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Unveiling Motion-Nature of Modulated Complex Wave

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***Abstract*:** This study aims to examine the motion-nature of complex wave with amplitude modulation and phase modulation. The Amplitude Modulated Complex Wave and Phase Modulated Complex wave have the potential to improve the localization of the time-frequency for signal analysis and since they are a function of time they tend to vary. In that regard, if the signal is nonstationary, the spectrum of a function (signal) is represented in time domain, and then the use of the constant coefficient of Fourier Series (FS) would imply a loss of data from the spectrum. Additionally, Modulated Complex wave has the potential to give a more accurate estimate of function within the local time domain. As such, the observations from the experiments indicate that the Amplitude Modulated Complex wave and Phase Modulated Complex wave help in analyzing the frequency spectrum as well as estimating its functions. Here MATLAB R2018a has been used to experiment the result.

***Keywords*:** Amplitude modulation, phase modulation, Fourier series, amplitude modulated complex wave, phase modulated complex wave, spectrum.

1. Introduction

A complex wave can be represented by a Fourier series [1–3]. A combination of simple sine waves with various frequencies, phases, and amplitudes can be used to represent any complex wave. The magnitude, phase and frequency of these different waves are known as the signal's spectrum. Spectrum Analysis [4, 5] is a technique for studying different time domain waveforms in order to determine their spectrum. In addition, the two spaces namely; the domain of time as well as its frequency can be used to explain a signal. Furthermore, the Fourier series helps in analyzing signals that are stationary [6]. A signal is known as stationary [7] if it does not change value over time in its frequency or spectral contents.

A complex wave $\Psi(t)$ can be expressed within the period $0 \leq t \leq T$ as follows:

$$\Psi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\frac{2\pi}{T}t) + b_n \sin(n\frac{2\pi}{T}t)\right) \quad (1)$$

$$\Psi(t) \sim \sum_{n=0}^{\infty} a_n \cos(n\frac{2\pi}{T}t) + \sum_{n=0}^{\infty} b_n \sin(n\frac{2\pi}{T}t) \quad (2)$$

$$\Psi(t) \sim \sum_{n=0}^{\infty} R_n \sin(n\frac{2\pi}{T}t + \theta_n) \quad (3)$$

Here, $R_n = \sqrt{a_n^2 + b_n^2}$.

Let,

$$J_N(t) = \sum_{n=0}^{N} R_n \sin(n\frac{2\pi}{T}t + \theta_n) \quad (4)$$

Depending on the instant amplitude of the modulating signal, the amplitude of the carrier wave varies; where the phase and frequency are retained as constant is defined as amplitude modulation. Similarly, in compliance with the instant amplitude of the modulating signal, the phase of the carrier wave varies where the amplitude and frequency are retained as constant is defined as Phase Modulation. If $\Psi(t)$ is continuous function then $J_N(t)$ expresses approximately $\Psi(t)$ within the period $0 \leq t \leq T$.

For instance, if a frequency domain is stationary [8] it is depicted by the symbol $\Psi(t)$ while the frequency itself is $n\omega$, where $\omega = \frac{1}{T}$. On the other hand, amplitude, R_n is the structure of a spectrogram while R_n^2 is the energy of the spectrum that corresponds to its frequency $n\omega$.

In that regard, if the nonstationary [9] signal is in between $[0, T]$ then the spectrogram will offer substandard results when conducting Fourier series. Meanwhile if a signal that is nonstationary has a spectrum $\Psi(t)$ then it is described as a dependent of time. However, if the Fourier coefficients of a_n and b_n exceeds $[0, T]$ it implies that the energy of the spectrum R_n^2 will not reveal the occurrence of the frequencies. It is for this reason that when one opts to use the coefficients constant that exceeds $[0, T]$ it would mean there will be a loss of signal from the local spectrum. This implies there would be no localization of time-frequency [10–12] for the Fourier series.

Fourier series is represented as the following way:

$$\Psi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\frac{2\pi}{T}t) + b_n \sin(n\frac{2\pi}{T}t)\right) [T = 2L]$$

The representation gives a clear physical meaning of the frequency of the signal as $n\omega$, which is calculated by multiplying the integer by the base frequency ω. In addition, the Fourier coefficients (amplitude) $R_n = \sqrt{a_n^2 + b_n^2}$ corresponds to the frequency and includes the phase shift $(\frac{2n\pi t}{T} + \theta_n)$.

