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Three Parameters Generalization of Lifetime Exponential Distribution-Power Series

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Abstract: Statistics science has presented frequency distribution for modeling the lifetime data such as gamma distribution. This study aims to introduce a new class of lifetime distributions having better behavior than other lifetime distributions. This class called as "Generalized Exponential- Power Series" (GEPS) derived from combining the generalized exponential distributions and power series distribution (cut at zero). We also intend to obtain properties of this distribution is one of the reliability functions (hazard function, survival function,. . .), torques, Maximum likelihood parameters estimators by EM algorithm. Hazard function of class GEPS distributions could have constant failure rate (CFR), increasing failure rate (IFR), decreasing failure rate (DFR), Bathtub shape (BT) depending on different parameters. Finally, we indicate how to use four sets of real data and capability of this new class of distributions.

Keywords: Failure rate, lifetime distribution, power series distributions, EM algorithm, maximum likelihood estimators

1. Introduction

Today, accelerated technological progress, development of super complicated goods, severe global competition and development of customers' expectations created new pressure on producers of manufacturers for producing the high quality product. Customers expect to purchase reliable and safe products. Therefore, systems, devices, machines and tools or instruments must have the ability of producing the reliable products and goods.

According to technical principles, the role played by such systems and tools is called as "***Reliability***" that is has a qualitative interpretation. Improving the reliability is a main part of discussion for improving the product's quality. There are many definitions about quality; however, the general discussion is that an unreliable product could not be a high quality product. Candra (. . . .yr..) stresses that "functionality is the quality on time".

Reliability mostly deals with non-negative random variables called as "***Lifetime***". Lifetime includes situations in which the time for occurrence of a specific outcome has been considered. A lifetime, as a random variable, is fully described by its distribution. Distributions that are used for modeling the lifetime called as "***Lifetime Distributions***". Lifetime becomes mostly clear by a failure, death or some other exhaustible events. Therefore, knowing the probability of failure in a component working in the next period (small enough) is highly important for analyzing the reliability of that component. Failure rate (or hazard function) expresses this probability. More precisely, failure rate or hazard rate is the possibility failing immediately after a given time of "t" assuming that the studied unit has been functioned until time "t". Failure rate of lifetime distributions could have different forms. Based on its type, this function could be constant failure rate (CFR), increasing failure rate (IFR), decreasing failure rate (DFR), Bathtub shape (BT) and upside-down bathtub shape (UBT). If the hazard function is increasing or ascending, the conditional probability of failure in infinitesimal timescales will be increased by time. For example, failure rate (mortality rate) in adults will be increased exponentially by time. Therefore, knowing and analyzing the form of failure rate plays important role in reliability and analyzing the lifetime data [1].

Lifetime distributions with descending failure rate have been just studied by different authors and researchers. Adamidis & Loukas [2] introduced a bi-parameter distribution obtained from combining the exponential distribution and geometrical distribution. Combining the exponential and Poison distributions, Kus [3] attained a lifetime distribution having decreasing failure rate. Chahkandi & Ganjali [4] obtained a new bi-parameters distribution that is a combination of power series and exponential distributions. This distribution has also a decreasing failure rate. Combining the negative binominal distribution and exponential distribution, Hajabi et al. [5] obtained a distribution having decreasing failure rate.

There are other distributions for lifetime variable having increasing failure rate such as Cancho et al. distribution [6]. This bi-parameter distribution with increasing failure rate has been obtained from combining exponential and poison distributions. Combining poison distribution cut at zero and exponential distribution, Rezaee & Tahmasbi [7] could also obtain a distribution with increasing failure rate.

In some cases, the failure rate of a lifetime distribution could have different forms. Gupta & Kundu [8] obtained a tri-parameter distribution with its failure rate depending on the studied parameter could have increasing, decreasing and or bathtub form. Silva et al. [9] have also obtained a new distribution with the same property. Combining Weibull distribution and geometrical distribution, Barreto- Souza et al. [10] attained to a distribution with decreasing, increasing and bathtub failure rate. Hemmati et al. [11] could also introduce a tri-parameters distribution with these three failure rates obtaining by combining Weibull distribution and Poison distribution. Bakouch et al. [12] could also introduce a new distribution with decreasing, increasing and or bathtub failure rate.

The purpose of this study is introducing a new class of tri-parameters lifetime distributions called as Generalized Exponential- Power Series (GEPS) distributions obtained from combining generalized exponential (GE) and power series (cut at zero) distributions. This new class of distributions have been presented by Mahmoudi & Jafari [13] comprising the Exponential- Power Series (EPS), Complementary Exponential- Geometric (CEG) and Poisson- Exponential (PE) models developed by Chahkandi & Ganjali [4], Louzada- Neto et al. [14] and Kundu [6]. This family also comprise lifetime distributions developed by Adamidis & Loukas, Kus as well as Tahmasbi & Rezaee as well.

The main objectives for introducing the class of GEPS distributions include:

1. This class of distributions provide a main model that could be used for different issues of modeling the lifetime data.

2. GEPS Distributions class provide reliable parametric fitness on skew data (that could not be fitted well by other distributions).

3. This class includes a few lifetime models as specific states.

2. Introduction to Distribution

Behind any statistical distribution is always at least one physical model. Physical model considerably assist fitting a proper distribution on a set of given data and sensibly interpreting that distribution. This section intends to introduce GEPS distributions with many applications could be considered for it. For this purpose, following physical interpretation has been presented.

Assume that a company has N parallel systems working independently in a given period such that N is a discrete random variable and is a member of power series distributions cut at zero with probability mass function Eq. (1) as below:

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},n=1,2,... \tag{1}$$

Also assume that any system has comprised from α units connected together in parallel. Failure times of units in ith system are indicated by $Z_{ia},\dots ,Z_{i2},Z_{i1}$ that are independent random variables and exponential co-distribution with scale parameter of β. Thus, for given N, there are X_N , ... ,X_1independent random variables and co-distribution (i.i.d) from generalized exponential distribution with cdf Eq. (2) that are the symbol of failure time (lifetime) N of related system.

$$G(x)=(1-e^{-\beta x})^{\alpha},x>0 \tag{2}$$

Because a parallel system works until when at least one of its units is intact (or more precisely, it will stop working when all of its units become out of order), the company will continue its activity until when at least one of its systems is working. On the other hand, the lifetime of related company (indicated with random variable X) is equal to lifetime of the system having the lifespan more than remaining; i.e. the lifetime of this company is equal to $X_{(n)}=\max\limits_{1\}i\}N} X_i$.

Now, we indent to model the lifetime of related company with GEPS distributions class. For this purpose, it is enough to attain the cumulative distribution function, X_n.

Conditional cumulative distribution function, $X_{(n)}\big|N=n$, includes:

$$\begin{aligned}G_{X_{(n)}|N=n}&=P(X_{(n)}\le x\big|N=n)\\&=(1-e^{-\beta x})^{n\alpha}\end{aligned} \tag{3}$$

That is a generalized exponential distribution with parameter α_n and β.

Related model is defined by lateral distribution function, X_n, (it should be noted that $X=X_n$), i.e.:

$$\begin{aligned}F(x)&=\sum_{n=1}^{\infty}P(N=n)P(X_{(n)}\le x\big|N=n)\\&=\sum_{n=1}^{\infty}\frac{a_n(\theta G(x))^n}{C(\theta)}=\frac{C(\theta G(x))}{C(\theta)}\\&=\frac{C(\theta(1-e^{-\beta x})^{\alpha})}{C(\theta)}\end{aligned} \tag{4}$$

where $\alpha,\beta>0,x>0$. This model emerges not only in the industrial usages but also in biological mechanisms. Equation Eq. (4) presents generalized exponential- power series distributions with parameters α, β and θ indicated by symbol GEPS (α, β, θ).

Remark 1: Put $X_1=\min\limits_{1\le i\le N}X_i$. Distribution function, X_1, equals with:

$$\begin{aligned}F_{X(1)}(x)&=1-\frac{C(\theta(1-G(x)))}{C(\theta)}\\&=1-\frac{C(\theta(1-(1-e^{-\beta x})^{\alpha}))}{C(\theta)}\end{aligned} \tag{5}$$

If α=1, cumulative distribution function will be $F_{X_1}(x)=1-\frac{C(\theta e^{-\beta x})}{C(\theta)}$ called as exponential- power series distribution by Chahkandi & Ganjali [4] and this family comprises also the lifetime distribution developed by Adamidis & Loukas [3] and Tahmasbi and Rezaee [15].

More, some structures of GEPS distributions class is expressed and by mentioning two examples, density function diagram and hazard function of this distribution will be observed.

3. Structural Properties of GEPS Distribution

3.1 Density Function

Assume that $X{\sim}GEPS(\alpha,\beta,\Theta)$. By differentiation of equation Eq. (4), the density function of this variable includes:

$$f(x)=\theta g(x)\frac{C'(\theta G(x))}{C(\theta)}=\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{C'(\theta(1-e^{-\beta x})^{\alpha})}{C(\theta)} \quad (6)$$

3.2 Hazard Function and Survival Function

Survival function and failure rate of GEPS distributions class include respectively:

$$S(x)=1-F(x)=1-\frac{C(\theta G(x))}{C(\theta)}=\frac{C(\theta)-C(\theta(1-e^{-\beta x})^{\alpha})}{C(\theta)},x>0 \quad (7)$$

and

$$h(x)=\frac{f(x)}{S(x)}=\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{C'(\theta(1-e^{-\beta x})^{\alpha})}{C(\theta)-C(\theta(1-e^{-\beta x})^{\alpha})} \quad (8)$$

More, for seeing the density function diagram and failure rate function of this distribution, we express two following examples:

Example 1: Put $C(\theta)=\theta+\theta^{20}$. If β=1 and θ=1, then,

$$f(x)=\frac{1}{2}\alpha e^{-x}(1-e^{-x})^{\alpha-1}[1+20(1-e^{-x})^{19\alpha}]$$

And

$$h(x)=\alpha e^{-x}(1-e^{-x})^{\alpha-1}\frac{1+20(1-e^{-x})^{19\alpha}}{2-(1-e^{-x})^{\alpha}-(1-e^{-x})^{20\alpha}}$$

Above Density function diagram and hazard function have been drawn in Figure 1 for a= $0.5,1,2$. For $\alpha=2$, this density has two modes with its values are 0/6931 and 3/5563.

Example 2: For $0<\theta<1$, put $C(\theta)=\sin^{-1}(\theta)$; thus, $C'(\theta)=\frac{1}{\sqrt{1-\theta^2}}$. Now density function and failure rate function based on β=1 include:

$$f(x)=\frac{\theta\alpha e^{-x}(1-e^{-x})^{\alpha-1}}{\sin^{-1}(\theta)\sqrt{1-(\theta(1-e^{-x})^{\alpha})^2}}$$

And

$$h(x)=\frac{\theta\alpha e^{-x}(1-e^{-x})^{\alpha-1}}{[\sin^{-1}(\theta)-\sin^{-1}(\theta(1-e^{-x})^{\alpha})]\sqrt{1-(\theta(1-e^{-x})^{\alpha})^2}}$$

Figure 2 indicates above hazard function and density function based on a= $0.5,1,2$ and θ= 0.9.

3.3 Quantile

Quantile ζ from GEPS distribution is equal with:

$$x_{\xi}=G^{-1}\left(\frac{C^{-1}(\xi C(\theta))}{\theta}\right) \quad (9)$$

Symbol $C^{-1}(\cdot)$ is the reverse function of $C(\cdot)$? (Generally, it isn't easy to find $C^{-1}(\cdot)$).

