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The Isomorphism Realized By Mixed Fractional Integrals In Hölder Classes

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***Abstract*:** As is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between Hölder spaces for functions one variables. We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained results extend the well-known theorem of Hardy-Littlewood for one-dimensional fractional integrals to the case of mixed Hölderness.

***Keywords*:** functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, mixed fractional integral, Hölder space.

1. Introduction

The classical result of G.Hardi and D.Littlewood (1928, see [1, §3]) is known that the fractional integral $\left(I^{\alpha}_{a+}f\right)(x) = \Gamma^{-1}(\alpha)\left(t_+^{\alpha-1} * f\right)(x),\ \ 0<\alpha<1$ maps isomorphically the space $H_0^{\lambda}([0,1])$ of Hölder order $\lambda \in (0,1)$ functions with a condition $f(0)=0$ on a similar space of a higher order $\lambda+\alpha$ provided that $\lambda+\alpha<1$. Further, this result was generalized in various directions: a space with a power weight, generalized Hölder spaces, spaces of the Nikolsky type, etc. A detailed review of these and some other similar results can be found in [1].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann - Liouville integral was studied in [2–6].

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y \frac{\varphi(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}, \tag{1}$$

$x,y>0$, Mixed fractional derivatives form Marchaud ([7–9])

$$\left(\mathrm{D}^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y) = \frac{\varphi(x,y)}{x^{\alpha}y^{\beta}\Gamma(1-\alpha)\Gamma(1-\beta)} + \\ + \frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int\limits_0^x\int\limits_0^y \frac{\varphi(x,y)-\varphi(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds, \tag{2}$$

where $x,y>0$, were not studied either in the usual Hölder space, or in the Hölder spaces defined by mixed differences. Meanwhile, there arise "points of interest" related to the investigation of the above mixed differences of fractional derivatives form Marchaud. For operators Eq. (1) and Eq. (2) in Hölder spaces of mixed order there arise some questions to be answered in relation to the usage of these or those differences in the definition of Hölder spaces. Such mapping properties in Hölder spaces of mixed order were not studied. This paper is aimed to fill in this gap. We deal with non-weighted spaces. Consider the operator Eq. (1) and Eq. (2) in a rectangle

$$Q = \{(x,y):\ \ 0<x<b,\ \ 0<y<d\}.$$

For a continuous function $\varphi(x,y)$ on $\mathbf{R}^2$ we introduce the notation

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y) = \varphi(x+h,y)-\varphi(x,y),$$
$$\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y) = \varphi(x,y+\eta)-\varphi(x,y),$$
$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y) = \varphi(x+h,y+\eta)-\varphi(x+h,y) \\ -\varphi(x,y+\eta)+\varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y) + \left(\overset{1,0}{\Delta}_h\varphi\right)(x,y) \\ + \left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+\varphi(x,y). \tag{3}$$

Everywhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurrences and even in the same line.

Definition 1: Let $\lambda,\gamma \in (0,1]$. We say that $\varphi \in H^{\lambda,\gamma}(Q)$, if

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)| \le C_1|x_1-x_2|^{\lambda}+|y_1-y_2|^{\gamma} \tag{4}$$

for all $(x_1,y_1),(x_2,y_2)\in Q$. Condition Eq. (4) is equivalent to the couple of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right| \le C_1|h|^{\lambda},$$
$$\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right| \le C_2|\eta|^{\lambda}$$

uniform with respect to another variable.

