Journal of Computer Science & Computational Mathematics, Volume 10, Issue 2, June 2020
DOI: 10.20967/jcscm.2020.02.001

Dual Complex Pell Quaternions

Fügen Torunbalcı Aydın

Department of Mathematical Engineering, Faculty of Chemical and Metallurgical Engineering, Yildiz Technical University, Istanbul, TURKEY
Corresponding author email: faydin@yildiz.edu.tr, ftorunay@gmail.com

Abstract: In this paper, dual complex Pell numbers and quaternions are defined. Also, some algebraic properties of dual-complex Pell numbers and quaternions which are connected with dual complex numbers and Pell numbers are investigated. Furthermore, the Honsberger identity, Binet's formula, Cassini's identity, Catalan's identity for these quaternions are given.

Keywords: dual number, dual complex number, Pell number, dual complex Pell number, Pell quaternion, dual complex Pell quaternion.

1. Introduction

The real quaternions were first described by Irish mathematician William Rowan Hamilton in 1843. Hamilton [1] introduced a set of real quaternions which can be represented as

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$
$$jk = -kj = i, \quad ki = -ik = j.$$

The real quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors. Horadam [2] defined complex Fibonacci and Lucas quaternions as follows

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

where F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Also, the imaginary quaternion units i, j, k have the following rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k,$$
$$jk = -kj = i, ki = -ik = j$$

The studies that follows is based on the work of Horadam [3–8].

In 1971, Horadam studied on the Pell and Pell-Lucas sequences and he gave Cassini-like formula as follows [9]:

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \tag{2}$$

and Pell identities

$$\begin{cases} P_r P_{n+1} + P_{r-1}P_n = P_{n+r}, \\ P_n(P_{n+1} + P_{n-1}) = P_{2n}, \\ P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n, \\ P_n^2 + P_{n+1}^2 = P_{2n+1}, \\ P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2), \\ P_{n+a}P_{n+b} - P_n P_{n+a+b} = (-1)^n P_n P_{n+a+b}, \\ P_{-n} = (-1)^{n+1}P_n. \end{cases} \tag{3}$$

and in 1985, Horadam and Mahon obtained Cassini-like formula as follows [10]

$$q_{n+1}q_{n-1} - q_n^2 = 8(-1)^{n+1}. \tag{4}$$

First the idea to consider Pell quaternions it was suggested by Horadam in paper [11].

In 2016, Çimen and İpek introduced the Pell quaternions and the Pell-Lucas quaternions and gived properties of them [12] as follows:

$$QP_n = \{QP_n = P_n e_0 + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3 \mid P_n \ n\text{-th Pell number}\}, \tag{5}$$

where

$${e_1}^2 = {e_2}^2 = {e_3}^2 = -1, e_1 e_2 = -e_2 e_1 = e_3,$$
$$e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_2.$$

In 2016, Anetta and Iwona introduced the Pell quaternions and the Pell octanions [13] as follows:

$$R_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} \tag{6}$$

where

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

In 2016, Torunbalcı Aydın and Yüce introduced the dual Pell quaternions [14] as follows:

$$P_D = \{D_n^P = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} \mid P_n \ n\text{-th Pell number}\}, \tag{7}$$

where

$$i^2 = j^2 = k^2 = ijk = 0,$$
$$ij = -ji = jk = -kj = ki = -ik = 0.$$

In 2017, Torunbalcı Aydın, Köklü and Yüce introduced generalized dual Pell quaternions [15] as follows:

$$Q_{\mathbb{D}} = \{\mathbb{D}^{\mathbf{P}}{}_n = \mathbb{P}_n + i\mathbb{P}_{n+1} + j\mathbb{P}_{n+2} + k\mathbb{P}_{n+3} \mid \mathbb{P}_n, \ n\text{- th Gen. Pell number}\}, \tag{8}$$

where

$$i^2 = j^2 = k^2 = ijk = 0,$$
$$ij = -ji = jk = -kj = ki = -ik = 0.$$

Furthermore, Torunbalcı Aydın, Köklü introduced the generalizations of the Pell sequence in 2017 [16] as follows:

$$\begin{cases} \mathbb{P}_0 = & q, \mathbb{P}_1 = p, \mathbb{P}_2 = 2p+q, pq \in \mathbb{Z} \\ \mathbb{P}_n = & 2\mathbb{P}_{n-1} + \mathbb{P}_{n-2},\ n \geq 2 \\ \quad or & \\ \mathbb{P}_n = & (p-2q)P_n + qP_{n+1} = pP_n + qP_{n-1} \end{cases} \tag{9}$$