3.4 Sorted Statistics

Assume that $X_1,\ldots,X_n$, is a random sample (i.i.d) of GEPS distributions class with parameters α, β and θ, ith distribution of sorted statistics of this example detailed as below:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{C'(\theta(1-e^{-\beta x})^{\alpha})[C(\theta(1-e^{-\beta x})^{\alpha})]^{i-1}[C(\theta)-C(\theta(1-e^{-\beta x})^{\alpha})]^{n-i}}{C^n(\theta)} \quad (10)$$

And distribution function corresponding to this distribution includes:

$$F_{i:n}(x)=\sum_{k=i}^{n}\binom{n}{k}\frac{[C(\theta(1-e^{-\beta x})^{\alpha})]^k[C(\theta)-C(\theta(1-e^{-\beta x})^{\alpha})]^{n-k}}{C^n(\theta)} \quad (11)$$

3.5 Momentum Generating Function

Assume that $X{\sim}GEPS(\alpha,\beta,\Theta)$, we have:

$$M_X(t)=\sum_{n=1}^{\infty}P(N=n)M_{Y(n)}(t) \quad (12)$$

Where, $M_{Y(n)}(t)$is the momentum generating function of $Y_{(n)}=\max(Y_1,...,Y_n)$. We know that $Y_{(n)}\sim GE(n\alpha,\beta)$, then:

$$M_{Y(n)}(t)=n\alpha B(1-\frac{t}{\beta},n\alpha)=\frac{\Gamma(n\alpha+1)\Gamma(1-\frac{t}{\beta})}{\Gamma(n\alpha+1-\frac{t}{\beta})},t<\beta \quad (13)$$

Finally, we have:

$$M_X(t)=\Gamma(1-\frac{t}{\beta})\sum_{n=1}^{\infty}P(N=n)\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}=\Gamma(1-\frac{t}{\beta})\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}=\Gamma(1-\frac{t}{\beta})\sum_{n=1}^{\infty}E\left(\frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\frac{t}{\beta})}\right),t<\beta$$

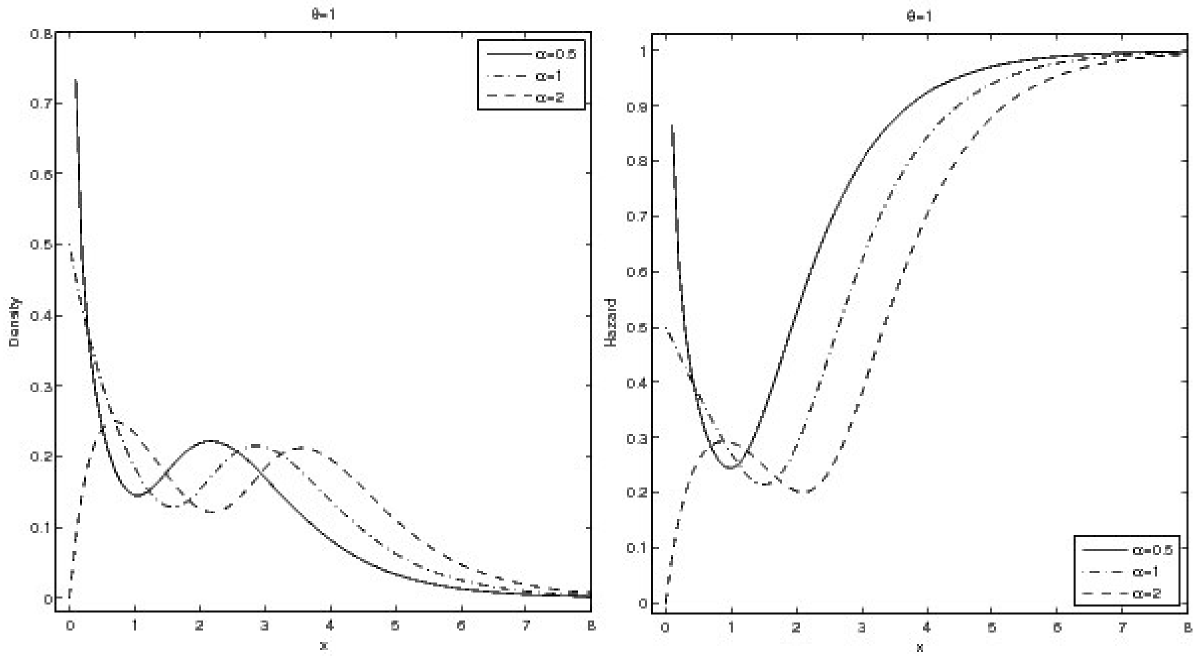

Figure 1. Diagram of Density Function and GEPS Hazard Function when $C(\theta)=\theta+\theta^{20}$

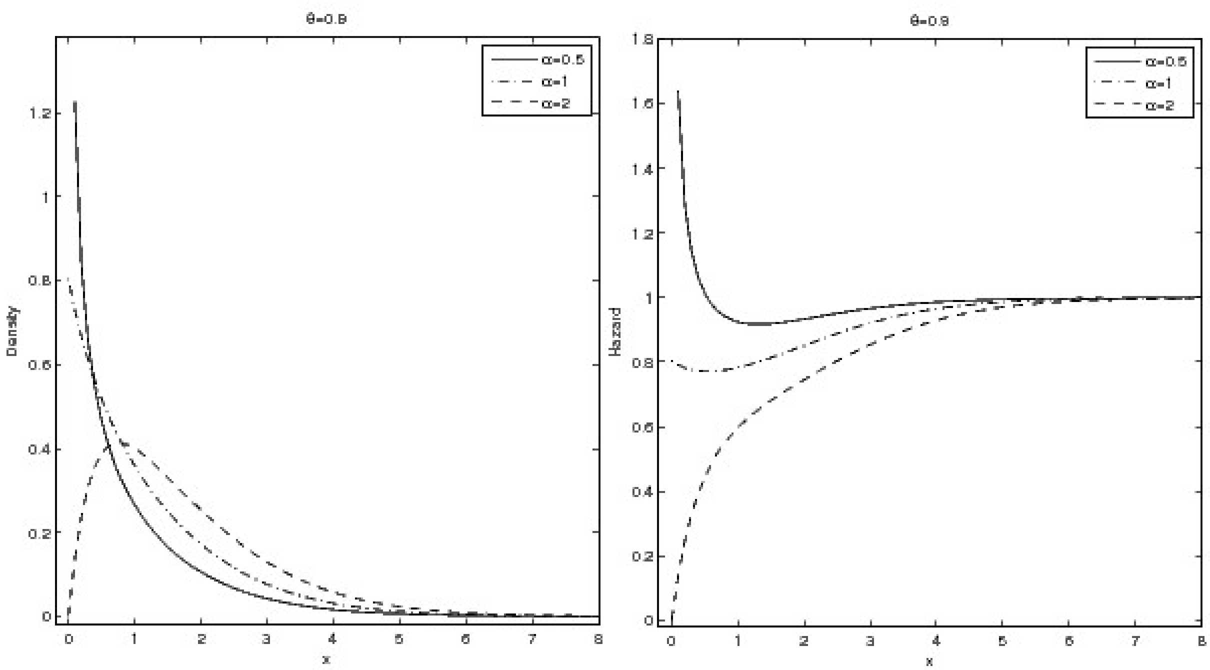

Figure 2. Diagram of Density Function and GEPS Hazard Function when $C(\theta)=\sin^{-1}(\theta)$

3.6 Central Torques

By direct calculation, we have:

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x)dx \\ &= \int_0^\infty x^k \sum_{n=1}^{\infty} P(N=n) g_{(n)}(x;n)dx \\ &= \sum_{n=1}^{\infty} P(N=n)E(Y_{(n)}^k) \end{aligned} \tag{14}$$

We know that $Y_{(n)} \sim GE(n\alpha,\beta)$, then we have:

$$E(Y_{(n)}) = \frac{1}{\beta}[\psi(n\alpha+1)-\psi(1)]$$

$$E(Y_{(n)}^2) = \frac{1}{\beta^2}\left[\psi'(1)-\psi'(n\alpha+1)+(\psi(n\alpha+1)-\psi(1))^2\right]$$

Thus,

$$E(X)=\sum_{n=1}^{\infty}P(N=n)E(Y_{(n)})$$
$$=\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\psi(n\alpha+1)-\frac{1}{\beta}\psi(1)$$

And

$$E(X^2)=\sum_{n=1}^{\infty}P(N=n)E(Y_{(n)}^2)$$
$$=\frac{1}{\beta^2}\psi'(1)-\frac{1}{\beta^2}\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\big[\psi'(n\alpha+1)$$
$$-(\psi(n\alpha+1)-\psi(1))^2\big]$$

Remember that $\sum_{n=1}^{\infty}P(N=n)=1$. Now, we will obtain kth torque of distribution $GEPS(\alpha,\beta,\Theta)$.

Because, $Y_{(n)}\sim GE(n\alpha,\beta)$, kth torque around the origin of this variable is as below:

$$E(Y_{(n)}^k)=\frac{n\alpha k!}{\beta^k}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{1}{(j+1)^{k+1}} \tag{15}$$

Finally, according to equalities Eq. (14) and Eq. (15), kth central torque of GEPS distribution include:

$$\mu_k=E(X^k)$$
$$=\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\frac{n\alpha k!}{\beta^k}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{1}{(j+1)^{k+1}}$$
$$=\frac{\alpha k!}{\beta^kC(\theta)}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{na_n\theta^n}{(j+1)^{k+1}}$$

4. Reviewing the Relation between GEPS Distribution and other Functions

This section introduces four distributions a member of generalized exponential - power series distributions with expressing some of their structural properties by using provided equations.

4.1 Generalized Exponential- Geometrical Distribution

Geometrical distribution (cut at zero) is a specific state of power series distributions with $a_n=1$ and $C(\theta)=\frac{\theta}{1-\theta}(0<\theta<1)$. Using Eq. (4), cumulative distribution function, Generalized- Exponential- Geometric Distribution (GEG) includes:

$$F(x)=\frac{(1-\theta)(1-e^{-\beta x})^{\alpha}}{1-\theta(1-e^{-\beta x})^{\alpha}},x>0 \tag{16}$$

This distribution has been presented by Gupta & Kundu [16].

4.1.1 Probability Density Function

We know that $C'(\theta)=\frac{1}{(1-\theta)^2}$. Therefore, according to Eq. (6), GEG Distribution Density Function equals with:

$$f(x)=\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{1-\theta}{[1-\theta(1-e^{-\beta x})^{\alpha}]^2} \tag{17}$$

It is to mention that above result could be directly obtained by differentiation from Eq. (16).

Remark 2: When $a=1$ and $\theta^*=1-\theta$, we have:

$$f(x)=\frac{\beta\theta^*e^{-\beta x}}{[e^{-\beta x}(1-\theta^*)+\theta^*]^2}$$

That has been introduced as geometric- complementary exponential distribution (GEG) by Louzada- Neto et al. [14].

4.1.2 Survival Function and Hazard Function

By direct calculation, the survival function and hazard function of this distribution respectively include:

$$S(x)=1-F(x)=\frac{1-(1-e^{-\beta x})^{\alpha}}{1-\theta(1-e^{-\beta x})^{\alpha}}$$

And

$$h(x)=\frac{\alpha\beta(1-\theta)e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}}{\left[1-(1-e^{-\beta x})^{\alpha}\right]\left[1-\theta(1-e^{-\beta x})^{\alpha}\right]} \tag{18}$$

Both above functions could be also obtained by equations Eq. (7) and Eq. (8).

Figures 3 and 4 indicate diagrams of density function and GEG failure rate for $\beta=1$ and some values of α and β.

Theorem 1: consider GEG Distribution Hazard Function in Eq. (18). For $\alpha\geq 1$, this Hazard Function is Increasing and for $0<a<1$ it is decreasing and bathtub shaped.

4.1.3 Quantiles

As stated in GEG Distribution, $C(\theta)=\frac{\theta}{1-\theta}$ $(0<\theta<1)$, then $C^{-1}(\theta)=\frac{\theta}{1+\theta}$ is reverse function of $C(\theta)$. Now, according to equation Eq. (9), quantile ζ from GEG distribution equals to:

$$x_{\xi}=G^{-1}\left(\frac{\xi\frac{\theta}{1-\theta}}{\theta(1+\xi\frac{\theta}{1-\theta})}\right)=G^{-1}\left(\frac{\xi}{1-\theta(1-\xi)}\right)$$

Where, $G^{-1}(y)=-\frac{1}{\beta}\log(1-y^{\frac{1}{\alpha}})$. One could use this expression for simulating the random data from GEG Distribution by producing random data from uniform distribution in range $(0,1)$.