By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f \in H_0^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

Let $\lambda = 0$ and /or $\gamma = 0$. We put $H^{0,0}(Q) = L^\infty(Q)$ and

$$H^{\lambda,0}(Q) = \left\{ \varphi \in L^\infty(Q) : \left| \left(\overset{1,0}{\Delta}_h \varphi \right)(x,y) \right| \le C_1 |h|^\lambda \right\}, \quad \lambda \in (0,1],$$

$$H^{0,\gamma}(Q) = \left\{ \varphi \in L^\infty(Q) : \left| \left(\overset{0,1}{\Delta}_\eta \varphi \right)(x,y) \right| \le C_2 |\eta|^\gamma \right\}, \quad \gamma \in (0,1].$$

Definition 2: We say that $\varphi(x,y) \in \widetilde{H}^{\lambda,\gamma}(Q)$, where $\lambda, \gamma \in (0,1]$, if

$$\varphi(x,y) \in H^{\lambda,\gamma} \text{and} \left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x,y) \right| \le C_3 |h|^\lambda |\eta|^\gamma. \quad (5)$$

We say that $\varphi(x,y) \in \widetilde{H}_0^{\lambda,\gamma}(Q)$ and $\varphi(x,y)\big|_{x=0,y=0} = 0$.

These spaces become Banach spaces under the standard definition of the

$$\|\varphi\|_{H^{\lambda,\gamma}} := \|\varphi\|_{C(Q)} + \sup_{x,x+h\in[0,b]} \sup_{y\in[0,d]} \frac{\left| \left(\overset{1,0}{\Delta}_h \varphi \right)(x,y) \right|}{|h|^\lambda} + \sup_{x\in[0,b]} \sup_{y,y+\eta\in[0,d]} \frac{\left| \left(\overset{0,1}{\Delta}_\eta \varphi \right)(x,y) \right|}{|\eta|^\gamma},$$

$$\|\varphi\|_{\tilde{H}^{\lambda,\gamma}} := \|\varphi\|_{H^{\lambda,\gamma}} + \sup_{x,x+h\in[0,b]} \sup_{y,y+\eta\in[0,d]} \frac{\left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x,y) \right|}{|h|^\lambda |\eta|^\gamma}.$$

Note that

$$\varphi \in H^{\lambda,\gamma} \Rightarrow \left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x,y) \right| \le C_\theta |h|^{\theta\lambda} |\eta|^{(1-\theta)\gamma} \quad (6)$$

for any $\theta \in [0,1]$, where $C_\theta = 2C_1^\theta C_2^{1-\theta}$, so that

$$\tilde{H}^{\lambda,\gamma}(Q) \hookrightarrow H^{\lambda,\gamma}(Q) \hookrightarrow \bigcap_{0\le\theta\le1} \tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q), \quad (7)$$

where $\hookrightarrow$ stands for the continuous embedding, and the norm for $\bigcap_{0\le\theta\le1} \tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$. Since $\theta \in [0,1]$ is arbitrary, it isn't hard to see that the inequality in Eq. (6) is equivalent (up to the constant factor C) to

$$\left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x,y) \right| \le C \min\{|h|^\lambda|, \, \eta|^\gamma\}. \quad (8)$$

2. A one-dimensional statements

The following statements are known [1]. We use the schemes of the proofs to make the presentation easier for two-dimensional case.

Lemma 1: If $f(x) \in H^{\lambda+\alpha}([0,b])$ and $0 < \lambda$, $0 < \lambda + \alpha < 1$, then

$$g(x) = \frac{f(x) - f(0)}{|x|^\alpha} \in H^\lambda([0,b]),$$

and

$$\|g\|_{H^\lambda} \le C \|f\|_{H^{\lambda+\alpha}},$$

where C doesn't depend from $f(x)$.

Proof: Let $h > 0$; $x, x+h \in [0,b]$. We consider the difference

$$|g(x+h) - g(x)| \le \frac{|f(x+h) - f(x)|}{(x+h)^\alpha} + |f(x) - f(0)| \frac{(x+h)^\alpha - x^\alpha}{x^\alpha (x+h)^\alpha}.$$

Since $f \in H^{\lambda,\gamma}$, we have

$$|f(x+h) - f(x)| \le C_1 h^{\lambda+\alpha}, \quad |f(x) - f(0)| \le C_2 x^{\lambda+\alpha}.$$