The Fourier series can be broken down into sinusoidal signals. However, the weakness of this analysis is that its amplitudes are constant. Hence, the amplitude of the Fourier series can be modulated to enhance the localization of the time-frequency [13, 14] for the Fourier series. But in order to carry out the modulation of the amplitude of the complex wave (FS), this can be presented in the manner of

$$\Psi(t) \sim \sum_{n=0}^{\infty} R_n(t) \sin(n\frac{2\pi}{T}t + \theta_n) \tag{5}$$

The above expression is considered the amplitude modulated complex wave. The Amplitude Modulation of complex wave (FS) can also be figured as

$$\Psi(t) \sim \sum_{n=0}^{\infty} R_n(t) \sin(n\frac{2\pi}{T}t + \theta_n) + v_N(t) \tag{6}$$

Where $v_N(t)$ indicates noise. When there is a combination of amplitude modulation and phase modulation signals it contributes to the Instantaneous Frequency (IF) and Instantaneous Amplitude (IA) [15, 16] study.

Thus, if IF and IA can be inferred as a branch of the Fourier series, the sinusoidal signal will be revealed as follows

$$J(t) = b\sin(2\pi ft + \theta) \tag{7}$$

Here, b indicates the amplitude of the signal, f indicates the frequency, θ indicates a phase difference.

Amplitude modulation and phase modulation has been defined by Van der Pol [17] in the following way

i. Amplitude modulation:

$$b(t) = b_0[1 + \mu g(t)] \tag{8}$$

In which the modulating signal is specified by $g(t)$.

ii. Phase modulation:

$$\theta(t) = \theta_0[1 + \lambda h(t)] \tag{9}$$

Therefore, Frequency modulation (FM):

$$J(t) = b(t)\sin\phi(t) \tag{10}$$

It is possible to define the multi-component FM signals as

$$J(t) = \sum_{n=0}^{N} J_n(t) + v_N(t) \tag{11}$$

where $J_n(t) = b_n(t)\sin\phi_n(t)$.

Suppose $\phi_n(t) = 2\pi n\omega t + \theta_n$, the multi-component signal is

$$s(t) = \sum_{n=0}^{N} b_n(t)\sin(2\pi n\omega t + \theta_n) + v_N(t) \tag{12}$$

The equation Eq. (12) is the same as the Fourier series [18, 19] with amplitude modulation Eq. (6).

For time-frequency measurement of signals, AMFS is very suitable.

Then frequency $\frac{1}{2\pi}\frac{\partial\phi_n(t)}{dt} = n\omega$ and amplitude $|b_n(t)|$; A perfect spectrum analyzer is designed in which the local spectrum information is transparent.

2. Implementation

2.1 Phase Modulation

Considering the modulating signal (e_m), is a pure sinusoidal wave and the carrier signal (e_c) having high frequency

$$e_m = E_m\cos\omega_m t \tag{13}$$

$$e_c = E_c\sin\omega_c t \tag{14}$$

Equations Eq. (13) and Eq. (14) ignore the early stages of modulating signals and carrier signals because of their not contributing to the modulation process for constant values. The stage of the carrier will not remain same after Phase Modulation. Then the carrier equation is featured as

$$e = E_c\sin\theta \tag{15}$$

The modulated carrier's instantaneous phase is θ. Thus, θ can be expressed as

$$\theta = \omega_c t + K_p e_m \tag{16}$$

The proportionality constant for phase modulation is K_p, putting the value of equation Eq. (13) in Eq. (16),

$$\theta = \omega_c t + K_p E_m\cos\omega_m t \tag{17}$$

The modulated index is $K_p E_m$ in Eq. (17), Let,

$$m_p = K_p E_m \tag{18}$$

m_p indicates the modulation index of the phase modulation, then from equation Eq. (17), we get,

$$\theta = \omega_c t + m_p\cos\omega_m t \tag{19}$$

Putting the values of θ in Eq. (15),

$$e = E_c\sin(\omega_c t + m_p\cos\omega_m t) \tag{20}$$

This is the phase modulation wave.