In 2017, Tokeşer, Ünal and Bilgici, introduced split Pell quaternions [17] as follows:

$$SP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}. \tag{10}$$

where

$$i^2 = -1, j^2 = k^2 = 1,$$
$$ij = -ji = k, jk = -jk = -i, ki = -ik = j.$$

In 2017, Catarino and Vasco introduced dual k-Pell quaternions and Octonions [18] as follows:

$$\widetilde{R_{k,n}} = \widetilde{P_{k,n}} e_0 + \widetilde{P_{k,n+1}} e_1 + \widetilde{P_{k,n+2}} e_2 + \widetilde{P_{k,n+3}} e_3, \tag{11}$$

where $\widetilde{P_{k,n}} = P_{k,n} + \varepsilon P_{k,n+1}$, $P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}$, $n \geq 2$

$$e_0 = 1,\ {e_i}^2 = -1,\ e_i e_j = -e_j e_i,\ i,j = 1,2,3,$$
$$\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0.$$

In 2018, Torunbalcı Aydın introduced bicomplex Pell and Pell-Lucas numbers [19] as follows:

$$BP_n = P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3} \tag{12}$$

and

$$BPL_n = Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3} \tag{13}$$

where $i^2 = -1, j^2 = -1, i\,j = ji$.

In the 19 th century Clifford invented a new number system by using the notation $(\varepsilon)^2 = 0, \varepsilon \neq 0$. This number system was called dual number system and the dual numbers were represented in the form $A = a + \varepsilon a^*$ with $a, a^* \in \mathbb{R}$ [20]. Afterwards, Kotelnikov (1895) and Study (1903) generalized first applications of dual numbers to mechanics [21, 22]. Besides mechanics, this concept has lots of applications in different areas such as algebraic geometry, kinematics, quaternionic formulation of motion in the theory of relativity. Majernik has introduced the multi-component number system [23]. There are three types of the four-component number systems which have been constructed by joining the complex, binary and dual two-component numbers. Later, Farid Messelmi has defined the algebraic properties of the dual-complex numbers in the light of this study [24]. There are many applications for the theory of dual-complex numbers. In 2017, Güngor and Azak have defined dual-complex Fibonacci and dual-complex Lucas numbers and their properties [25]. Dual-complex numbers [24] w can be expressed in the form as

$$\mathbb{DC} = \{w = z_1 + \varepsilon z_2 \mid z_1, z_2 \in \mathbb{C} \text{ where } \varepsilon^2 = 0, \varepsilon \neq 0\}. \tag{14}$$

Here if $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$, then any dual-complex number can be written

$$w = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2 \tag{15}$$
$$i^2 = -1,\ \varepsilon \neq 0,\ \varepsilon^2 = 0,\ (i\varepsilon)^2 = 0.$$

Addition, substraction and multiplication of any two dual-complex numbers w_1 and w_2 are defined by

$$\begin{aligned} w_1 \pm w_2 &= (z_1 + \varepsilon z_2) \pm (z_3 + \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4), \\ w_1 \times w_2 &= (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon(z_2 z_4 + z_2 z_3). \end{aligned} \tag{16}$$

On the other hand, the division of two dual-complex numbers are given by

$$\begin{aligned} \frac{w_1}{w_2} &= \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} \\ \frac{(z_1 + \varepsilon z_2)(z_3 - \varepsilon z_4)}{(z_3 + \varepsilon z_4)(z_3 - \varepsilon z_4)} &= \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{z_3^2}. \end{aligned} \tag{17}$$

If $Re(w_2) \neq 0$,then the division $\frac{w_1}{w_2}$ is possible. The dual-complex numbers are defined by the basis $\{1, i, \varepsilon, i\varepsilon\}$. Therefore, dual-complex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four-component numbers. But real and dual quaternions are non commutative, whereas, dual-complex numbers are commutative. The real and dual quaternions form a division algebra, but dual-complex numbers form a commutative ring with characteristics 0. Moreover, the multiplication of these numbers gives the dual-complex numbers the structure of 2-dimensional complex Clifford Algebra and 4-dimensional real Clifford Algebra. The base elements of the dual-complex numbers satisfy the following commutative multiplication scheme (Table 1).

Table 1. Multiplication scheme of dual-complex numbers

x	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	-1	$i\varepsilon$	$-\varepsilon$
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

Five different conjugations can operate on dual-complex numbers [24] as follows:

$$
\begin{aligned}
w &= x_1+ix_2+\varepsilon y_1+i\varepsilon y_2,\\
w^{*1} &= (x_1-ix_2)+\varepsilon(y_1-iy_2)=(z_1)^*+\varepsilon(z_2)^*,\\
w^{*2} &= (x_1+ix_2)-\varepsilon(y_1+iy_2)=z_1-\varepsilon z_2,\\
w^{*3} &= (x_1-ix_2)-\varepsilon(y_1-iy_2)=z_1^*-\varepsilon z_2^*,\\
w^{*4} &= (x_1-ix_2)(1-\varepsilon\frac{y_1+iy_2}{x_1+ix_2})=(z_1)^*(1-\varepsilon\frac{z_2}{z_1}),\\
w^{*5} &= (y_1+iy_2)-\varepsilon(x_1+ix_2)=z_2-\varepsilon z_1.
\end{aligned} \tag{18}
$$