Using these inequalities we obtain

$$|g(x+h) - g(x)| \le C_1 \frac{h^{\lambda+\alpha}}{(x+h)^\alpha} + C_2 x^\lambda \frac{(x+h)^\alpha - x^\alpha}{(x+h)^\alpha} = G_1 + G_2.$$

For G_1 we have

$$G_1 = C_1 h^\lambda \left(\frac{h}{x+h} \right)^\alpha \le C h^\lambda.$$

Let's estimate $G2$, here we shall consider two cases: $x \le h$ and $x > h$. In the first case, we use inequality $\left|\sigma_1^\mu - \sigma_2^\mu\right| \le |\sigma_1 - \sigma_2|^\mu$, $(\sigma \ne \sigma_2)$ and obtain

$$G_2 \le C_2 \frac{x^\lambda h^\alpha}{(x+h)^\alpha} \le C h^\lambda.$$

In second case, using $(1+t)^\alpha - 1 \le \alpha t$, $t > 0$ we have

$$G_2 = C_2 \frac{x^{\lambda+\alpha}}{(x+h)^\alpha} \left| \left(1 + \frac{h}{x}\right)^\alpha - 1 \right| \le C h x^{\lambda-1} \le C h^\lambda,$$

which completes the proof. □

The Marchaud fractional differentiation operator has a form:

$$\left(\mathrm{D}_{0+}^\alpha f\right)(x) = \frac{f(x)}{x^\alpha \Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt, \quad (9)$$

where $0 < \alpha < 1$.

Theorem 1: If $f(x) \in H^{\lambda+\alpha}([0,b])$, $0 < \alpha + \lambda < 1$ that

$$\left(\mathrm{D}_{0+}^\alpha f\right)(x) = \frac{f(0)}{x^\alpha \Gamma(1-\alpha)} + \psi(x), \quad (10)$$

where $\psi(x) \in H^\lambda([0,b])$ and $\psi(0) = 0$, thus $\|\psi\|_{H^\lambda} \le C\|f\|_{H^{\lambda+\alpha}}$.

Proof: We present Eq. (9) as

$$\left(\mathrm{D}_{0+}^\alpha f\right)(x) = \frac{f(0)}{x^\alpha \Gamma(1-\alpha)} + \frac{f(x) - f(0)}{x^\alpha \Gamma(1-\alpha)}$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt,$$

receive equality Eq. (10), where

$$\psi(x)=\psi_1(x)+\psi_2(x) =\frac{f(x)-f(0)}{x^\alpha\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt.$$

Here $\psi_1(x)\in H^\lambda([0,b])$ by Lemma 1. It is enough to show $\psi_2(x)\in H^\lambda([0,b])$.

Let $h>0;\ \ x,x+h\in[0,b]$. Let's consider the difference

$$\psi_2(x+h)-\psi_2(x)=\int_0^x \frac{f(x+h)-f(x)}{(x+h-t)^{1+\alpha}}dt + \int_x^{x+h} \frac{f(x+h)-f(t)}{(x+h-t)^{1+\alpha}}dt + \int_0^x (f(x)-f(t))\left[\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{1+\alpha}}\right]dt = I_1+I_2+I_3.$$

Since $f\in H^{\lambda+\alpha}$, then we have for I_1

$$|I_1|\le Ch^{\lambda+\alpha}\int_0^x (t+h)^{-1-\alpha}dt\le C_1h^\lambda.$$

Let's estimate I_2. We have

$$|I_2|\le C\int_x^{x+h}(x+h-t)^{\lambda-1}dt=C_2h^\lambda.$$

For I_3,

$$|I_3|\le C\int_0^{\frac{x}{h}} t^\lambda\left|\frac{1}{(1+t)^{1+\alpha}}-\frac{1}{t^{1+\alpha}}\right|dt\le C_3h^\lambda.$$