2.2 Amplitude Modulation

Amplitude modulation is the action of transmitting wave over the amplitude of an uninterrupted highfrequency carrier. It is possible to characterize the modulated AM waveform by

$$x(t) = [A_c + m(t)]\cos(2\pi f_c t) \tag{21}$$

where A_c is the carrier amplitude, $m(t)$ is the message signal as desired, and f_c is the carrier frequency.

The modulation $R_n(t)$ can be carried out in two phases $R_n(t) = A_n^0 + A_n(t)$. The first step will be to define, A_n^0, followed by the second step which is to estimate $A_n(t)$. This can be further simplified as a piecewise constant.

By the following step, the equation Eq. (6) is recognized.

Step 1: calculating $R_0(t) = A_0^0 = a_0$ and $\theta_0 = \frac{\pi}{2}$

$$\therefore a_0 = \frac{1}{T}\int_0^T \Psi(t)dt \tag{22}$$

Step 2: Calculating A_1^0 and θ_1. Let

$$\Psi_1(t) = \Psi(t) - a_0 \tag{23}$$

$$\Psi_2(t) = \Psi_1(t) - c_1 \cos\frac{2\pi t}{T} - d_1 \sin\frac{2\pi t}{T} \tag{24}$$

We gain from using the least square form,

$$c_1 = \frac{2}{T}\int_0^T \Psi_1(t)\cos\frac{2\pi t}{T}dt \tag{25}$$

$$d_1 = \frac{2}{T}\int_0^T \Psi_1(t)\sin\frac{2\pi t}{T}dt \tag{26}$$

Then $A_1^0 = \sqrt{c_1^2 + d_1^2}$, $\theta_1 = \tan^{-1}(\frac{c_1}{d_1})$.
Step 3:
Evaluating $A_1(t)$
Assume $\theta_1 < 0$
Within $\left[\frac{T(-\theta_1)}{2\pi}, \frac{T(\pi-\theta_1)}{2\pi}\right]$, let $A_1(t)$ be a constant $A_1(1)$ in $\left[\frac{T(\pi-\theta_1)}{2\pi}, \frac{T(2\pi-\theta_1)}{2\pi}\right]$.
Let $A_1(t)$ be another constant $A_1(2)$.

$$\Psi_3(t) = \Psi_2(t) - A_1(t)\sin(\frac{2\pi t}{T} + \theta_1) \tag{27}$$

Then

$$\begin{aligned}\int_0^T \Psi_3^2(t)dt &= \int_0^T \Psi_2^2(t)dt \\ &+ \sum_{i=1}^{2} A_1^2(i)\int_{\frac{T((i-1)\pi-\theta_1)}{2\pi}}^{\frac{T(i\pi-\theta_1)}{2\pi}} \sin^2(\frac{2\pi t}{T} + \theta_1)dt \\ &- 2\sum_{i=1}^{2} A_1(i)\int_{\frac{T((i-1)\pi-\theta_1)}{2\pi}}^{\frac{T(i\pi-\theta_1)}{2\pi}} \Psi_2 \sin(\frac{2\pi t}{T} + \theta_1)dt\end{aligned} \tag{28}$$

To construct $\int_0^T \Psi_3^2(t)dt = \min$ it through the least square process, we get

$$A_1(i) = \frac{4}{T}\int_{\frac{T((i-1)\pi-\theta_1)}{2\pi}}^{\frac{T(i\pi-\theta_1)}{2\pi}} \Psi_2(t)\sin(\frac{2\pi t}{T} + \theta_1)dt,\ (i = 1, 2) \tag{29}$$

Step $2n$: Calculating A_n^0 and θ_n.
Let,

$$\Psi_{2n}(t) = \Psi_{2n-1}(t) - c_n \cos\frac{2n\pi t}{T} - d_n \sin\frac{2n\pi t}{T} \tag{30}$$