Therefore, the norm of the dual-complex numbers is defined as

$$
\begin{aligned}
N_w^{*1} &= \|w\times w^{*1}\| = \sqrt{|z_1|^2+2\varepsilon Re(z_1z_2{}^*)},\\
N_w^{*2} &= \|w\times w^{*2}\| = \sqrt{z_1^2},\\
N_w^{*3} &= \|w\times w^{*3}\| = \sqrt{|z_1|^2-2i\varepsilon Im(z_1z_2{}^*)},\\
N_w^{*4} &= \|w\times w^{*4}\| = \sqrt{|z_1|^2},\\
N_w^{*5} &= \|w\times w^{*5}\| = \sqrt{z_1z_2+\varepsilon(z_2^2-z_1{}^2)}.
\end{aligned} \tag{19}
$$

In this paper, the dual-complex Pell numbers and quaternions will be defined. The aim of this work is to present in a unified manner a variety of algebraic properties of the dual-complex Pell quaternions as well as both the dual-complex numbers and dual-complex Pell numbers. In particular, using five types of conjugations, all the properties established for dual-complex numbers and dual-complex Pell numbers are also given for the dual-complex Pell quaternions. In addition, the Honsberger identity, the d'Ocagne's identity, Binet's formula, Cassini's identity, Catalan's identity for these quaternions are given.

2. The dual-complex Pell numbers

The dual-complex Pell and Pell-Lucas numbers can be defined by the basis $\{1,i,\varepsilon,i\varepsilon\}$, where i, j and ij satisfy the conditions

$$i^2=-1, \varepsilon\neq 0,\ \varepsilon^2=0,\ (i\varepsilon)^2=0.$$

as follows

$$
\begin{aligned}
\mathbb{DC}P_n &=(P_n+iP_{n+1})+\varepsilon(P_{n+2}+iP_{n+3})\\
&=P_n+iP_{n+1}+\varepsilon P_{n+2}+i\varepsilon P_{n+3}
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
\mathbb{DC}PL_n &=(PL_n+iPL_{n+1})+\varepsilon(PL_{n+2}+iPL_{n+3})\\
&=PL_n+iPL_{n+1}+\varepsilon PL_{n+2}+i\varepsilon PL_{n+3}.
\end{aligned} \tag{21}
$$

With the addition, substraction and multiplication by real scalars of two dual-complex Pell numbers, the dual-complex Pell number can be obtained again. Then, the addition and subtraction of the dual-complex Pell numbers are defined by

$$
\begin{aligned}
\mathbb{DC}P_n\pm\mathbb{DC}P_m =&(P_n\pm P_m)+i(P_{n+1}\pm P_{m+1})\\
&+\varepsilon(P_{n+2}\pm P_{m+2})+i\varepsilon(P_{n+3}\pm P_{m+3}).
\end{aligned} \tag{22}
$$

The multiplication of a dual-complex Pell number by the real scalar λ is defined as

$$\lambda\mathbb{DC}P_n=\lambda P_n+i\lambda P_{n+1}+\varepsilon\lambda P_{n+2}+i\varepsilon\lambda P_{n+3}. \tag{23}$$

By using (Table 1) the multiplication of two dual-complex Pell numbers is defined by

$$
\begin{aligned}
\mathbb{DC}P_n\times\mathbb{DC}P_m =\ & (P_nP_m-P_{n+1}P_{m+1})\\
&+i(P_{n+1}P_m+P_nP_{m+1})\\
&+\varepsilon(P_nP_{m+2}-P_{n+1}P_{m+3}\\
&+P_{n+2}P_m-P_{n+3}P_{m+1})\\
&+i\varepsilon(P_{n+1}P_{m+2}+P_nP_{m+3}\\
&+P_{n+3}P_m+P_{n+2}P_{m+1})\\
=\ & \mathbb{DC}P_m\times\mathbb{DC}P_n.
\end{aligned} \tag{24}
$$