4.1.4 Sorted Statistics

Considering $X_1,\ldots,X_n$ is a random sample from GEG (α,β,θ) distribution. The density function of ith sorted statistics of this sample includes:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}$$
$$\frac{(1-\theta)^i(1-e^{-\beta x})^{\alpha(i-1)}[1-(1-e^{-\beta x})^{\alpha}]^{n-i}}{[1-\theta(1-e^{-\beta x})^{\alpha}]^{n+1}}$$

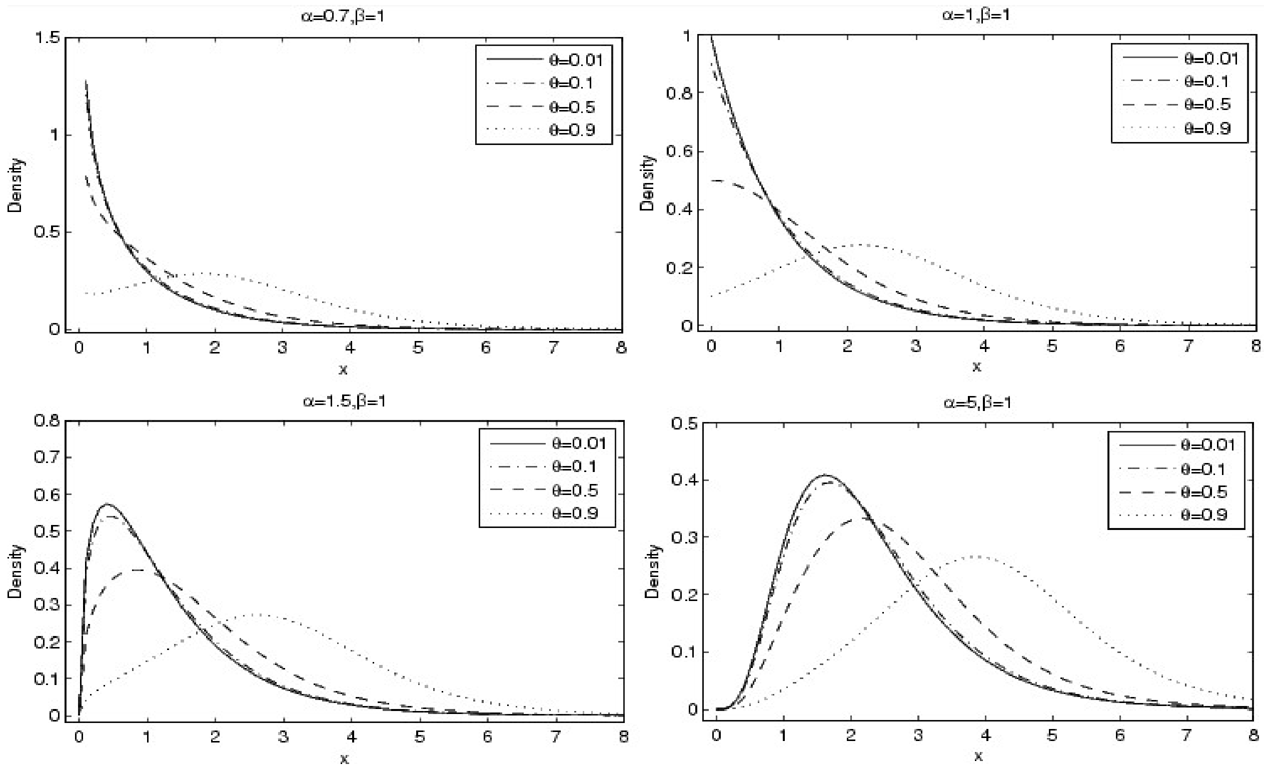

Figure 3. GEG Distribution Density Diagram

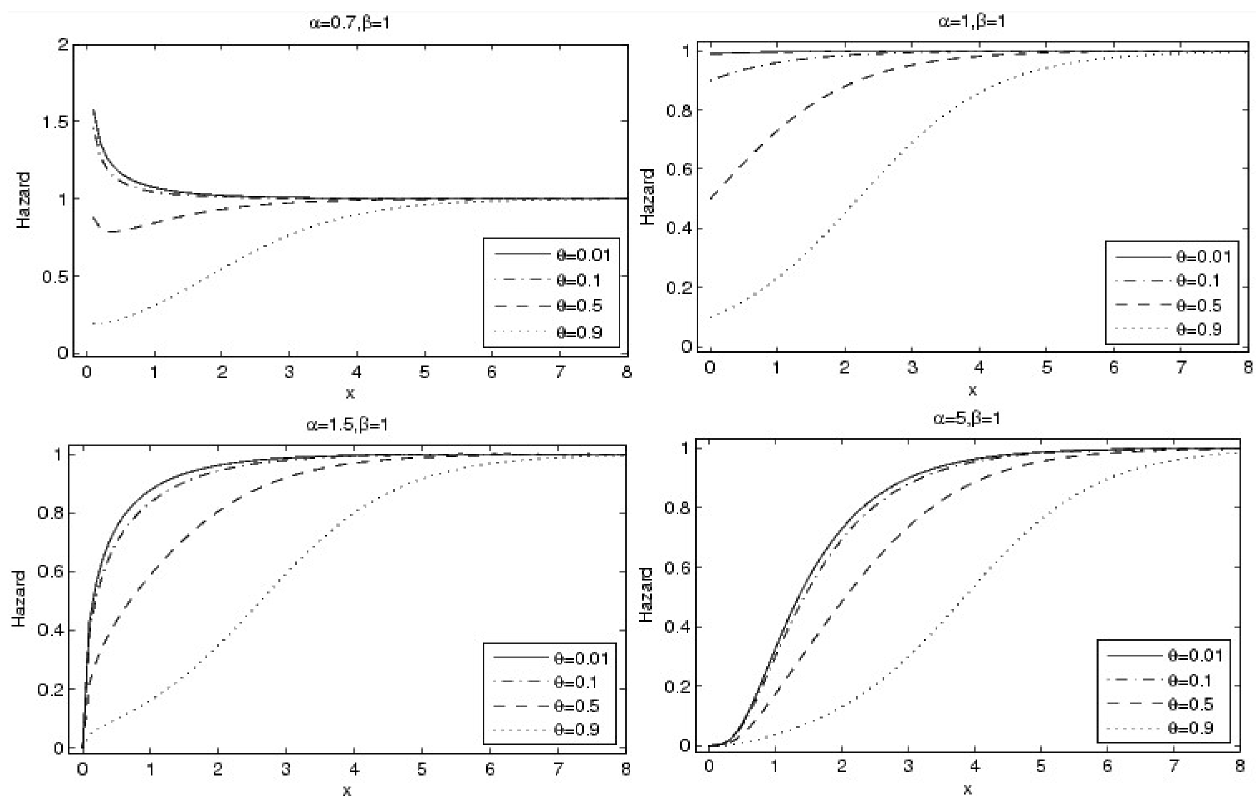

Figure 4. GEG Distribution Hazard Function Diagram

And corresponding distribution function equals to:

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} \frac{(1-\theta)^k (1-e^{-\beta x})^{k\alpha} [1-(1-e^{-\beta x})^{\alpha}]^{n-k}}{[1-\theta(1-e^{-\beta x})^{\alpha}]^n}$$

4.1.5 Momentum Generating Function

GEG Momentum Generating Function includes:

$$M_X(t) = \frac{(1-\theta)\Gamma(1-\frac{t}{\beta})}{\theta}\sum_{n=1}^{\infty}\theta^n\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}$$

4.1.6 Central Momentums

Central momentums of this distribution include:

$$E(X^k) = \frac{\alpha(1-\theta)k!}{\beta^k\theta}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{n\theta^n}{(j+1)^{k+1}}$$

4.2 Generalized Exponential- Poisson Distribution

Poisson distribution (cut at zero) is a specific state of power series distributions with $a_n = \frac{1}{n!}$ and $C(\theta) = e^{\theta}-1(\theta>0)$. Now, using cumulative distribution function Eq. (4), Generalized Exponential- Poisson Distribution (GEP) is given as below:

$$F(x) = \frac{e^{\theta(1-e^{-\beta x})^{\alpha}}-1}{e^{\theta}-1}, x>0$$

This distribution has been provided by Barreto- Souza and Cribari-Neto [17].

4.2.1 Probability Density Function

It is clear that $C'(\theta) = e^{\theta}$; therefore, according to Eq. (6), GEP Density Distribution is obtained as below:

$$f(x) = \theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{e^{\theta(1-e^{-\beta x})^{\alpha}}}{e^{\theta}-1}$$

Remark 3: When α=1, we have:

$$f(x) = \theta\beta e^{-\beta x}\frac{e^{\theta(1-e^{-\beta x})}}{e^{\theta}-1} = \theta\beta\frac{e^{-\beta x-\theta e^{-\beta x}}}{1-e^{\theta}}$$

(The final fraction is obtained by dividing the numerator and denominator on e^{θ}) introduced by Cancho et al. [6] as Poisson- Exponential Distribution (PE).

4.2.2 Survival Function and Hazard Function

According to equations Eq. (7) and Eq. (8), survival function and hazard function of this distribution respectively are:

$$S(x) = \frac{e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}}{e^{\theta}-1}$$

And

$$h(x) = \theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{e^{\theta(1-e^{-\beta x})^{\alpha}}}{e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}} \quad (19)$$

Figures 5 and 6 indicate diagrams of density function and GEP failure rate for $\beta=1$ and some values of α and θ.

Theorem 2: Consider **GEP** Distribution Hazard Function in Eq. (19). This hazard function is increasing for $a=1$, and is decreasing and bathtub shape for $0<a<1$

4.2.3 Quantiles

In GEP distribution, it has been stated that $C(\theta) = e^{\theta}-1$, thus, $C^{-1}(\theta) = \log(1+\theta)$. Then, according to equation Eq. (9), quantile ζ from GEP distribution equals with:

$$x_{\xi} = G^{-1}\left(\frac{\log(1+\xi C(\theta))}{\theta}\right) = G^{-1}\left(\frac{\log(1+\xi(e^{\theta}-1))}{\theta}\right)$$

Where, $G^{-1}(y) = -\frac{1}{\beta}\log(1-y^{\frac{1}{\alpha}})$ and log is the symbol of natural logarithm in base "e". Quantiles of this distribution are used for random data simulation from GEP distribution.

4.2.4 Sorted Statistics

According to Eq. (10), probability density function of i^{th} sorted statistics of a n-times random sample of GEP distribution equals to:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}$$
$$\frac{e^{\theta(1-e^{-\beta x})^{\alpha}}\left[e^{\theta(1-e^{-\beta x})^{\alpha}}-1\right]^{i-1}\left[e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}\right]^{n-i}}{(e^{\theta}-1)^{n}}$$

And according to Eq. (11) distribution function of this statistics equals with:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}$$
$$\frac{e^{\theta(1-e^{-\beta x})^{\alpha}}\left[e^{\theta(1-e^{-\beta x})^{\alpha}}-1\right]^{i-1}\left[e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}\right]^{n-i}}{(e^{\theta}-1)^{n}}$$

4.2.5 Momentum Generating Function

For finding the momentum generating function of GEP, it is enough to put $a_n = \frac{1}{n!}$ and $C(\theta) = e^{\theta}-1$, we will have:

$$M_X(t) = \frac{\Gamma(1-\frac{t}{\beta})}{e^{\theta}-1}\sum_{n=1}^{\infty}\frac{\theta^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}$$

4.2.6 Central Momentums

K^{th} momentum of this distribution is as below:

$$E(X^k) = \frac{\alpha k!}{\beta^k(e^{\theta}-1)}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{\theta^n}{(n-1)!(j+1)^{k+1}}$$

4.3 Generalized Exponential- Binominal Distribution

Binominal distribution (cut at zero) is a specific state of power series distributions with $a_n = \binom{m}{n}$ and $C(\theta) = (\theta+1)^m-1(\theta>0)$ that m $(n\le m)$ is number of repeats. Using cumulative distribution function Eq. (4), Generalized Exponential- Binominal Distribution (GEB) Cumulative Distribution Function includes:

$$F(x) = \frac{[\theta(1-e^{-\beta x})^{\alpha}+1]^m-1}{(\theta+1)^m-1}, x>0$$

This distribution has been presented by Bakouch et al. [12].