Through the least square method,

$$c_n = \frac{2}{T}\int_0^T \Psi_{2n-1}(t)\cos\frac{2n\pi t}{T}dt \tag{31}$$

$$d_n = \frac{2}{T}\int_0^T \Psi_{2n-1}(t)\sin\frac{2n\pi t}{T}dt \tag{32}$$

Then $A_n^0 = \sqrt{c_n^2 + d_n^2}$, $\theta_n = \tan^{-1}(\frac{c_n}{d_n})$.
Step $2n+1$: Estimating $A_n(t)$
Assume $\theta_n < 0$
Within $\left[\frac{T((i-1)\pi-\theta_n)}{2n\pi}, \frac{T(i\pi-\theta_n)}{2n\pi}\right]$.
Let, $A_n(t)$ be constant $A_n(i)$, $(i = 1, 2, \ldots, n)$,

$$\Psi_{2n+1}(t) = \Psi_{2n}(t) - A_n(t)\sin(\frac{2n\pi t}{T} + \theta_n) \tag{33}$$

then

$$\begin{aligned}\int_0^T \Psi_{2n+1}^2(t)dt &= \int_0^T \Psi_{2n}^2(t)dt \\ &+ \sum_{i=1}^{n} A_n^2(i)\int_{\frac{T((i-1)\pi-\theta_n)}{2n\pi}}^{\frac{T(i\pi-\theta_n)}{2n\pi}} \sin^2(\frac{2n\pi t}{T} + \theta_n)dt \\ &- 2\sum_{i=1}^{n} A_n(i)\int_{\frac{T((i-1)\pi-\theta_n)}{2n\pi}}^{\frac{T(i\pi-\theta_n)}{2n\pi}} \Psi_{2n} \sin(\frac{2n\pi t}{T} + \theta_n)dt\end{aligned} \tag{34}$$

To construct $\int_0^T \Psi_{2n+1}^2(t)dt = \min$ it through the least square process, we get

$$A_n(i) = \frac{4n}{T}\int_{\frac{T((i-1)\pi-\theta_n)}{2n\pi}}^{\frac{T(i\pi-\theta_n)}{2n\pi}} \Psi_{2n}(t)\sin(\frac{2n\pi t}{T} + \theta_n)dt, \tag{35}$$

Phase by phase, we will be able to get

$$\Psi_{2n+1}(t) = \Psi(t) - a_0 - \sum_{n=1}^{n}(A_n^0 + A_n(t))\sin(\frac{2n\pi t}{T} + \theta_n) \tag{36}$$

That is

$$\Psi_{2n+1}(t) = \Psi(t) - \sum_{n=0}^{n} R_n(t)\sin(\frac{2n\pi t}{T} + \theta_n) \tag{37}$$

Therefore, Amplitude Modulated Complex wave can be realized as $\Psi(t) \sim \sum_{n=0}^{\infty} R_n(t)\sin(\frac{2n\pi t}{T} + \theta_n)$ given the proof that $\lim_{n\to\infty}\int_0^T \Psi_{2n+1}^2(t)dt = 0$.

Besides, when carrying out an estimation of $R_n(t) = A_n^0 + A_n(t)$, A_n^0 is a constant that corresponds with the residual signal $\Psi_{2n-1}(t)$ in which the functions $\sin 2\pi n\omega t$ and $\cos 2\pi n\omega t$ are of the orthogonal space. Thus to get the orthogonal projections, in the residual signal $\Psi_{2n-1}(t)$ and $\Psi_{2n}(t)$ the original terms $A_k^0(t)\sin(\frac{2k\pi t}{T} + \theta_k)$ and $A_k(t)\sin(\frac{2k\pi t}{T} + \theta_k)$ $(k \le n)$ gets erased from $\Psi(t)$.

3. Result Analysis of signal in Time-frequency localization

Localization of time-frequency is expressed using the amplitude modulated Fourier series coefficients. Therefore given the local time domain $[T_1, T_2]$, if $R_n(t) \cong 0$, it implies that the signal component $\sin(2\pi n\omega t + \theta_n)$ is nonexistent. Besides the bigger the value of $|R_n(t)|$ in the local time domain, the larger spectrum energy from the corresponding frequency $n\omega$.