Also, there exits five conjugations as follows:

$$\mathbb{DC}P_n^{*1}=P_n-iP_{n+1}+\varepsilon P_{n+2}-i\varepsilon P_{n+3}, \quad \text{complex-conjugation} \tag{25}$$

$$\mathbb{DC}P_n^{*2}=P_n+iP_{n+1}-\varepsilon P_{n+2}-i\varepsilon P_{n+3}, \quad \text{dual-conjugation} \tag{26}$$

$$\mathbb{DC}P_n^{*3}=P_n-iP_{n+1}-\varepsilon P_{n+2}+i\varepsilon P_{n+3}, \quad \text{coupled-conjugation} \tag{27}$$

$$\mathbb{DC}P_n^{*4}=(P_n-iP_{n+1}).\varepsilon(1-\frac{P_{n+2}+iP_{n+3}}{P_n+iP_{n+1}}), \quad \text{dual-complex-conjugation} \tag{28}$$

$$\mathbb{DC}P_n^{*5}=P_{n+2}+iP_{n+3}-\varepsilon P_n-i\varepsilon P_{n+1}, \quad \text{anti-dual-conjugation.} \tag{29}$$

In this case, we can give the following relations:

$$\mathbb{DC}P_n(\mathbb{DC}P_n)^{*1}=P_{2n+1}+2\varepsilon P_{2n+3}, \tag{30}$$

$$\mathbb{DC}P_n(\mathbb{DC}P_n)^{*2}=-q_nq_{n+1}+2iP_nP_{n+1}, \quad q_n,\ n\text{-th mod. Pell number} \tag{31}$$

$$\mathbb{DC}P_n(\mathbb{DC}P_n)^{*3}=P_{2n+1}-4i\varepsilon(-1)^n, \tag{32}$$

$$\mathbb{DC}P_n(\mathbb{DC}P_n)^{*4}=P_{2n+1}, \tag{33}$$

$$\mathbb{DC}P_n+(\mathbb{DC}P_n)^{*1}=2(P_n+\varepsilon P_{n+2}), \tag{34}$$

$$\mathbb{DC}P_n+(\mathbb{DC}P_n)^{*2}=2(P_n+iP_{n+1}), \tag{35}$$

$$\mathbb{DC}P_n+(\mathbb{DC}P_n)^{*3}=2(P_n+i\varepsilon P_{n+3}), \tag{36}$$

$$
\begin{aligned}
(P_n+iP_{n+1})(\mathbb{DC}P_n)^{*4} &= (P_{2n+1}-\varepsilon P_{2n+3}+2i\varepsilon(-1)^n)\\
&= (P_n-iP_{n+1})(\mathbb{DC}P_n)^{*2},
\end{aligned} \tag{37}
$$

$$\varepsilon\mathbb{DC}P_n+(\mathbb{DC}P_n)^{*5}=P_{n+2}+iP_{n+3}, \tag{38}$$

$$\mathbb{DC}P_n-\varepsilon(\mathbb{DC}P_n)^{*5}=P_n+iP_{n+1}. \tag{39}$$

The norm of the dual-complex Pell numbers $\mathbb{DC}P_n$ is defined in five different ways as follows

$$\begin{aligned}N_{\mathbb{DC}P_n^{*1}} &= \|\mathbb{DC}P_n \times (\mathbb{DC}P_n)^{*1}\|^2 \\ &= (P_n^2 + P_{n+1}^2) + 2\varepsilon(P_nP_{n+2} + P_{n+1}P_{n+3}) \\ &= P_{2n+1} + 2\varepsilon P_{2n+3}, \end{aligned} \tag{40}$$

$$\begin{aligned}N_{\mathbb{DC}P_n^{*2}} &= \|\mathbb{DC}P_n \times (\mathbb{DC}P_n)^{*2}\|^2 \\ &= |(P_n^2 - P_{n+1}^2) + 2iP_nP_{n+1}| \\ &= |-q_nq_{n+1} + 2iP_nP_{n+1}|, \end{aligned} \tag{41}$$

$$\begin{aligned}N_{\mathbb{DC}P_n^{*3}} &= \|\mathbb{DC}P_n \times (\mathbb{DC}P_n)^{*3}\|^2 \\ &= (P_n^2 + P_{n+1}^2) + 2i\varepsilon(P_nP_{n+3} - P_{n+1}P_{n+2}) \\ &= P_{2n+1} - 4i\varepsilon(-1)^n, \end{aligned} \tag{42}$$

$$\begin{aligned}N_{\mathbb{DC}P_n^{*4}} &= \|\mathbb{DC}P_n \times (\mathbb{DC}P_n)^{*4}\|^2 \\ &= P_n^2 + P_{n+1}^2 = P_{2n+1}. \end{aligned} \tag{43}$$