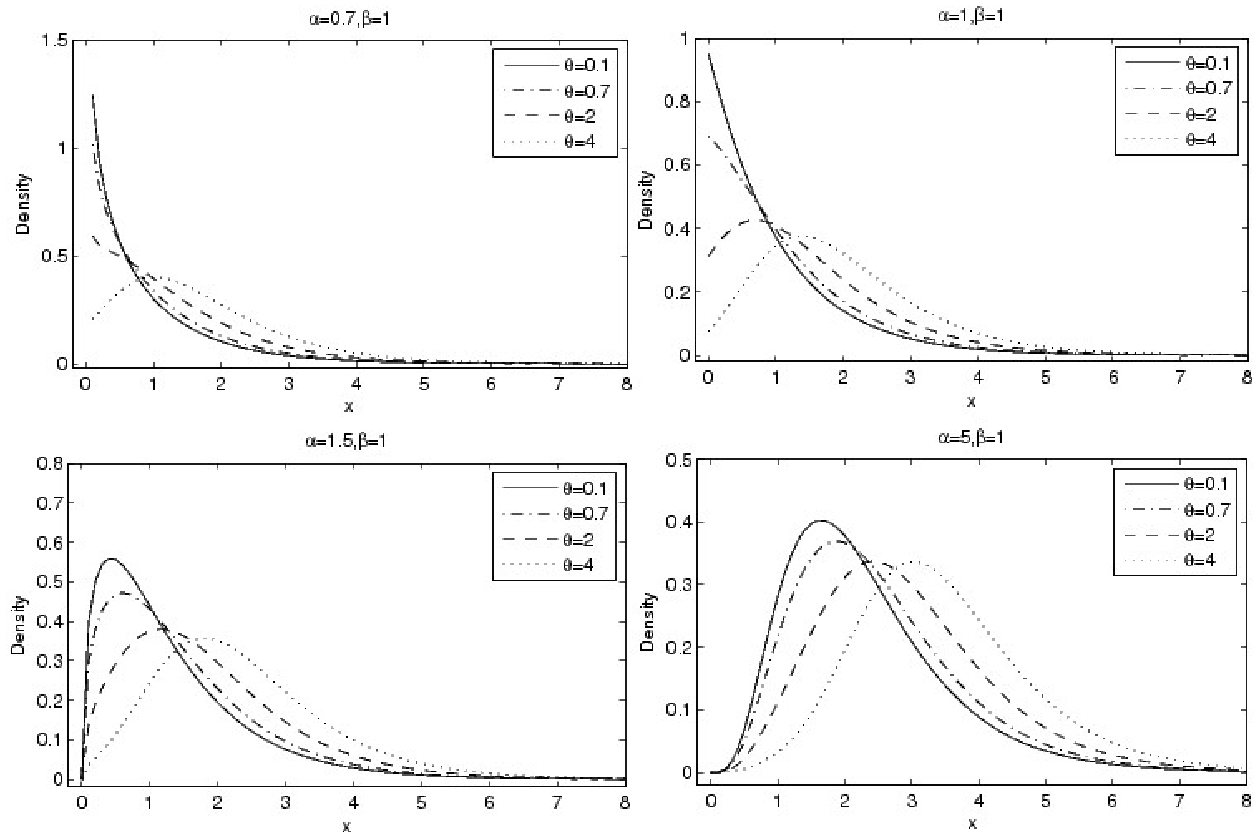

Figure 5. GEP Distribution Density Function Diagram

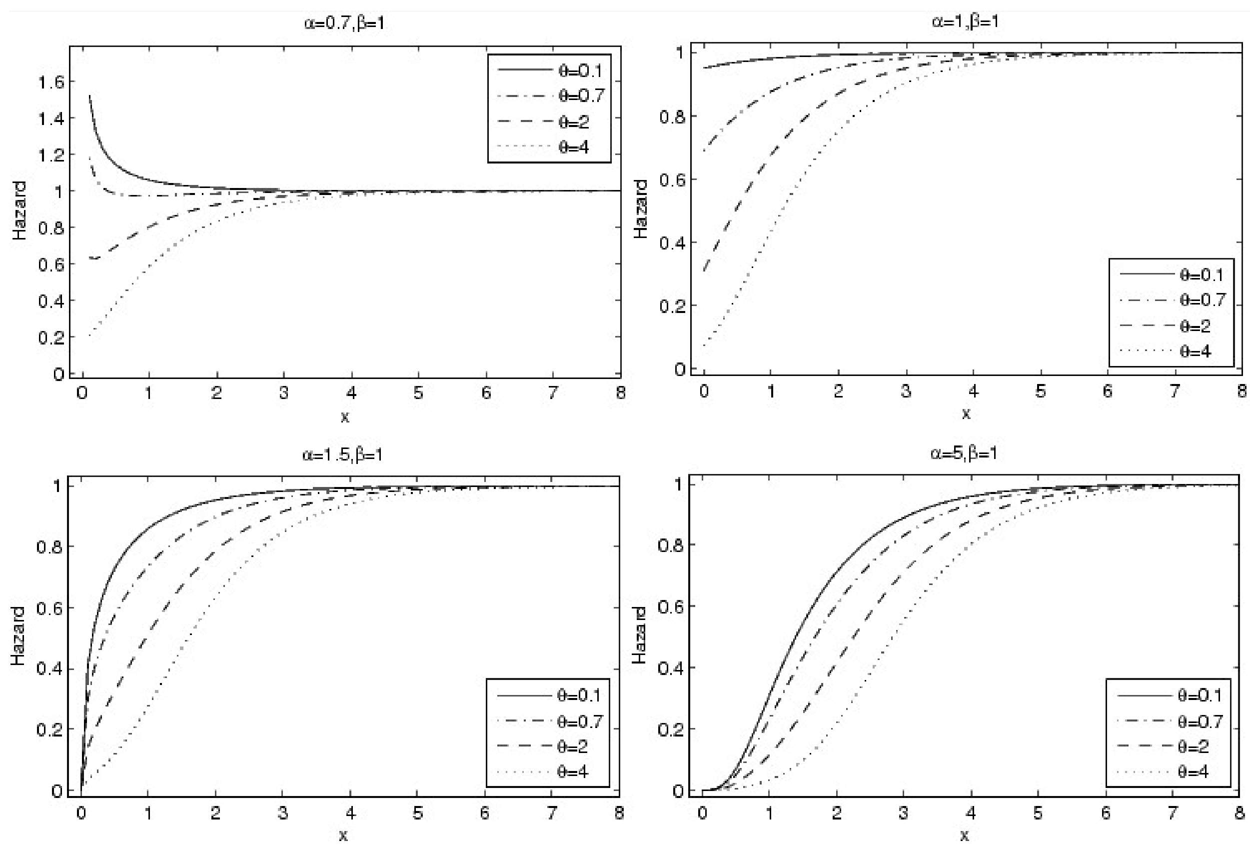

Figure 6. GEP Distribution Hazard Function Diagram

4.3.1 Probability Density Function

Because derivative $C(\theta)$ equals with $C'(\theta)=m(\theta+1)^{m-1}$, according to equation Eq. (6), GEB Distribution Density Function is as below:

$$f(x)=m\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{[\theta(1-e^{-\beta x})^{\alpha}+1]}{(\theta+1)^m-1}^{m-1}$$

4.3.2 Survival Function and Hazard Function

By using equation Eq. (7) and Eq. (8), survival function and hazard rate function of this distribution respectively is:

$$S(x)=\frac{(\theta+1)^m-[\theta(1-e^{-\beta x})^{\alpha}+1]^m}{(\theta+1)^m-1}$$

And

$$\begin{aligned}h(x)=&m\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\\&\frac{[\theta(1-e^{-\beta x})^{\alpha}+1]^{m-1}}{(\theta+1)^m-[\theta(1-e^{-\beta x})^{\alpha}+1]^m}\end{aligned} \tag{20}$$

Figures 7 and 8 indicate diagrams of density function and GEB Distribution Failure Rate for β= 1 and some values of α and θ.

Theorem 3: Consider **GEP** Distribution Hazard Function in Eq. (20). This hazard function is increasing for $a=1$, and is decreasing and bathtub shape for $0<a<1$.

4.3.3 Quantiles

In GEP, we know that $C(\theta)=(\theta+1)^m-1$, thus, its reverse is $C^{-1}(\theta)=\sqrt[m]{\theta+1}-1$. Consequently, according to equation Eq. (9), quantile ζ from GEP distribution equals with:

$$\begin{aligned}x_{\xi}=&G^{-1}\left(\frac{\sqrt[m]{\xi C(\theta)+1}-1}{\theta}\right)\\=&G^{-1}\left(\frac{\sqrt[m]{\xi((\theta+1)^m-1)+1}-1}{\theta}\right)\end{aligned}$$

Where, $G^{-1}(\cdot)$ is the reverse of distribution function GE(α,β).

4.3.4 Sorted Statistics

Assume that $X_1,\ldots,X_n$ is a random sample of GEB(α,β,θ) distribution. According to equations Eq. (10) and Eq. (11), probability density function and distribution function of i^{th} sorted statistics of this random sample respectively is:

$$\begin{aligned}f_{i:n}(x)=&\frac{n!}{(i-1)!(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\\&\frac{e^{\theta(1-e^{-\beta x})^{\alpha}}\left[e^{\theta(1-e^{-\beta x})^{\alpha}}-1\right]^{i-1}\left[e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}\right]^{n-i}}{(e^{\theta}-1)^n}\end{aligned}$$

and

$$\begin{aligned}F_{i:n}(x)=&\sum_{k=i}^{n}\binom{n}{k}\\&\left[(\theta(1-e^{-\beta x})^{\alpha}+1)^m-1\right]^k\\&\frac{\left[(\theta+1)^m-(\theta(1-e^{-\beta x})^{\alpha}+1)^m\right]^{n-k}}{((\theta+1)^m-1)^n}\end{aligned}$$

4.3.5 Momentum Generating Function

The momentum generating function equals to:

$$M_X(t)=\frac{\Gamma(1-\frac{t}{\beta})}{(\theta+1)^m-1}\sum_{n=1}^{\infty}\binom{m}{n}\theta^n\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}$$

4.3.6 Central Momentums

Central momentums of this distribution are:

$$\begin{aligned}E(X^k)=&\frac{\alpha k!}{\beta^k[(\theta+1)^m-1]}\\&\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{\binom{m}{n}n\theta^n}{(j+1)^{k+1}}\end{aligned}$$

4.4 Generalized Exponential- Binominal Distribution

Logarithmic distribution (cut at zero) is a specific state of power series distributions with $a_n=\frac{1}{n}$and $C(\theta)=-\log(1-\theta)(0<\theta<1)$. According to Eq. (4) Generalized Exponential- Logarithmic Distribution (GEL) cumulative distribution function equals with:

$$F(x)=\frac{\log(1-\theta(1-e^{-\beta x})^{\alpha})}{\log(1-\theta)},x>0$$

4.4.1 Probability Density Function

It could be easily seen that $C'(\theta)=\frac{1}{1-\theta}$, then according to Eq. (6), GEL distribution density function equals to:

$$f(x)=-\frac{\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}}{[1-\theta(1-e^{-\beta x})^{\alpha}]\log(1-\theta)}$$

Remark 4: GEL Distribution Density Function based on α=1 equals to:

$$f(x)=-\frac{\theta\beta e^{-\beta x}}{[1-\theta(1-e^{-\beta x})]\log(1-\theta)}$$

That has been developed as exponential-logarithmic distribution (EL) by Tahmasbi & Rezaee [15].

4.4.2 Survival Function and Hazard Function

According to equations Eq. (7) and , survival function and hazard rate function of this distribution respectively is:

$$S(x)=-\frac{\log\left(\frac{1-\theta(1-e^{-\beta x})^{\alpha}}{1-\theta}\right)}{\log(1-\theta)}$$

And

$$h(x)=\frac{\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}}{[1-\theta(1-e^{-\beta x})^{\alpha}]\log\left(\frac{1-\theta(1-e^{-\beta x})^{\alpha}}{1-\theta}\right)} \tag{21}$$

Figures 9 and 10 indicate diagrams of density function and GEB Distribution Failure Rate for β= 1 and some values of α and θ.