Finally, it remains to note that $\psi_2(0)=0$, since

$$|\psi_2(x)|\le C\int_0^x t^{\lambda-1}dt. \qquad \square$$

3. Main result

Theorem 2: Let $\varphi(x,y)\in H^{\lambda,\gamma}(Q),\ \ 0\le\lambda,\gamma\le1,\ \ 0<\alpha,\beta<1$. Then for the mixed fractional integral operator Eq. (1) the representation

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)=\frac{\varphi(0,0)}{\Gamma(1+\alpha)\Gamma(1+\beta)}x^\alpha y^\beta + \frac{y^\beta}{\Gamma(1+\beta)}A_1(x)+\frac{x^\alpha}{\Gamma(1+\alpha)}A_2(y)+A(x,y) \tag{11}$$

holds, where

$$A_1(x)=\frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t,0)-\varphi(0,0)}{(x-t)^{1-\alpha}}dt,$$

$$A_2(y)=\frac{1}{\Gamma(\beta)}\int_0^y \frac{\varphi(0,s)-\varphi(0,0)}{(y-s)^{1-\beta}}ds,$$

$$A(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \frac{\left(\overset{1,1}{\Delta}_{t,s}\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}dtds,$$

and

$$|A_1(x)|\le C_1x^{\alpha+\lambda},\ \ |A_2(y)|\le C_2y^{\beta+\gamma}, \tag{12}$$

$$|A(x,y)|\le C\min_{0\le\theta\le1}x^{\alpha+\theta\lambda}y^{\beta+(1-\theta)\gamma}=C =x^\alpha y^\beta\min\left\{x^\lambda,y^\gamma\right\}. \tag{13}$$

Theorem 3: Let $0\le\lambda,\ \gamma\le1$. Then the mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from $H^{\lambda,\gamma}_0(Q)$ into $H^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Theorem 4: The mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from the space $\widetilde{H}^{\lambda,\gamma}_0(Q),\ \ 0\le\lambda,\gamma\le1$ into the space $\widetilde{H}^{\alpha+\lambda,\beta+\gamma}_0(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Theorem 5: Let $f(x,y)\in\widetilde{H}^{\lambda,\gamma}(Q),\ \ \alpha<\lambda\le1,\ \ \beta<\gamma\le1$. Then for the mixed fractional differential operator Eq. (2) the representation

$$\left(\mathrm{D}^{\alpha,\beta}_{0+,0+}f\right)(x,y)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[\frac{f(0,0)}{x^\alpha y^\beta}+ +\frac{\psi_1(x)}{y^\beta}+\frac{\psi_2(y)}{x^\alpha}+\psi(x,y)\right] \tag{14}$$

and

$$|\psi_1(x)|\le C_1x^{\lambda-\alpha},\ \ |\psi_2(y)|\le C_2y^{\gamma-\beta}, \tag{15}$$

$$|\psi(x,y)|\le Cx^{\lambda-\alpha}y^{\gamma-\beta}, \tag{16}$$

where

$$\psi_1(x)=\frac{f(x,0)-f(0,0)}{x^\alpha}+\alpha\int_0^x \frac{f(x,0)-f(t,0)}{(x-t)^{1+\alpha}}dt,$$

$$\psi_2(y)=\frac{f(0,y)-f(0,0)}{y^\beta}+\beta\int_0^y \frac{f(0,y)-f(0,s)}{(y-s)^{1+\beta}}ds,$$

$$\psi(x,y)=\frac{\left(\overset{1,1}{\Delta}_{x,y}\varphi\right)(0,0)}{x^\alpha y^\beta}+\frac{\alpha}{y^\beta}\int_0^x\left(\overset{1,1}{\Delta}_{x-t,y}\varphi\right)(t,0)\frac{dt}{(x-t)^{1+\alpha}} +\frac{\beta}{x^\alpha}\int_0^y\left(\overset{1,1}{\Delta}_{x,y-s}\varphi\right)(0,s)\frac{ds}{(y-s)^{1+\beta}}$$

$$+\alpha\beta\int_0^x\int_0^y \frac{\left(\overset{1,1}{\Delta}_{x-t,y-s}\,\varphi\right)(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds.$$

Theorem 6: Let $f(x,y)\in H^{\lambda+\alpha,\gamma+\beta}(Q),\ \alpha<\lambda\le 1,\ \beta<\gamma\le 1$. Then the operator $\mathrm{D}^{\alpha,\beta}_{0+,0+}$ continuously maps $H^{\lambda+\alpha,\gamma+\beta}(Q)$ into $H^{\lambda,\gamma}(Q)$.