3. The dual-complex Pell and Pell-Lucas quaternions

In this section, firstly the dual-complex Pell quaternions will be defined. The dual-complex Pell quaternions and the dual-complex Pell-Lucas quaternions are defined by using the dual-complex Pell numbers and the dual-complex Pell-Lucas numbers respectively, as follows

$$\begin{aligned}\mathbb{DC}^{P_n} = \{&Q_{P_n} = P_n + iP_{n+1} + \varepsilon P_{n+2} + i\varepsilon P_{n+3} \mid P_n, \\ &n - \text{th Pell number}\} \end{aligned} \tag{44}$$

and

$$\begin{aligned}\mathbb{DC}^{PL_n} = \{&Q_{PL_n} = Q_n + iQ_{n+1} + \varepsilon Q_{n+2} + i\varepsilon Q_{n+3} \mid Q_n, \\ &n - \text{th Pell-Lucas number}\} \end{aligned} \tag{45}$$

where

$$i^2 = -1, \varepsilon \neq 0, \varepsilon^2 = 0, (i\varepsilon)^2 = 0.$$

Let Q_{P_n} and Q_{P_m} be two dual-complex Pell quaternions such that

$$Q_{P_n} = P_n + iP_{n+1} + \varepsilon P_{n+2} + i\varepsilon P_{n+3} \tag{46}$$

and

$$Q_{P_m} = P_m + iP_{m+1} + \varepsilon P_{m+2} + i\varepsilon P_{m+3} \tag{47}$$

Then, the addition and subtraction of two dual-complex Pell quaternions are defined in the obvious way,

$$\begin{aligned}Q_{P_n} \pm Q_{P_m} &= (P_n + iP_{n+1} + \varepsilon P_{n+2} + i\varepsilon P_{n+3}) \\ &\quad \pm (P_m + iP_{m+1} + \varepsilon P_{m+2} + i\varepsilon P_{m+3}) \\ &= (P_n \pm P_m) + i(P_{n+1} \pm P_{m+1}) \\ &\quad + \varepsilon(P_{n+2} \pm P_{m+2}) \\ &\quad + i\varepsilon(P_{n+3} \pm P_{m+3}). \end{aligned} \tag{48}$$

Multiplication of two dual-complex Pell quaternions is defined by

$$\begin{aligned}Q_{P_n} \times Q_{P_m} &= (P_n + iP_{n+1} + \varepsilon P_{n+2} + i\varepsilon P_{n+3}) \\ &\quad (P_m + iP_{m+1} + \varepsilon P_{m+2} + i\varepsilon P_{m+3}) \\ &= (P_nP_m - P_{n+1}P_{m+1}) \\ &\quad + i(P_{n+1}P_m + P_nP_{m+1}) \\ &\quad + \varepsilon(P_nP_{m+2} - P_{n+1}P_{m+3} \\ &\qquad + P_{n+2}P_m - P_{n+3}P_{m+1}) \\ &\quad + i\varepsilon(P_{n+1}P_{m+2} + P_nP_{m+3} \\ &\qquad + P_{n+3}P_m + P_{n+2}P_{m+1}) \\ &= Q_{P_m} \times Q_{P_n}. \end{aligned} \tag{49}$$

The scaler and the dual-complex vector parts of the dual-complex Pell quaternion (Q_{P_n}) are denoted by

$$S_{Q_{P_n}} = P_n \quad \text{and} \quad V_{Q_{P_n}} = iP_{n+1} + \varepsilon P_{n+2} + i\varepsilon P_{n+3}. \tag{50}$$

Thus, the dual-complex Pell quaternion Q_{P_n} is given by $Q_{P_n} = S_{Q_{P_n}} + V_{Q_{P_n}}$.

The five types of conjugation given for the dual-complex Pell numbers are the same within the dual-complex Pell quaternions. Furthermore, the conjugation properties for these quaternions are given by the relations in Eq. (25)-Eq. (29). In the following theorem, some properties related to the dual-complex Pell quaternions are given.

Theorem 1: Let Q_{P_n} be the dual-complex Pell quaternion. In this case, we can give the following relations:

$$Q_{P_n} + 2Q_{P_{n+1}} = Q_{P_{n+2}} \tag{51}$$

$$Q_{PL_n} + 2Q_{PL_{n+1}} = Q_{PL_{n+2}} \tag{52}$$

$$Q_{P_{n+1}} + Q_{P_{n-1}} = Q_{PL_n} \tag{53}$$

$$Q_{P_{n+2}} - Q_{P_{n-2}} = 2Q_{PL_n} \tag{54}$$

$$Q_{P_{n+2}} + Q_{P_{n-2}} = 6Q_{P_n} \tag{55}$$

$$\begin{aligned}&(Q_{P_n})^2 + (Q_{P_{n+1}})^2 = Q_{P_{2n+1}} - 2P_{2n+3} + iP_{2n+2} \\ &+ \varepsilon(-11P_{2n+3} + 2P_{2n+1} + 4i\varepsilon(P_{2n+4} + P_{n+2}^2), \end{aligned} \tag{56}$$

$$\begin{aligned}&(Q_{P_{n+1}})^2 - (Q_{P_{n-1}})^2 = Q_{P_{2n}} + (P_{2n} - 2P_{2n+2}) \\ &+ 3iP_{2n+1} - \varepsilon(P_{2n+2} + 8P_{2n+3}) + i\varepsilon(3P_{2n+3} - 2P_{2n} \\ &+ 12P_{n+1}P_{n+2} - 4P_{n+1}P_{n-1}), \end{aligned} \tag{57}$$

$$Q_{P_n} - i(Q_{P_{n+1}})^{*3} - \varepsilon Q_{P_{n+2}} - i\varepsilon Q_{P_{n+3}} = 2(-P_{n+1} + \varepsilon P_{n+4}). \tag{58}$$