Theorem 4: Consider **GEL** Distribution Hazard Function in Eq. (20). This hazard function is increasing for $a=1$, and is decreasing and bathtub shape for $0<a<1$

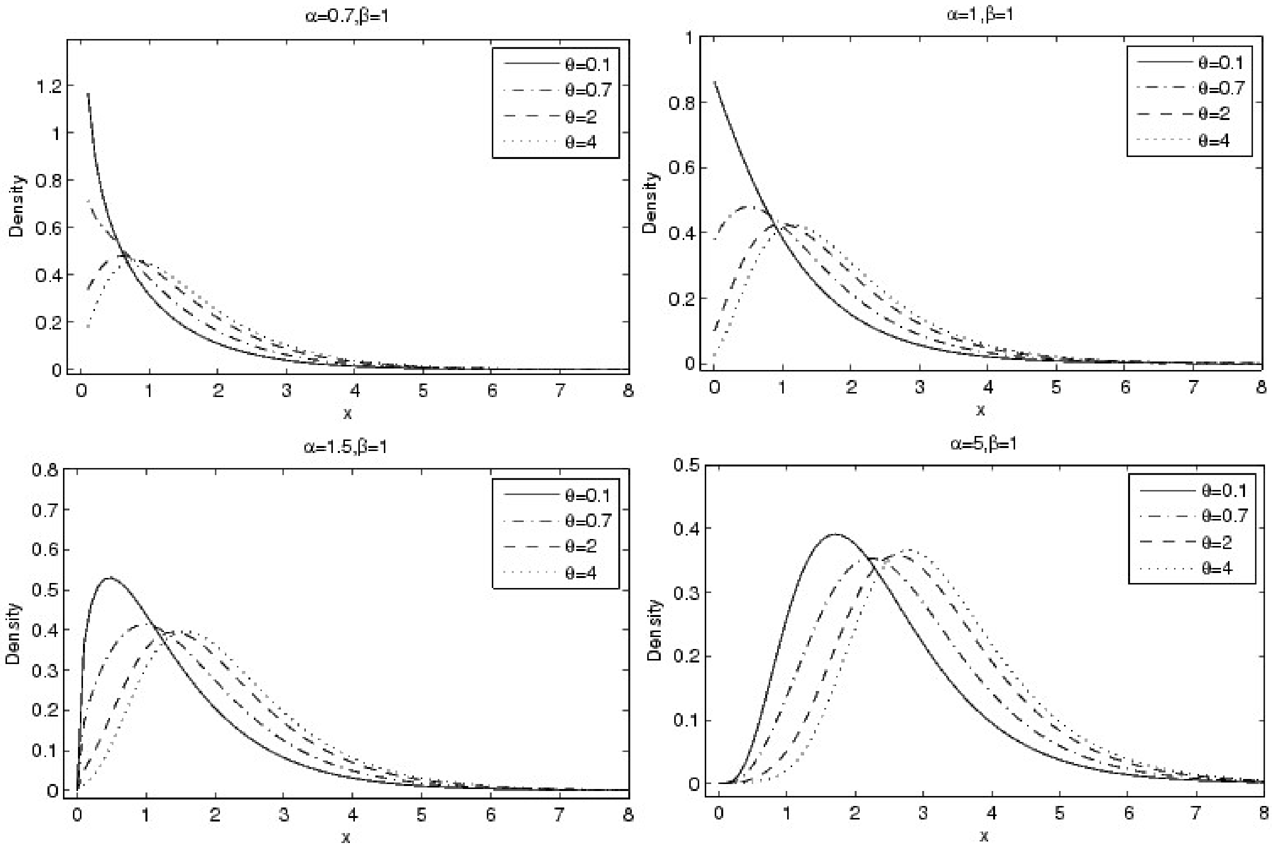

Figure 7. GEB Distribution Density Function Diagram

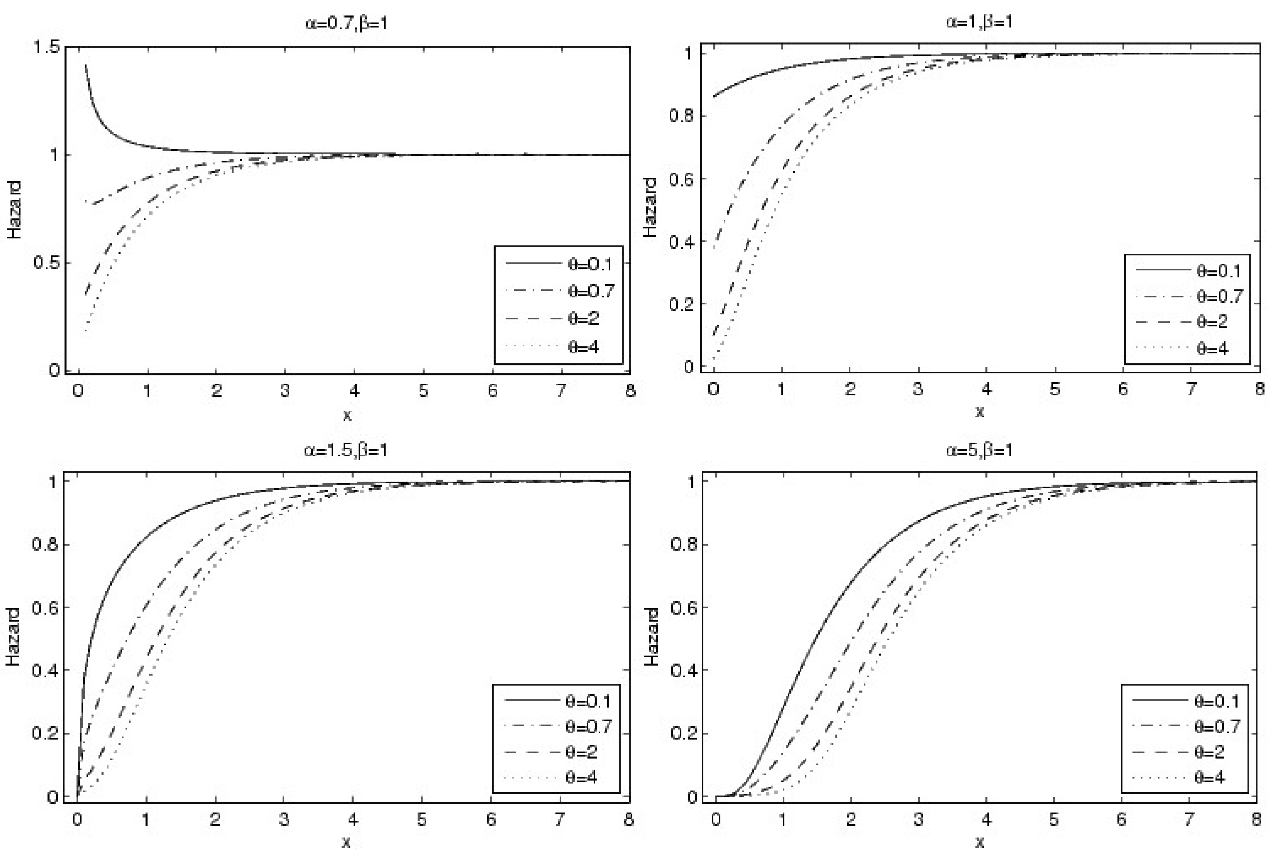

Figure 8. GEB Distribution Hazard Function Diagram

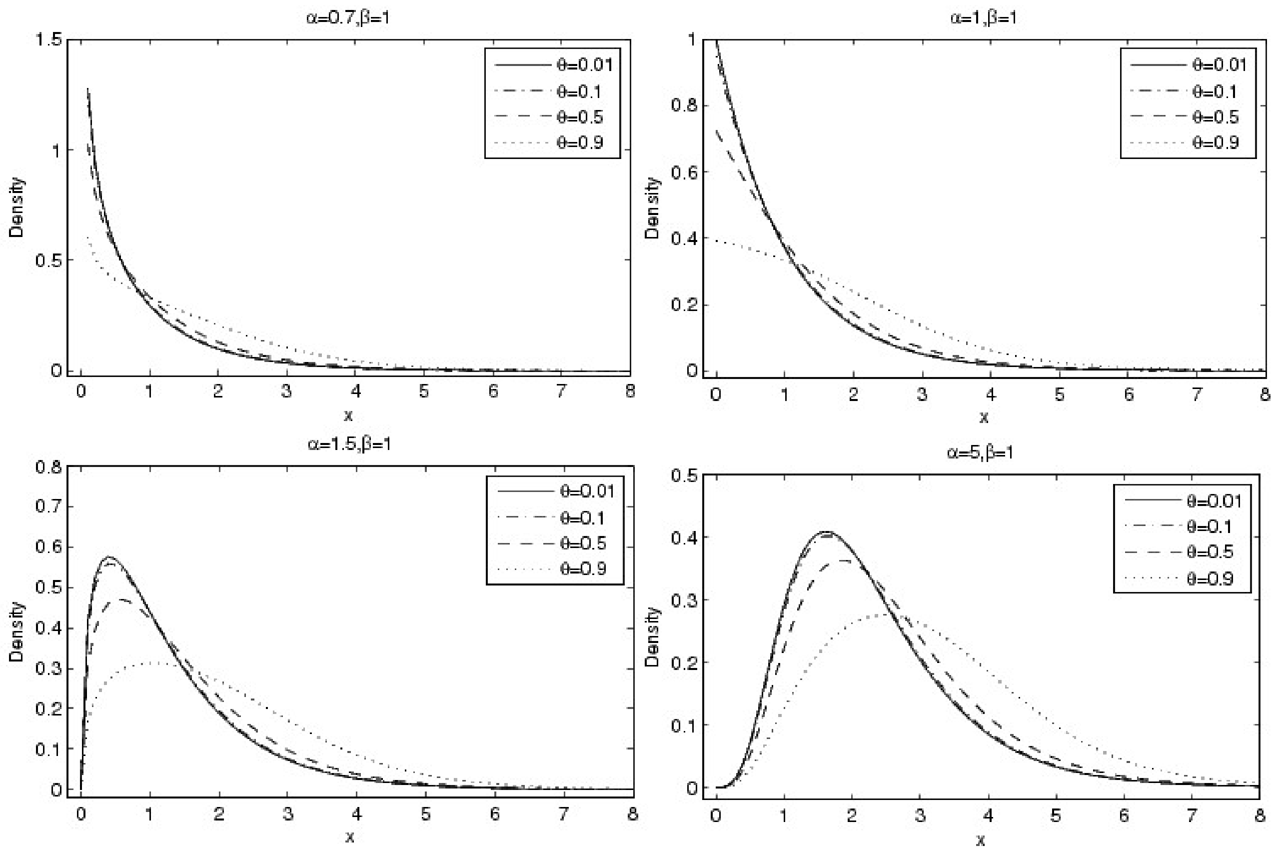

Figure 9. GEL Distribution Density Function Diagram

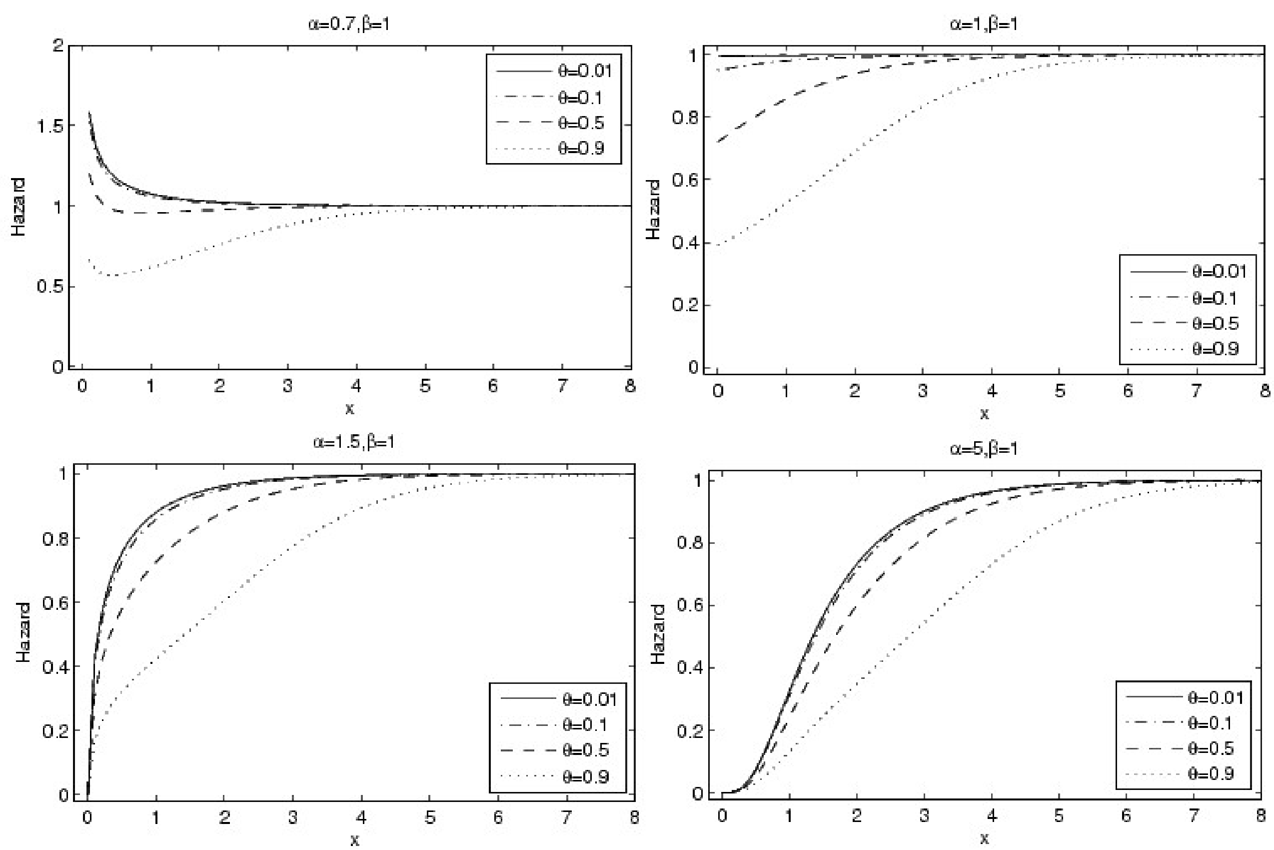

Figure 10. GEL Distribution Hazard Function Diagram

4.4.3 Quantiles

For GEL Distribution we saw that $C(\theta)=-\log(1-\theta)$, then, its reverse is $C^{-1}(\theta)=1-e^{-\theta}$. Now, quantile ζ from this distribution equals with:

$$x_\xi = G^{-1}\left(\frac{1-e^{-\xi C(\theta)}}{\theta}\right) = G^{-1}\left(\frac{1-e^{\xi\log(1-\theta)}}{\theta}\right) \\ = G^{-1}\left(\frac{1-(1-\theta)^\xi}{\theta}\right)$$

where, $G^{-1}(y)=-\frac{1}{\beta}\log(1-y^{\frac{1}{\alpha}})$ is the logarithm symbol with base of Napier number. This expression is applicable for simulating the random data from GEL distribution by creating random numbers in the range of (0,1).

4.4.4 Sorted Statistics

Probability density function of ith sorted statistics includes:

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1} \\ \frac{e^{\theta(1-e^{-\beta x})^\alpha}\left[e^{\theta(1-e^{-\beta x})^\alpha}-1\right]^{i-1}\left[e^{\theta}-e^{\theta(1-e^{-\beta x})^\alpha}\right]^{n-i}}{(e^\theta-1)^n}$$

Also, corresponding distribution function equals to:

$$F_{i:n}(x) \\ = \sum_{k=i}^{n}\binom{n}{k}\frac{\left[e^{\theta(1-e^{-\beta x})^\alpha}-1\right]^{k}\left[e^{\theta}-e^{\theta(1-e^{-\beta x})^\alpha}\right]^{n-k}}{\left[e^{\theta(1-e^{-\beta x})^\alpha}-1\right]^{n}}$$

4.4.5 Momentum Generating Function

The momentum generating function of GEL distribution equals to:

$$M_X(t) = \frac{\Gamma(1-\frac{t}{\beta})}{-\log(1-\theta)}\sum_{n=1}^{\infty}\frac{\theta^n}{n}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\frac{t}{\beta})}$$

4.4.6 Central Momentums

K^{th} momentums of this distribution is:

$$E(X^k) = -\frac{\alpha k!}{\beta^k\log(1-\theta)} \\ \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{n\alpha-1}{j}(-1)^j\frac{\theta^n}{(j+1)^{k+1}}$$

5. Estimating the parameters

In this section, we find the estimation for parameters of GEPS distributions class and present an asymptotic confidence interval for these parameters. by Using EM algorithm, we also solve the likelihood equations.

5.1 Maximum Likelihood Estimators

Assume that $X_1,\ldots,X_n$ is a random sample of GEPS (α, β, θ) Distributions and $T=(\alpha,\beta,\theta)^T$ is the parameters vector. In this case, likelihood function includes:

$$L(\Theta,\mathbf{x}) = \prod_{i=1}^{n}\theta\alpha\beta e^{-\beta x_i}(1-e^{-\beta x_i})^{\alpha-1}\frac{C'(\theta(1-e^{-\beta x_i})^\alpha)}{C(\theta)} \\ = (\theta\alpha\beta)^n e^{-\beta\sum_{i=1}^{n}x_i}\frac{\prod_{i=1}^{n}(1-e^{-\beta x_i})^{\alpha-1}C'(\theta(1-e^{-\beta x_i})^\alpha)}{C^n(\theta)}$$

Thus, likelihood function logarithm will be:

$$\ell = \log L(\Theta,\mathbf{x}) = n\log(\theta\alpha\beta)-\beta\sum_{i=1}^{n}x_i \\ +\sum_{i=1}^{n}\log[(1-e^{-\beta x_i})^{\alpha-1}C'(\theta(1-e^{-\beta x_i})^\alpha)]-n\log(C(\theta)) \\ = n\log(\theta)+n\log(\alpha)+n\log(\beta)-n\beta\bar{x} \\ +(\alpha-1)\sum_{i=1}^{n}\log(1-e^{-\beta x_i})+\sum_{i=1}^{n}\log(C'(\theta(1-e^{-\beta x_i})^\alpha)) \\ -n\log(C(\theta))$$

Where, $\bar{x}=\frac{\sum_{i=1}^{n}x_i}{n}$. Now, by putting $p_i=1-e^{-\beta x_i}$, we have:

$$\ell = n\log(\theta)+n\log(\alpha)+n\log(\beta)-n\beta\bar{x} \\ +(\alpha-1)\sum_{i=1}^{n}\log(p_i)+\sum_{i=1}^{n}\log(C'(\theta {p_i}^\alpha))-n\log(C(\theta))$$

Score function is equal to $U(x,\Theta)=(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta})^T$, that

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\log(p_i)+\sum_{i=1}^{n}\frac{\theta p_i^\alpha\log(p_i)C''(\theta p_i^\alpha)}{C'(\theta p_i^\alpha)} \\ \frac{\partial\ell}{\partial\beta} = \frac{n}{\beta}-n\bar{x}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i(1-p_i)}{p_i} \\ +\sum_{i=1}^{n}\frac{\theta\alpha x_i(1-p_i){p_i}^{\alpha-1}C''(\theta {p_i}^\alpha)}{C'(\theta {p_i}^\alpha)} \\ \frac{\partial\ell}{\partial\theta} = \frac{n}{\theta}+\sum_{i=1}^{n}\frac{{p_i}^\alpha C''(\theta {p_i}^\alpha)}{C'(\theta {p_i}^\alpha)}-\frac{nC'(\theta)}{C(\theta)}$$

Considering two following derivatives will make the above partial derivatives more intelligible:

$$\frac{\partial p_i^\alpha}{\partial\alpha} = p_i^\alpha\log(p_i)$$

$$\frac{\partial {p_i}^\alpha}{\partial\beta} = \frac{\partial(1-e^{-\beta x_i})^\alpha}{\partial\beta} = \alpha x_i e^{-\beta x_i}(1-e^{-\beta x_i})^{\alpha-1} \\ = \alpha x_i(1-p_i){p_i}^{\alpha-1}$$

Likelihood maximum Θ, indicated by $\hat{T}$, would be estimated by solving the non-linear equations system $U(x,\Theta)=0$.

5.2 Interval Estimation

In this section we will find asymptotic confidence interval for parameters of GEPS Distributions class. Attaining this goal requires finding the observed Fisher Matrix. A matrix observed for class of GEPS distributions is one matrix 3×3 obtained by following form:

$$I_n(\Theta) = -\begin{bmatrix} \alpha\alpha & I_{\alpha\beta} & I_{\alpha\theta} \\ I_{\alpha\beta} & I_{\beta\beta} & I_{\beta\theta} \\ I_{\alpha\theta} & I_{\beta\theta} & I_{\theta\theta} \end{bmatrix}$$

Elements of this matrix include:

$$\begin{aligned} I_{\alpha\alpha} =& \frac{\partial^2 \ell}{\partial \alpha^2} \\ =& -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{\theta^2 p_i^{2\alpha} \log^2(p_i) C'''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{\theta p_i^{\alpha} \log^2(p_i) C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta^2 p_i^{2\alpha} \log^2(p_i) [C''(\theta p_i^{\alpha})]^2}{[C'(\theta p_i^{\alpha})]^2} \end{aligned}$$

$$\begin{aligned} I_{\alpha\beta} =& \frac{\partial^2 \ell}{\partial \beta \partial \alpha} \\ =& \sum_{i=1}^{n} \frac{x_i(1-p_i)}{p_i} \\ &+ \sum_{i=1}^{n} \frac{\theta^2 \alpha x_i (1-p_i) p_i^{2\alpha-1} \log(p_i) C'''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{\theta \alpha x_i (1-p_i) p_i^{\alpha-1} \log(p_i) C'(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{\theta x_i (1-p_i) p_i^{\alpha-1} C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta^2 \alpha x_i (1-p_i) p_i^{2\alpha-1} \log(p_i) [C''(\theta p_i^{\alpha})]^2}{[C'(\theta p_i^{\alpha})]^2} \end{aligned}$$

$$\begin{aligned} I_{\alpha\theta} =& \frac{\partial^2 \ell}{\partial \theta \partial \alpha} \\ =& \sum_{i=1}^{n} \frac{\theta p_i^{2\alpha} \log(p_i) C'''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{p_i^{\alpha} \log(p_i) C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta p_i^{2\alpha} \log(p_i) [C''(\theta p_i^{\alpha})]^2}{[C'(\theta p_i^{\alpha})]^2} \end{aligned}$$

$$\begin{aligned} I_{\beta\beta} =& \frac{\partial^2 \ell}{\partial \beta^2} \\ =& -\frac{n}{\beta^2} - (\alpha-1) \sum_{i=1}^{n} \frac{x_i^2(1-p_i)}{p_i} \\ &- (\alpha-1) \sum_{i=1}^{n} \frac{x_i^2 (1-p_i)^2}{p_i^2} \\ &+ \sum_{i=1}^{n} \frac{\theta^2 \alpha^2 x_i^2 (1-p_i)^2 p_i^{2\alpha-2} C'''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{\theta \alpha^2 x_i^2 (1-p_i)^2 p_i^{\alpha-2} C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta \alpha x_i^2 (1-p_i) p_i^{\alpha-1} C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta \alpha x_i^2 (1-p_i)^2 p_i^{\alpha-2} C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta^2 \alpha^2 x_i^2 (1-p_i)^2 p_i^{2\alpha-2} [C''(\theta p_i^{\alpha})]^2}{[C'(\theta p_i^{\alpha})]^2} \end{aligned}$$

$$\begin{aligned} I_{\beta\theta} =& \frac{\partial^2 \ell}{\partial \theta \partial \beta} \\ =& \sum_{i=1}^{n} \frac{\theta \alpha x_i (1-p_i) p_i^{2\alpha-1} C'''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &+ \sum_{i=1}^{n} \frac{\alpha x_i (1-p_i) p_i^{\alpha-1} C''(\theta p_i^{\alpha})}{C'(\theta p_i^{\alpha})} \\ &- \sum_{i=1}^{n} \frac{\theta \alpha x_i (1-p_i) p_i^{\alpha-1} [C''(\theta p_i^{\alpha})]^2}{[C'(\theta p_i^{\alpha})]^2} \end{aligned}$$