We will not prove these theorems. Their proofs can be found from [6, 9].

Theorem 7: Let $f(x,y)\in \widetilde{H}^{\lambda+\alpha,\gamma+\beta}(Q),\ \alpha<\lambda\le 1,\ \beta<\gamma\le 1$. Then the operator $\mathrm{D}^{\alpha,\beta}_{0+,0+}$ continuously maps $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}(Q)$ into $\widetilde{H}^{\lambda,\gamma}(Q)$.

Proof: Let $f(x,y)\in \widetilde{H}^{\lambda+\alpha,\gamma+\beta}(Q)$. Then we have $\varphi(x,y)=\left(\mathrm{D}^{\alpha,\beta}_{0+,0+}f\right)(x,y)=\psi(x,y)$, where $\psi(x,y)$ is the function from Eq. (14). The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable.

Let $h,\eta>0;\ \ x,x+h\in[0,b],\ \ y,y+\eta\in[0,d]$. We consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta}\,\psi\right)(x,y)=\sum_{k=1}^{25}\Psi_k:=\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\,f\right)(x,y)}{(x+h)^{\alpha}(y+\eta)^{\beta}}$$

$$+\frac{\left(\overset{1,1}{\Delta}_{h,y}\,f\right)(x,0)}{(x+h)^{\alpha}}\left[y^{-\beta}-(y+\eta)^{-\beta}\right]$$

$$+\frac{\left(\overset{1,1}{\Delta}_{x,\eta}\,f\right)(0,y)}{(y+\eta)^{\beta}}\left[x^{-\alpha}-(x+h)^{-\alpha}\right]$$

$$+\left(\overset{1,1}{\Delta}_{x,y}\,f\right)(0,0)\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\left[y^{-\beta}-(y+\eta)^{-\beta}\right]$$

$$+\beta(x+h)^{-\alpha}\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,y+\eta-s}\,f\right)(x,s)}{(y+\eta-s)^{1+\beta}}ds$$

$$+\beta(x+h)^{-\alpha}\int_0^{y}\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\,f\right)(x,y)}{(y+\eta-s)^{1+\beta}}ds$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x,y+\eta-s}\,f\right)(0,s)}{(y+\eta-s)^{1+\beta}}ds$$

$$+\beta(x+h)^{-\alpha}\int_0^{y}\left(\overset{1,1}{\Delta}_{h,y-s}\,f\right)(x,s)$$
$$\left[(y-s)^{-1-\beta}-(y+\eta-s)^{-1-\beta}\right]ds$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_0^{y}\frac{\left(\overset{1,1}{\Delta}_{x,\eta}\,f\right)(0,y)}{(y+\eta-s)^{1+\beta}}ds$$

$$+\alpha(y+\eta)^{-\beta}\int_x^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta}\,f\right)(t,y)}{(x+h-t)^{1+\alpha}}dt$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_0^{y}\left(\overset{1,1}{\Delta}_{x,y-s}\,f\right)(0,s)$$
$$\left[(y-s)^{-1-\beta}-(y+\eta-s)^{-1-\beta}\right]ds$$

$$+\alpha(y+\eta)^{-\beta}\int_0^{x}\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\,f\right)(x,y)}{(x+h-t)^{1+\alpha}}dt$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_x^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y}\,f\right)(t,0)}{(x+h-t)^{1+\alpha}}dt$$