Proof: Eq. (51)-Eq. (55): It is easily proved using Eq. (44), Eq. (45). Eq. (56): By using Eq. (44) we get,

$$\begin{aligned}&(Q_{P_n})^2 + (Q_{P_{n+1}})^2 = (P_{2n+1} - P_{2n+3}) + 2iP_{2n+2} \\ &\quad + 2\varepsilon(P_{2n+1} - 5P_{2n+3}) + 2i\varepsilon(\frac{5}{2}P_{2n+4} + 2P_{n+2}^2) \\ &= (P_{2n+1} + iP_{2n+2} + \varepsilon P_{2n+3} + i\varepsilon P_{2n+4}) \\ &\quad - P_{2n+3} + iP_{2n+2} - \varepsilon(P_{2n+1} - 11P_{2n+3}) \\ &\quad + 4i\varepsilon(P_{2n+4} + P_{n+2}^2) \\ &= Q_{P_{2n+1}} - P_{2n+3} + iP_{2n+2} - \varepsilon(P_{2n+1} - 11P_{2n+3}) \\ &\quad + 4i\varepsilon(P_{2n+4} + P_{n+2}^2). \end{aligned}$$

Eq. (57): By using Eq. (44) we get,

$$\begin{aligned}(Q_{P_{n+1}})^2-(Q_{P_{n-1}})^2 &= 2(P_{2n}-P_{2n+2})+4iP_{2n+1}\\ &-8\varepsilon P_{2n+3}+2i\varepsilon(6P_{n+1}P_{n+2}-2P_{n+1}P_{n-1}\\ &+2P_{2n+3}-P_{2n})\\ &=(P_{2n}+iP_{2n+1}+\varepsilon P_{2n+2}+i\varepsilon P_{2n+3})\\ &+(P_{2n}-2P_{2n+2})+3iP_{2n+1}-\varepsilon(8P_{2n+3}+P_{2n+2})\\ &+i\varepsilon(3P_{2n+3}-2P_{2n}+12P_{n+1}P_{n+2}-4P_{n+1}P_{n-1})\\ &=Q_{P_{2n}}-(P_{2n}-2P_{2n+2})+3iP_{2n+1}\\ &-\varepsilon(8P_{2n+3}+P_{2n+2})+i\varepsilon(3P_{2n+3}-2P_{2n}\\ &+12P_{n+1}P_{n+2}-4P_{n+1}P_{n-1}).\end{aligned}$$

Eq. (58): By using Eq. (44) and Eq. (27) we get,

$$\begin{aligned}&Q_{P_n}-i(Q_{P_{n+1}})^{*3}-\varepsilon Q_{P_{n+2}}-i\varepsilon Q_{P_{n+3}}\\ &=(P_n-P_{n+2})+2\varepsilon P_{n+4}=-2(P_{n+1}+\varepsilon P_{n+4}).\end{aligned}$$ □

Theorem 2: For $n,m\geq 0$ the Honsberger identity for the dual-complex Pell quaternions Q_{P_n} and Q_{P_m} is given by

$$\begin{aligned}Q_{P_n}Q_{P_m}+Q_{P_{n+1}}Q_{P_{m+1}} &= Q_{P_{n+m+1}}-P_{n+m+3}\\ &+iP_{n+m+2}+\varepsilon(P_{n+m+3}-2P_{n+m+5})+3i\varepsilon P_{n+m+4}.\end{aligned} \tag{59}$$

Proof: Eq. (59): By using Eq. (44) we get,

$$\begin{aligned}Q_{P_n}Q_{P_m}+Q_{P_{n+1}}Q_{P_{m+1}} &= [(P_nP_m-P_{n+2}P_{m+2})\\ &+i[(P_nP_{m+1}+P_{n+1}P_{m+2})+(P_{n+1}P_m+P_{n+2}P_{m+1})\\ &+\varepsilon[(P_nP_{m+2}+P_{n+1}P_{m+3})-(P_{n+1}P_{m+3}+P_{n+2}P_{m+4})\\ &+(P_{n+2}P_m+P_{n+3}P_{m+1})-(P_{n+3}P_{m+1}+P_{n+4}P_{m+2})]\\ &+i\varepsilon[(P_nP_{m+3}+P_{n+1}P_{m+4})+(P_{n+1}P_{m+2}+P_{n+2}P_{m+3})\\ &+(P_{n+2}P_{m+1}+P_{n+3}P_{m+2})+(P_{n+3}P_m+P_{n+4}P_{m+1})]\\ &=-2(P_{n+m+2})+2iP_{n+m+2}+2\varepsilon(P_{n+m+3}-P_{n+m+5})\\ &+4i\varepsilon P_{n+m+4}\\ &=Q_{P_{n+m+1}}-P_{n+m+3}+iP_{n+m+2}\\ &+\varepsilon(P_{n+m+3}-P_{n+m+5})+3i\varepsilon P_{n+m+4}.\end{aligned}$$