By selecting a sample enough great from GEPS Distributions class, MLE for Θ is asymptote based having normal distribution with Θ mean and variance- covariance matrix is equal to expected Fisher matrix $J_n(\Theta)^{-1}$; i.e. $\hat{\Theta} \sim N_3(\Theta, J_n(\Theta)^{-1})$. By taking the expectation from observed matrix, the elements of expected Fisher matrix are obtained. More precisely, $J_n(\Theta) = E[I_n(\Theta)]$. Under complete conditions for parameters in the parameter space (but not on the border of parameter space), $\sqrt{n}(\hat{\Theta} - \Theta)$has an asymptote distribution of $N_3(0, J(\Theta)^{-1})$that $J(\Theta) = \lim_{n\to\infty} \frac{1}{n} I_n(\Theta)$ is the unit matrix. If Θ is replaced with a sample mean of matrix in the point $J(\Theta) = \lim_{n\to\infty} \frac{1}{n} I_n(\Theta)$, this asymptote behavior remains valid. On the other hand, in cases that θ could not be calculated and if sample size is large enough, the reverse of Fisher Matrix observed in point $\hat{\Theta}$, an asymptote variance-covariance matrix will calculate the Maximum Likelihood Estimators; i.e.:

$$I_n(\hat{\Theta})^{-1} = \begin{bmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\beta}) & Cov(\hat{\alpha}, \hat{\theta}) \\ Cov(\hat{\alpha}, \hat{\beta}) & Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\theta}) \\ Cov(\hat{\alpha}, \hat{\theta}) & Cov(\hat{\beta}, \hat{\theta}) & Var(\hat{\theta}) \end{bmatrix}$$

Is the asymptote multivariable normal distribution for $\hat{\Theta}$ that is as $\hat{\Theta} \sim N_3(\Theta, I_n(\hat{\Theta})^{-1})$, it is used for calculating the asymptote confidence interval for parameters and inference for them [18]. An asymptote confidence interval $100(1-\gamma)\%$ for any parameter of GEPS Distribution detailed as below:

$$\alpha \in \left(\hat{\alpha} - Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\alpha})}, \hat{\alpha} + Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\alpha})}\right)$$

$$\beta \in \left(\hat{\beta} - Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta})}, \hat{\beta} + Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta})}\right)$$

$$\theta \in \left(\hat{\theta} - Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\theta})}, \hat{\theta} + Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\theta})}\right)$$

Where, $Var(\hat{\alpha})$, $Var(\hat{\beta})$ and $Var(\hat{\theta})$are the components on the main diagonal of matrix $I_n(\hat{\Theta})^{-1}$ and $Z_{\frac{\gamma}{2}}$ is quantile of standard normal distribution.

5.3 Estimation by EM Algorithm

EM algorithm is a very powerful tool for finding the maximum likelihood estimators in the incomplete data problems. This a repeating method algorithm that is converging with a relatively slow speed absolutely with ML estimation. In this sub-section, by using EM algorithm, we find maximum likelihood estimators of parameters of GEPS Distributions class. EM algorithm has two steps. Following states these two steps for finding the ML estimators for GEPS Distributions parameters.

Step E. Assume that $\Theta^{(r)} = (\alpha^{(r)}, \beta^{(r)}, \theta^{(r)})$ is the current estimation in ?\$r\$th repeat for Θ estimation; then, for performing E step from EM circle, we require mathematical expectation of $(Z|X;\Theta^{(r)})$. Initially, we find joint density function (X,Z) that $z \in N$is the member of power series densities family and $X = \max_{1\le i\le N} X_i$. Thereafter, we calculate the conditional distribution $g_{Z|X}(z|x;\Theta)$ and by using it, we obtain required mathematical expectation. Distribution function $X|Z=z$ equals to:

$$G_{X|Z}(x|z;\alpha,\beta) = (1-e^{-\beta x})^{z\alpha}$$

By differentiating from above equation, the density function corresponding to this distribution function will be obtained:

$$g_{X|Z}(x|z;\alpha,\beta) = z\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{z\alpha-1}$$

Therefore, joint density function (X,Z) includes:

$$\begin{aligned} g_{X,Z}(x,z;\Theta) =& P(Z=z) g_{X|Z}(x|z;\alpha,\beta) \\ =& \frac{a_z\theta^z}{C(\theta)} z\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{z\alpha-1} \end{aligned}$$

Where, $\alpha,\beta,\theta > 0$and x>0 and $\Theta = (\alpha,\beta,\theta)$are parameters vector. thus, according to equations by applying the Bayes' theorem, we have:

$$\begin{aligned} g_{Z|X}(z|x;\Theta) =& \frac{g_{X,Z}(x,z;\Theta)}{f(x)} \\ =& \frac{\frac{a_z\theta^z}{C(\theta)} z\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{z\alpha-1}}{\theta\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\frac{C'(\theta(1-e^{-\beta x})^{\alpha})}{C(\theta)}} \\ =& \frac{a_z\theta^{z-1} z(1-e^{-\beta x})^{\alpha(z-1)}}{C'(\theta(1-e^{-\beta x})^{\alpha})} \end{aligned}$$

Now, because

$$\begin{aligned} C'(\theta) + \theta C''(\theta) &= \sum_{z=1}^{\infty} z a_z \theta^{z-1} + \theta \sum_{z=1}^{\infty} z(z-1) a_z \theta^{z-2} \\ &= \sum_{z=1}^{\infty} z^2 a_z \theta^{z-1} \end{aligned}$$

Related mathematical expectation equals to:

$$\begin{aligned} E(Z|X=x) &= \sum_{z=1}^{\infty} z g_{Z|X}(z|x;\Theta) \\ &= \sum_{z=1}^{\infty} z \frac{a_z\theta^{z-1} z(1-e^{-\beta x})^{\alpha(z-1)}}{C'(\theta(1-e^{-\beta x})^{\alpha})} \\ &= \frac{1}{C'(\theta(1-e^{-\beta x})^{\alpha})} \sum_{z=1}^{\infty} a_z z^2 [\theta(1-e^{-\beta x})^{\alpha}]^{z-1} \\ &= \frac{C'(\theta(1-e^{-\beta x})^{\alpha}) + \theta(1-e^{-\beta x})^{\alpha} C''(\theta(1-e^{-\beta x})^{\alpha})}{C'(\theta(1-e^{-\beta x})^{\alpha})} \\ &= 1 + \frac{\theta(1-e^{-\beta x})^{\alpha} C''(\theta(1-e^{-\beta x})^{\alpha})}{C'(\theta(1-e^{-\beta x})^{\alpha})} \end{aligned}$$

Consequently mathematical expectation $(Z|X;\Theta^{(r)})$ equals to:

$$\begin{aligned} &E(Z|X;\Theta^{(r)}) \\ &= 1 + \frac{\theta^{(r)}(1-e^{-\beta^{(r)}x})^{\alpha^{(r)}} C''(\theta^{(r)}(1-e^{-\beta^{(r)}x})^{\alpha^{(r)}})}{C'(\theta^{(r)}(1-e^{-\beta^{(r)}x})^{\alpha^{(r)}})} \end{aligned} \quad (22)$$

Thus, step E will be completed.

Step M. EM loop completes with M step that is the same as maximum likelihood of complete data on θ obtained by replacing Zs with their conditional mathematical expectation (presented in Eq. (22)). For maximizing the likelihood function of complete data, we have:

$$\begin{aligned} L^*(\mathbf{x},\mathbf{z};\Theta) &= \prod_{i=1}^{n} g_{X,Z}(x_i,z_i;\Theta) \\ &= \prod_{i=1}^{n} \frac{a_{z_i}\theta^{z_i}}{C(\theta)} z_i\alpha\beta e^{-\beta x_i}(1-e^{-\beta x_i})^{z_i\alpha-1} \\ &= \frac{(\prod_{i=1}^{n} a_{z_i} z_i)\theta^{\sum_{i=1}^{n} z_i}}{C^n(\theta)} (\alpha\beta)^n e^{-\beta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1-e^{-\beta x_i})^{z_i\alpha-1} \end{aligned}$$

By removing the fixed or constant expressions that parameters having no role in them, the logarithm of likelihood function of complete data is proportional to:

$$\begin{aligned} \ell^*(\mathbf{x},\mathbf{z};\Theta) \propto& \sum_{i=1}^{n} z_i \log(\theta) + n\log(\alpha) + n\log(\beta) - n\beta\bar{x} \\ &+ \sum_{i=1}^{n} (z_i\alpha - 1)\log(1-e^{-\beta x_i}) - n\log(C(\theta)) \end{aligned}$$

By differentiating from above equation than parameters α and β and θ, we obtain the score function components $U_c(y;\Theta) = (\frac{\partial \ell^*}{\partial \alpha}, \frac{\partial \ell^*}{\partial \beta}, \frac{\partial \ell^*}{\partial \theta})^T$ including:

$$\frac{\partial \ell^*}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} z_i \log(1 - e^{-\beta x_i})$$

$$\frac{\partial \ell^*}{\partial \beta} = \frac{n}{\beta} - n\bar{x} + \sum_{i=1}^{n} \frac{(z_i\alpha - 1)x_i e^{-\beta x_i}}{1 - e^{-\beta x_i}} = \frac{n}{\beta} - n\bar{x} + \sum_{i=1}^{n} \frac{x_i(z_i\alpha - 1)}{e^{\beta x_i} - 1}$$

$$\frac{\partial \ell^*}{\partial \theta} = \frac{\sum_{i=1}^{n} z_i}{\theta} - \frac{C'(\theta)}{C(\theta)}$$

From non-linear equations system $U_c(y;\Theta) = 0$, repeated loop of log EM includes:

$$\hat{\alpha}^{(t+1)} = \frac{-n}{\sum_{i=1}^{n} \hat{z}_i^{(t)} \log(1 - e^{-\hat{\beta}^{(t)} x_i})}$$

$$\frac{n}{\hat{\beta}^{(t+1)}} - n\bar{x} + \sum_{i=1}^{n} \frac{x_i(\hat{z}_i^{(t)}\hat{\alpha}^{(t)} - 1)}{e^{\hat{\beta}^{(t+1)} x_i} - 1} = 0$$

$$\hat{\theta}^{(t+1)} = \frac{C(\hat{\theta}^{(t+1)})}{nC'(\hat{\theta}^{(t+1)})} \sum_{i=1}^{n} \hat{z}_i^{(t)}$$

That $\hat{\alpha}^{(t+1)}$, $\hat{\beta}^{(t+1)}$ and $\hat{\theta}^{(t+1)}$ are numerical and $\hat{z}_i^{(t)}$ is conditional expectation of z_is in t^{th} repeat; i.e.

$$\hat{z}_i^{(t)} = 1 + \frac{\hat{\theta}^{(t)}(1 - e^{-\hat{\beta}^{(t)} x_i})^{\hat{\alpha}^{(t)}} C''(\hat{\theta}^{(t)}(1 - e^{-\hat{\beta}^{(t)} x_i})^{\hat{\alpha}^{(t)}})}{C'(\hat{\theta}^{(t)}(1 - e^{-\hat{\beta}^{(t)} x_i})^{\hat{\alpha}^{(t)}})}, \quad \forall i = 1, ..., n$$

We repeat the repeat cycle of algorithm EM to the extent that the distance between two frequent estimators is negligible (for example the desired value).

6. Intuitive Examples

In this section, we review the analysis or fitness of GEG, GEP and GEL models on two sets of actual data and fit two Weibull and generalized exponential two-parameter distribution for comparing with GEPS models on these data.

We estimate the parameters of any model by using EM algorithm; then, we draw the diagram of cumulative distribution function, density function and P-P Plot of each model using estimated parameters and will explain them. Finally, we will run Kolmogorov-Smirnov (K-S) test for parameters of any model and analyze the fitness of any model.

GEG, GEP, GEL, Weibull and GE Distributions Density Functions respectively include:

$$f_{GEG}(x) = \alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1} \frac{1-\theta}{[1 - \theta(1 - e^{-\beta x})^{\alpha}]^2}, \quad \alpha, \beta > 0, 0 < \theta < 1$$

$$f_{GEP}(x) = \theta\alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1} \frac{e^{\theta(1-e^{-\beta x})^{\alpha}}}{e^{\theta} - 1}, \alpha, \beta, \theta > 0$$

$$f_{GEL}(x) = -\frac{\theta\alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}}{[1 - \theta(1 - e^{-\beta x})^{\alpha}]\log(1-\theta)}, \alpha, \beta > 0, 0 < \theta < 1$$

$$f_{Wei}(x) = \alpha\beta^{\alpha} x^{\alpha-1} e^{-(\beta x)^{\alpha}}, \alpha, \beta > 0$$

$$f_{GE}(x) = \alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}, \alpha, \beta > 0$$

More, we analyze two sets of actual data in term of two applicable examples.