$$+\alpha(y+\eta)^{-\beta}\int_0^{x}\left(\overset{1,1}{\Delta}_{x-t,\eta}\,f\right)(t,y)$$
$$\left[(x-t)^{1-\alpha}-(x+h-t)^{-1-\alpha}\right]dt$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_0^{x}\frac{\left(\overset{1,1}{\Delta}_{h,y}\,f\right)(x,0)}{(x+h-t)^{1+\alpha}}dt$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_0^{x}\left(\overset{1,1}{\Delta}_{x-t,y}\,f\right)(t,0)$$
$$\left[(x-t)^{-1-\alpha}+(x+h-t)^{-1-\alpha}\right]dt$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\,f\right)(x,y)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}$$

$$+\int_0^x\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,y+\eta-s}\,f\right)(x,s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{h,y-s}\,f\right)(x,s)}{(x+h-t)^{1+\alpha}}$$
$$\left[(y-s)^{-1-\beta}-(y+\eta-s)^{-1-\beta}\right]dtds$$

$$+\int_x^{x+h}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta}\,f\right)(t,y)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}$$

$$+\int_x^{x+h}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y-s}\,f\right)(t,s)}{(x+h-t)^{1+\alpha}}$$
$$\left[(y-s)^{-1-\beta}-(y+\eta-s)^{-1-\beta}\right]dtds$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x-t,\eta}\,f\right)(t,y)}{(y+\eta-s)^{1+\beta}}$$

$$\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dtds$$
$$+\int_0^x\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x-t,y+\eta-s}f\right)(t,s)}{(y+\eta-s)^{1+\beta}}$$
$$\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dtds$$
$$+\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{x-t,y-s}f\right)(t,s)\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]$$
$$\left[(y-s)^{-1-\beta}-(y+\eta-s)^{-1-\beta}\right]dtds$$
$$+\int_x^{x+h}\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y+\eta-s}f\right)(t,s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}.$$

The validity of this representation may be checked directly. Since $f(x,y)\in\widetilde{H}^{\lambda,\gamma}(Q)$, we have