where the identity $P_nP_m+P_{n+1}P_{m+1}=P_{n+m+1}$ is used [9, 26, 27]. □

Theorem 3: For $n,m\geq 0$ the d'Ocagne's identity for the dual-complex Pell quaternions Q_{P_n} and Q_{P_m} is given by

$$Q_{P_m}Q_{P_{n+1}}-Q_{P_{m+1}}Q_{P_n}=2(-1)^nP_{m-n}(1+i+6\varepsilon+6i\varepsilon). \tag{60}$$

Proof: Eq. (60): By using Eq. (44) we get,

$$\begin{aligned}Q_{P_m}Q_{P_{n+1}}-Q_{P_{m+1}}Q_{P_n} &= [(P_mP_{n+1}-P_{m+1}P_n)\\ &-(P_{m+1}P_{n+2}-P_{m+2}P_{n+1})]+i[(P_mP_{n+2}-P_{m+1}P_{n+1})\\ &+(P_{m+1}P_{n+1}-P_{m+2}P_n)]+\varepsilon[(P_mP_{n+3}-P_{m+1}P_{n+2})\\ &-(P_{m+1}P_{n+4}-P_{m+2}P_{n+3})+(P_{m+2}P_{n+1}-P_{m+3}P_n)\\ &-(P_{m+3}P_{n+2}-P_{m+4}P_{n+1})]+i\varepsilon[(P_mP_{n+4}-P_{m+1}P_{n+3})\\ &+(P_{m+1}P_{n+3}-P_{m+2}P_{n+2})+(P_{m+2}P_{n+2}-P_{m+3}P_{n+1})\\ &+(P_{m+3}P_{n+1}-P_{m+4}P_n)]\\ &=(-1)^n2(P_{m-n})+i(-1)^n(P_{m-n+1}-P_{m-n-1})\\ &+2\varepsilon(-1)^n(P_{m-n+2}+P_{m-n-2})\\ &+i\varepsilon[(-1)^n(P_{m-n+3}-P_{m-n-3})+(P_{m-n-1}-P_{m-n+1})]\\ &=2(-1)^nP_{m-n}(1+i+6\varepsilon+6i\varepsilon).\end{aligned}$$

where the identity $P_nP_m+P_{n+1}P_{m+1}=P_{n+m+1}$ and $P_{n+3}-P_{n-3}=14P_n$ are used [9, 26, 27]. □

Theorem 4: Let Q_{P_n} be the dual-complex Pell quaternion.Then, we have the following identities

$$\sum_{s=1}^{n}Q_{P_s}=\frac{1}{4}[Q_{PL_{n+1}}-Q_{PL_1}], \tag{61}$$

$$\sum_{s=0}^{p}Q_{P_{n+s}}=\frac{1}{4}[Q_{PL_{n+p+1}}-Q_{PL_{n+1}}], \tag{62}$$

$$\sum_{s=1}^{n}Q_{P_{2s-1}}=\frac{1}{2}(Q_{P_{2n}}-Q_{P_0}), \tag{63}$$

$$\sum_{s=1}^{n}Q_{P_{2s}}=\frac{1}{2}[Q_{P_{2n+1}}-Q_{P_1}]. \tag{64}$$

Proof: Eq. (61):

$$\begin{aligned}\sum_{s=1}^{n}Q_{P_s} &= \sum_{s=1}^{n}P_s+i\sum_{s=1}^{n}P_{s+1}+\varepsilon\sum_{s=1}^{n}P_{s+2}+i\varepsilon\sum_{s=1}^{n}P_{s+3}\\ &=\frac{1}{2}[(P_n+P_{n+1}-P_1-P_0)\\ &+i(P_{n+1}+P_{n+2}-P_2-P_1)\\ &+\varepsilon(P_{n+2}+P_{n+3}-P_3-P_2)\\ &+i\varepsilon(P_{n+3}+P_{n+4}-P_4-P_3)]\\ &=\frac{1}{2}[Q_{P_n}+Q_{P_{n+1}}-Q_{P_1}-Q_{P_0}]\\ &=\frac{1}{4}[Q_{PL_{n+1}}-Q_{PL_1}].\end{aligned}$$