Example 1: Analysis of singular signals

Let,

$$\Psi_1(t) = \begin{cases} 1; & 0 \le t < 3 \\ 0; & 3 \le t < 6 \\ 1; & 6 \le t \le 9 \end{cases} \quad (38)$$

At the point $t = 3$ and $t = 6$, $\Psi_1(t)$ is singular. The elevated frequency at those points is also infinite. Extending

$$\Psi_1(t) = \sum_{n=0}^{N} R_n(t)\sin(2\pi n\omega t + \theta_n) + v_N(t)$$

drawing each term through Amplitude Modulated Complex Wave and $R_n(t)\sin(2\pi n\omega t + \theta_n)$ (connected with the curve of $(n+1)$th), from $R_n(t)\sin(2\pi n\omega t + \theta_n)$ (when n is bigger) we can see that in the interval $[0, 3-\delta_n]$, $[3+\delta_n, 6-\delta_n]$ and $[6+\delta_n, 18]$ ($\lim_{n\to\infty}\delta_n = 0$) the modulated signal $R_n(t)\sin(2\pi n\omega t + \theta_n)$ near zero, that is $R_n(t) \cong 0$; Near to a point only $t = 3$ and $t = 6$, $R_n(t) \neq 0$, there is a elevated frequency part $R_n(t)\sin(2\pi n\omega t + \theta_n)$.

Let time reflect the x-axis, the frequency represented by the y-axis (unit is ω) and the zaxis represents the absolute value of amplitude, that is, $|R_n(t)|$, The local time frequency information of $\Psi_1(t)$ is shown more clearly in figure 3

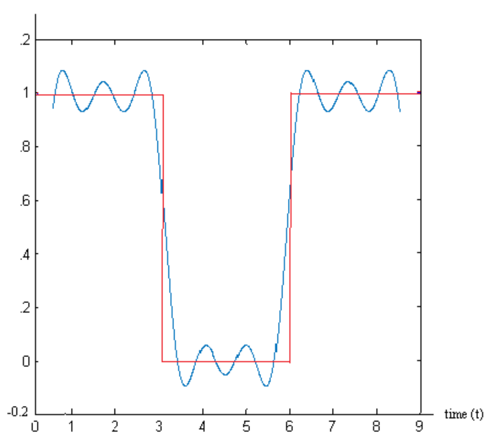

Figure 1. $\Psi_1(t)$ can be approximated by $J_7(t)$

4. Function approximation

The approximate speed between complex wave and Amplitude Modulated Complex wave is compared in this section.

By the equation no 2 and 3, suppose,

$$J_N(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\frac{2\pi}{T}t) + \sum_{n=1}^{\infty} b_n \sin(n\frac{2\pi}{T}t),$$

$\Psi(t)$ can be approximated by $J_N(t)$.

By equation 7, complex wave approximation, Say, $J_N^*(t) = \sum_{n=1}^{N} R_n(t)]sin(\frac{2n\pi t}{T} + \theta_n)$, it is possible to estimate $\Psi(t)$ with $J_N^*(t)$.

Suppose, $N = 7$, Figures 1 and 2 indicate that $\Psi_1(t)$ (as shown in the example Eq. (38)) is estimated by $J_7(t)$ and

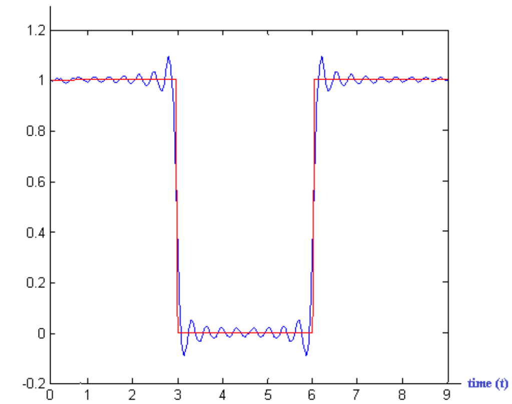

Figure 2. $\Psi_1(t)$ can be approximated by $J_7^*(t)$

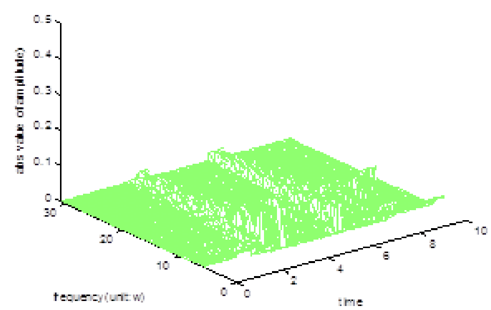

Figure 3. $\Psi_1(t)$ time-frequency analysis by Amplitude Modulated Complex wave