$$\left|\left(\overset{0,1}{\Delta_{h,\eta}\psi}\right)(x,y)\right|\le\sum_{k=1}^{25}|\Psi_k|\le C\left[\frac{h^\lambda\eta^\gamma}{(x+h)^\alpha(y+\eta)^\beta}\right.$$
$$+\frac{h^\lambda y^\gamma}{(x+h)^\alpha}\left[y^{-\beta}-(y+\eta)^{-\beta}\right]$$
$$+\frac{x^\lambda\eta^\gamma}{(y+\eta)^\beta}\left[x^{-\alpha}-(x+h)^{-\alpha}\right]$$
$$+x^\lambda y^\gamma\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\left[y^{-\beta}-(y+\eta)^{-\beta}\right]$$
$$+h^\lambda(x+h)^{-\alpha}\int_y^{y+\eta}(y+\eta-s)^{\gamma-1}ds$$
$$+h^\lambda\eta^{\gamma+\beta}(x+h)^{-\alpha}\int_0^y(\eta+s)^{-1-\beta}ds$$
$$+x^\lambda\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_y^{y+\eta}(y+\eta-s)^{\gamma-1}ds$$
$$+h^\lambda(x+h)^{-\alpha}\int_0^{\frac{y}{\eta}}s^\gamma\left[s^{-1-\beta}-(1+s)^{-1-\beta}\right]ds$$
$$+x^\lambda y^{\beta+\gamma}\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_0^y(\eta+s)^{-1-\beta}ds$$
$$+\eta^\gamma(y+\eta)^{-\beta}\int_x^{x+h}(x+h-t)^{\lambda-1}dt$$
$$+x^\lambda\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int_0^{\frac{y}{\eta}}y^\gamma\left[s^{-1-\beta}-(s+1)^{-1-\beta}\right]ds$$
$$+h^{\alpha+\lambda}\eta^\gamma(y+\eta)^{-\beta}\int_0^x(h+t)^{-1-\alpha}dt$$
$$+y^\gamma\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_x^{x+h}(x+h-t)^{\lambda-1}dt$$
$$+\eta^\gamma(y+\eta)^{-\beta}\int_0^{\frac{x}{h}}t^\lambda\left[t^{1-\alpha}-(1+t)^{-1-\alpha}\right]dt$$
$$+h^{\alpha+\lambda}y^\gamma\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_0^x(h+t)^{-1-\alpha}dt$$
$$+y^\gamma\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int_0^{\frac{x}{h}}t^\lambda\left[t^{-1-\alpha}-(1+t)^{-1-\alpha}\right]dt$$
$$+h^{\alpha+\lambda}\eta^{\beta+\gamma}\int_0^x\int_0^y(h+t)^{-1-\alpha}(\eta+s)^{-1-\beta}dtds$$
$$+h^{\lambda+\alpha}\int_0^x\int_y^{y+\eta}(h+t)^{-1-\alpha}(y+\eta-s)^{\gamma-1}dtds$$
$$+h^{\alpha+\lambda}\int_0^x\int_0^{\frac{y}{\eta}}s^\gamma(h+t)^{-1-\alpha}\left[s^{-1-\beta}-(1+s)^{-1-\beta}\right]dtds$$
$$+\eta^{\gamma+\beta}\int_x^{x+h}\int_0^y(x+h-t)^{\lambda-1}(\eta+s)^{-\beta-1}dtds$$
$$+\int_x^{x+h}\int_0^{\frac{y}{\eta}}(x+h-t)^{\lambda-1}s^\gamma\left[s^{-1-\beta}-(1+s)^{-1-\beta}\right]dtds$$
$$+\eta^{\gamma+\beta}\int_0^{\frac{x}{h}}\int_0^y(\eta+s)^{-1-\beta}t^\lambda\left[t^{-1-\alpha}-(1+t)^{-1-\alpha}\right]dtds$$
$$+\int_0^{\frac{x}{h}}\int_y^{y+\eta}(y+\eta-s)^{\lambda-1}t^\lambda\left[t^{-1-\alpha}-(1+t)^{-1-\alpha}\right]dtds$$
$$+\int_0^{\frac{x}{h}}\int_0^{\frac{y}{\eta}}t^\lambda s^\gamma\left[t^{-1-\alpha}-(1+t)^{-1-\alpha}\right]\left[s^{-1-\beta}-(1+s)^{-1-\beta}\right]dtds$$
$$+\int_x^{x+h}\int_y^{y+\eta}(y+\eta-s)^{\gamma-1}(x+h-t)^{\lambda-1}dtds.$$

After which every term is estimated in the standard way, and we get

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\psi\right)(x,y)\right|\le C_3h^\lambda y^\gamma.$$

This completes the proof. □

Theorem 8 (Main theorem): The mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ isomorphically maps the space $\widetilde{H}^{\lambda,\gamma}_0(Q)$, $0\le\lambda,\gamma\le1$ onto the space $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\alpha+\lambda<1$ and $\beta+\gamma<1$.

Proof: We should consider, as usual the following three parts of the proof:

1. Action of the mixed fractional integral operator from the space $\widetilde{H}_0^{\lambda,\gamma}(Q)$ to the space $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$;
2. Action of the mixed fractional differentiation operator from the space $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ to the space $\widetilde{H}_0^{\lambda,\gamma}(Q)$;
3. The possibility to represent any function $f(x,y) \in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ as $\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)$ with the density in $\widetilde{H}_0^{\lambda,\gamma}(Q)$.

Because of Eq. (1) the parts 1) -2) are covered by Theorems 1 and 6. The part 3) is treated in the standard way in case $0 < \alpha, \beta < 1$ by using the possibility of similar representation with the density from $L_{\overline{p}}(\mathbf{R}^2)$, $\overline{p} = (p_1, p_2)$. See [1, Theorem 24.4]. □

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