Example 3: First data set includes number of frequent failures of air conditioning system in any of 13 airline Boeing 720 fleet. This information including 213 observations were analyzed for the first time by Proschan [19] followed by investigations by Dahiya & Gurland [20] and Adamidis & Loukas [2]. These data indicated in Table 1 and Table 2 indicates the estimations for parameters of any model and K-S test statistics together with probability value of this test. The smaller the K-S statistics, the greater the probability of K-S test and null hypothesis of this test, indicting the fitness of related model on data in a greater significance level will be confirmed. By comparing the figures of K-S column in Table 2, we find out that GEL distribution has maximum fitness and GEP distribution has minimum fitness than other distributions. These subjects could be also perceived by observing the diagram of density functions and fitness distribution functions. Figure 11 reflects the density function diagram of fitted models against rectangular diagram of data. From rectangular diagram of data, one could perceive that these data skewed rightwards. Therefore, the tail of fitted diagrams has been also drawn rightwards. Figure 12 indicates the P-P plot of any model that ignorable difference of points and fitness line given in any model confirms the suitable fitness of these models on third data set.

Example 4: Second data set has been developed by Fonseca & Franca indicating the influence of soil fertility and characteristics of biological fixation of N_2. They have measured the concentration of phosphorous in the leaves of 128 plants. Recently, Silva et al. [21] have analyzed these data by Gompertz Poisson GP) and Chen Poisson (CP) distributions and three parameters Weibull-geometry (WG) model. Table 3 indicates set of related data and Table 4 reflects the estimation for parameters of any model, K-S statistics and probability value of K-S test. Because the probability values of K-S test in all models exceeds from 0.10, it is no reason for rejecting the null hypothesis of this test in 10% significance level. More precisely, in 10% significance level, one could accept that distribution of these data follows from GEPS models and two bi-parametric Weibull and GE models.

Figure 13 indicates the diagram of density functions and distribution functions of five studied models. By observing both diagrams, we find out that the curve of Weibull model

Table 1. Data for number of frequent failures of air conditioning system in 13 aviation Boing 720 fleet.

184	447	169	9	65	100	37	58	14	413	181	33	29	41	15	194
41	61	186	60	10	90	34	22	7	62	57	67	18	34	18	31
130	62	23	44	26	76	310	156	25	118	29	59	44	84	79	20
59	386	29	21	70	12	502	29	48	57	74	208	102	101	56	118
35	104	15	33	182	176	246	47	239	220	104	56	320	55	326	26
11	120	12	5	20	42	21	246	71	225	47	26	14	120	7	87
18	141	4	11	51	97	95	52	16	1	90	16	11	14	11	71
191	18	63	39	111	46	216	31	54	82	206	106	16	1	80	77
5	5	46	88	79	188	197	15	3	39	22	72	102	44	50	54
438	2	104	3	603	270	12	9	359	14	13	23	30	97	210	136
230	43	91	85	5	98	7	100	18	487	493	130	12	35	283	5
163	18	68	142	14	3	261	23	153	27	70	208	24	14	201	36
27	152	134	59	67	32	54	57	14	209	130	3	254	50	22	36
											34	61	66	230	14

Table 2. Estimating the maximum likelihood of parameters, statistics and probability value of K-S test from fitness of five distributions on first data set.

Distribution	MLE			K-S	p-value
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$		
GEG	0.9234	0.0102	0.0012	0.0635	0.3424
GEP	1.0342	0.0115	0.0050	0.0635	0.3418
GEL	0.9800	0.0113	0.0080	0.0510	0.6173
GE	0.9034	0.0112	-	0.0586	0.4401
Weibull	0.9220	0.0111	-	0.0537	0.5520

Table 3. Data for phosphor concentration in the leaves of plants

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23
0.24	0.20	0.08	0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09
0.10	0.12	0.12	0.10	0.09	0.17	0.19	0.21	0.18	0.26	0.19	0.17	0.18
0.20	0.24	0.19	0.21	0.22	0.17	0.08	0.08	0.06	0.09	0.22	0.23	0.22
0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12	0.15	0.08	0.12	0.09	0.14
0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13	0.11	0.11	0.11
0.08	0.22	0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15	0.17
0.14	0.12	0.18	0.14	0.18	0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11
0.11	0.14	0.07	0.07	0.19	0.17	0.18	0.16	0.19	0.15	0.07	0.09	0.17
		0.10	0.08	0.15	0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13

Table 4. Estimating the maximum likelihood of parameters, statistics and probability value of K-S test from fitness of five distributions on second data set.

Distribution	MLE			K-S	p-value
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$		
GEG	10.4304	23.9716	0.4400	0.0817	0.3414
GEP	11.0104	22.1600	0.4005	0.0837	0.3137
GEL	13.7219	24.5029	0.4850	0.0900	0.2359
GE	10.6059	21.1328	-	0.0797	0.3710
Weibull	2.8185	6.3098	-	0.1078	0.1526

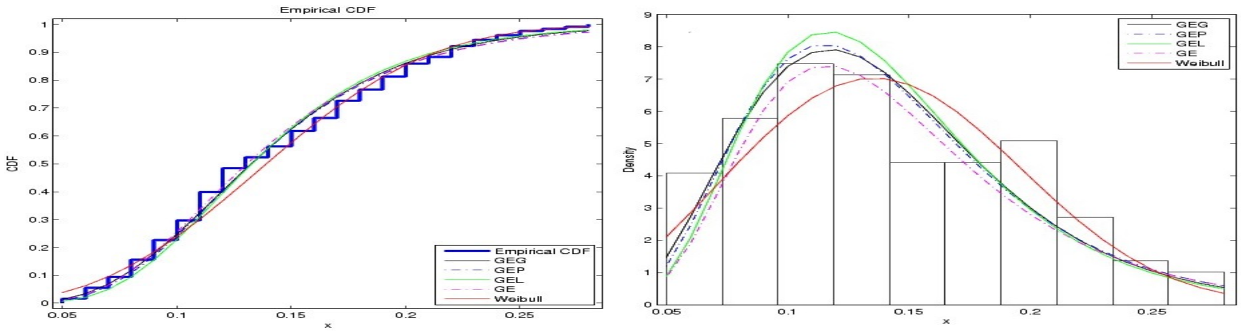

Figure 11. Diagram of density functions and distribution functions of fitted models on second data set

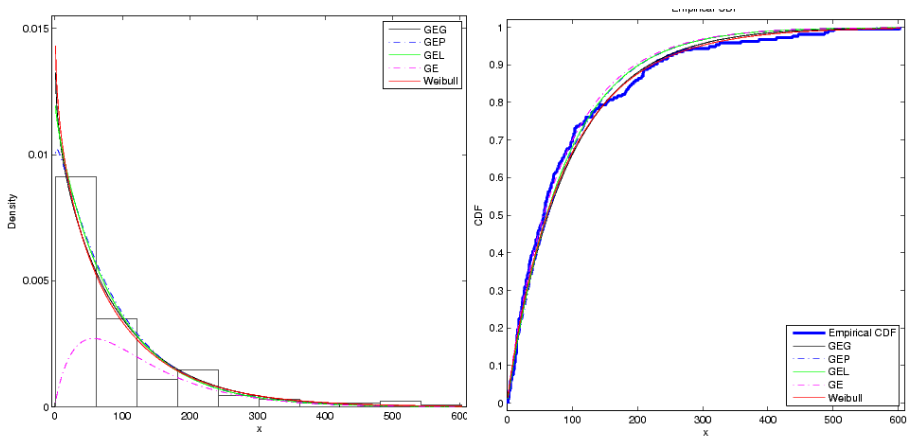

Figure 12. P-P plot of first data set for fitted models

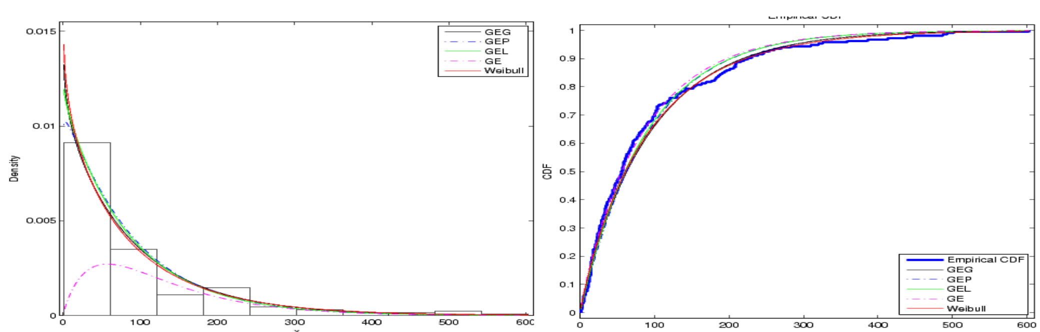

Figure 13. Diagram of density functions and distribution functions of fitted models on first data set

has a considerable difference with diagram of four other models. This indicates that Weibull distribution has lower fitness among these five models. This could be also inferred from comparing the difference of points drawn and fitted line in P-P Plots drawn in Figure 14.

7. Conclusion

According to calculations for Kolmogrov- Smirnov test, we saw that probability for five models, i.e. GEG, GEP, GEL, GE and Weibull in the applicable examples as provided above exceeds from figure 0.1; it means that five studied models is fitted on this set of actual data. By comparing the figures of P-value column and K-S column in tables, we could find out the excellence of tri-parameter GEPS models than bi-parameter GE and Weibull models. For more sensible comparison of studied models, the diagrams for density functions of models have been drawn against histogram of data and distribution functions of models drawn against experimental distribution of data and P-P Plot, and results are completely conformed to the results from tables of K-S test.

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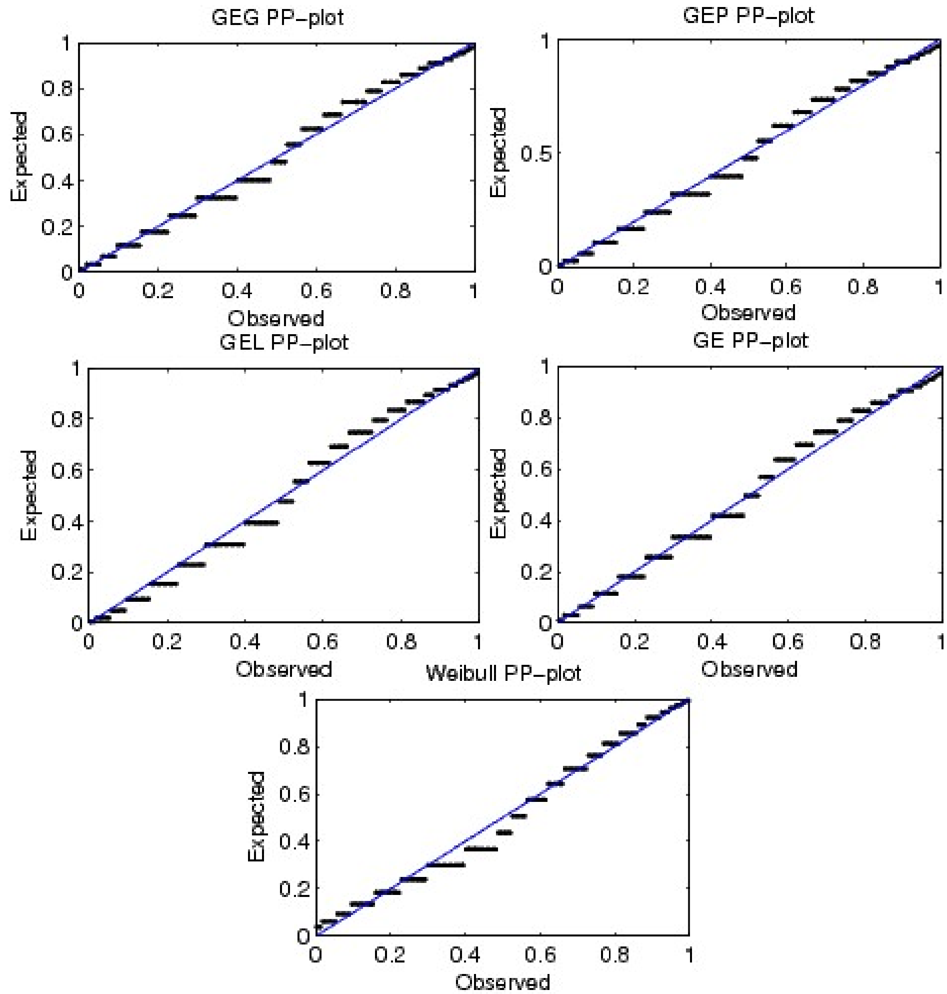

Figure 14. P-P Plot of second data set for fitted models

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