ). Suppose (X_t,δ_t) is type 1 invertible for some $t \in T$. Let (u_t,e) be an inverting pair of (X_t,δ_t). Then by theorem 4, $u_t' \leq u_t$. Now consider $p_t^{-1}(u_t) = v$, clearly $v \in \delta$ and $v' \leq v$ so that e is an inverting map for v. Hence (X,δ) is type 1 invertible.
Following is an easy consequence of the above theorem:
Corollary 2: The product of a family of type 1 invertible L-topological spaces is type 1 invertible.
Remark 1: From Remark 5.6 in [10], it follows that the converse of theorem 8 and theorem 9 are not true. From example 5.7 in [10] it follows that the type 2 invertibility of all the coordinate spaces need not imply the type 2 invertibility of the product space. Conversely, (X,δ) can be type 2 invertible even if none of the coordinate spaces are so. But if all the coordinate spaces are invertible, the situation is different!
Theorem 10: If the product topology (X,δ) of a family of invertible L-ts' (X_t,δ_t), $t \in T$ is type 2 invertible, then each (X_t,δ_t) is type 2 invertible.
Proof 4: Suppose (X,δ) be type 2 invertible. If possible assume that (X_k,δ_k) is not type 2 invertible for some $k \in T$. Then there exists an inverting L-subset u_k of (X_k,δ_k) such that e is not an inverting map for u_k. Let θ_k be an inverting map for u_k. Now define $\theta : X \to X$ by $\theta(x) = y$ where $y = (y_t)$ such that $y_t = x_t,\ t \neq k,\ y_k = \theta_k(x_k)$. Clearly $\theta \neq e$ is an L-homeomorphism of (X,δ). Now consider $v = p_k^{-1}(u_k)$, clearly (v,θ) is an inverting pair of (X,δ) and e is not an inverting map for v, a contradiction. Hence (X_k,δ_k) is type 2 invertible.

But with type 2 complete invertibility, we have the following characterization:
Theorem 11: The product L-topology (X,δ) of a family of L-ts' (X_t,δ_t), $t \in T$ is type 2 completely invertible if and only if (X_t,δ_t) is type 2 completely invertible for each $t \in T$.
Proof 5: Let (X_t,δ_t), $t \in T$ be a family of type 2 completely invertible L-ts. Let X be the Cartesian product of $\{X_t,\ t \in T\}$ and let p_t be the projection of the product X into the t^{th} coordinate set X_t. Let $\mathcal{B}$ be the canonical base for the product topology (X,δ). Let $u \in \mathcal{B}$ and $u \neq \underline{0},\ \underline{1}$, then $u = \bigwedge_{t\in K} p_t^{-1}(u_t) :\ u_t \in \delta_t$ where $K \in [T]^{<\omega}$. Since $u_t \geq u_t'$, $\forall u_t \in \delta_t$ and $u_t \neq \underline{0}$, $t \in T$, we have $u \geq u'$ so that (u,e) is an inverting pair of (X,δ). Then by theorem 5,

(X,δ) is type 2 completely invertible.

Conversely suppose that (X,δ) is type 2 completely invertible. If possible assume that (X_t,δ_t) is not type 2 completely invertible for some $t\in T$. Then by theorem 6, there exists a $u_t\in\delta_t$ and $u_t\neq\underline{0}$ such that $u_t'(x_t)\not\leq u_t(x_t)$ for some $x_t\in X_t$. Now consider $v\in X$ defined by $v=p_t^{-1}(u_t)$. Clearly $v\in\delta$ and (v,e) is not an inverting pair of (X,δ) so that (X,δ) is not type 2 completely invertible, a contradiction. Hence for each $t\in T$, (X_t,δ_t) is type 2 completely invertible.

In [10], it is observed that the product of two completely invertible fuzzy topological spaces need not be completely invertible. The following example shows that the product space may be completely invertible even if none of the coordinate spaces is invertible.

Example 1: Let $X_1=\{1,2\}$, $X_2=\{3,4\}$ and $L=\{0,a,b,1\}$ be the diamond-type lattice such that $a'=b$ and $b'=a$. Consider $u_1\in L^{X_1}$ defined by $u_1(1)=a$ and $u_1(2)=b$ and $u_2\in L^{X_2}$ defined by $u_2(3)=b$ and $u_2(4)=a$. Clearly (X_1,δ_1) where $\delta_1=\{\underline{0},\underline{1},u_1\}$ and (X_2,δ_2) where $\delta_2=\{\underline{0},\underline{1},u_2\}$ are L-ts which are not invertible. Let X be the Cartesian product of X_1 and X_2 and p_1 and p_2 be the corresponding projection functions. Let $u=p_1^{-1}(u_1)$ and $v=p_2^{-1}(u_2)$. Then clearly $\delta=\{\underline{0},\underline{1},u,v\}$. Let $\theta:X\to X$ defined by $\theta((1,3))=(2,3)$, $\theta((1,4))=(2,4)$, $\theta((2,3))=(1,3),\theta((2,4))=(1,4)$. Clearly θ is an L-homeomorphism of (X,δ) and it is an inverting map for both u and v. Hence (X,δ) is completely invertible.

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