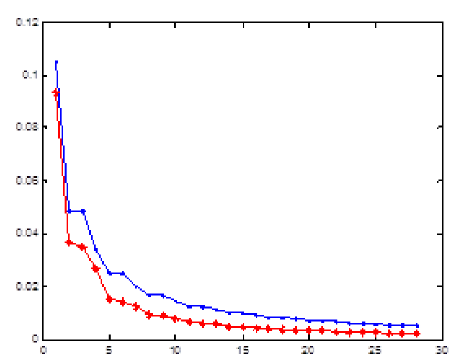

Figure 4. approximating speed between complex wave and Amplitude Modulated Complex wave

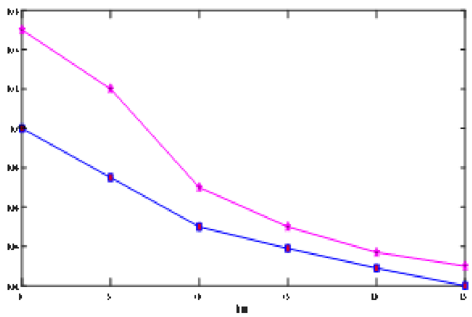

Figure 5. approximating speed between complex wave and Phase Modulated Complex Wave

$J_7^*(t)$ accordingly. Clearly, Amplitude Modulation of complex wave can more accurately calculate $\Psi_1(t)$ in the local time domain $[0,3-\delta]$, $[3+\delta,6-\delta]$ and $6+\delta,9]$. The grounds is that the Amplitude Modulated Complex wave coefficients $R_n(t)$ is very much a function of time t; the local value of $\Psi_1(t)$ can be more accurately calculated.

Let $\xi(N) = \frac{\int_0^T v_N^2(t)dt}{\int_0^T \Psi^2(t)dt}$.

Here, $v_N(t) = \Psi(t) - J_N(t)$ and $\xi^*(N) = \frac{\int_0^T v_N^{*2}(t)dt}{\int_0^T \Psi^2(t)dt}$ where $v_N^*(t) = \Psi(t) - J'_N(t)$.

It can be proved that $\lim_{N\to\infty} \xi^*(N) = 0$, $\lim_{N\to\infty} \xi(N) = 0$ is obviously. In Fig 4, more specifically, it can be shown that Amplitude Modulated Complex wave's approximate speed is higher than Complex wave compared with $\xi(N)$ and $\xi^*(N)$; where $\xi(N)$ is measured as complex wave without modulation and $\xi^*(N)$ is measured as modulated complex wave. In Fig 5, more specifically, it can be shown that Phase Modulated Complex wave's approximate speed is higher than Complex wave.

5. Conclusion

From the analysis, it shows that the Amplitude Modulated Complex wave & Phase modulated complex wave help in analyzing frequency of the wave and approximating function. It is also a useful tool for analyzing signals, processing images, and control of systems. In that regard, the theory and application of Modulated Complex wave wavelet analysis must be further developed to be internationally recognized as a minute tool for analyzing time-frequency. Also by drawing a comparison with the wavelet analysis, findings indicate that the Amplitude Modulated Complex Wave has its advantages. For instance, the components of Amplitude Modulated Complex wave $R_n(t)\sin(2\pi n\omega t+\theta_n)$ are said to be simple, of which the frequency $n\omega$ is clear. Future studies should therefore focus on combining the Amplitude Modulated Complex wave with an analysis of wavelet in signal analysis as well as control. On the other hand, Amplitude Modulated Complex wave has its shortcomings. The first limitation is that its function $R_n(t)$ is not continuous. Its other weakness is that the frequency $n\omega$; which corresponds to the component $R_n(t)\sin(2\pi n\omega t+\theta_n)$. It is combined with a high frequency of others. In that regard, future work should focus on improving the shortcomings of Amplitude & Phase Modulated Complex wave.

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