Eq. (62): Hence, we can write

$$\begin{aligned}\sum_{s=0}^{p} Q_{P_{n+s}} &= \sum_{s=0}^{p} P_{n+s} + i\sum_{s=0}^{p} P_{n+s+1} \\ &\quad + \varepsilon\sum_{s=0}^{p} P_{n+s+2} + i\varepsilon\sum_{s=0}^{p} P_{n+s+3} \\ &= \frac{1}{2}[(P_{n+p+1} + P_{n+p} - P_{n+1} - P_n) \\ &\quad + i(P_{n+p+2} + P_{n+p+1} - P_{n+2} - P_{n+1}) \\ &\quad + \varepsilon(P_{n+p+3} + P_{n+p+2} - P_{n+3} - P_{n+2}) \\ &\quad + i\varepsilon(P_{n+p+4} + P_{n+p+3} - P_{n+4} - P_{n+3})] \\ &= \frac{1}{2}[Q_{P_{n+p+1}} + Q_{P_{n+p}} - Q_{P_{n+1}} - Q_{P_n}] \\ &= \frac{1}{4}[Q_{PL_{n+p+1}} - Q_{PL_{n+1}}].\end{aligned}$$

Eq. (63): Hence, we can write

$$\begin{aligned}\sum_{s=1}^{n} Q_{P_{2s-1}} &= \sum_{s=1}^{n} P_{2s-1} + i\sum_{s=1}^{n} P_{2s} + \varepsilon\sum_{s=1}^{n} P_{2s+1} \\ &\quad + i\varepsilon\sum_{s=1}^{n} P_{2s+2} \\ &= (P_1 + P_3 + \ldots + P_{2n-1}) \\ &\quad + i(P_2 + P_4 + \ldots + P_{2n}) \\ &\quad + \varepsilon(P_3 + P_5 + \ldots + P_{2n+1}) \\ &\quad + i\varepsilon(P_4 + P_6 + \ldots + P_{2n+2}) \\ &= \frac{1}{2}[(P_{2n} - P_0) + i(P_{2n+1} - P_1) \\ &\quad + \varepsilon(P_{2n+2} - P_2) + i\varepsilon(P_{2n+3} - P_3)] \\ &= \frac{1}{2}[P_{2n} + iP_{2n+1} + \varepsilon P_{2n+2} + i\varepsilon P_{2n+3}] \\ &\quad - \frac{1}{2}[P_0 + iP_1 + \varepsilon P_2 + i\varepsilon P_3] \\ &= \frac{1}{2}(Q_{P_{2n}} - Q_{P_0}).\end{aligned}$$

Eq. (64): Hence, we obtain

$$\begin{aligned}\sum_{s=1}^{n} Q_{P_{2s}} &= (P_2 + P_4 + \ldots + P_{2n}) \\ &\quad + i(P_3 + P_5 + \ldots + P_{2n+1}) \\ &\quad + \varepsilon(P_4 + P_6 + \ldots + P_{2n+2}) \\ &\quad + i\varepsilon(P_5 + P_7 + \ldots + P_{2n+3}) \\ &= \frac{1}{2}[(P_{2n+1} - P_1) + i(P_{2n+2} - P_2) \\ &\quad + \varepsilon(P_{2n+3} - P_3) \\ &\quad + i\varepsilon(P_{2n+4} - P_4)] \\ &= \frac{1}{2}[P_{2n+1} + iP_{2n+2} + \varepsilon P_{2n+3} + i\varepsilon P_{2n+4}] \\ &\quad - \frac{1}{2}[P_1 + iP_2 + \varepsilon P_3 + i\varepsilon P_4] \\ &= \frac{1}{2}[Q_{P_{2n+1}} - Q_{P_1}].\end{aligned}$$

□

Theorem 5 (Binet's Formula): Let Q_{P_n} be the dual-complex Pell quaternion. For $n \geq 1$, Binet's formula for these quaternions is as follows:

$$Q_{P_n} = \frac{1}{\alpha - \beta}\left(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n\right) \tag{65}$$

where

$\hat{\alpha} = 1 + i\alpha + \varepsilon\alpha^2 + i\varepsilon\alpha^3, \alpha = 1 + \sqrt{2}$

and

$\hat{\beta} = 1 + i\beta + \varepsilon\beta^2 + i\varepsilon\beta^3, \beta = 1 - \sqrt{2}$.

Proof: Binet's formula of the dual-complex Pell quaternion is the same as Binet's formula of the Pell quaternion [9]. □

Theorem 6 (Cassini's Identity): Let Q_{P_n} be the dual-complex Pell quaternion. For $n \geq 1$, Cassini's identity for Q_{P_n} is as follows:

$$Q_{F_{n-1}}Q_{F_{n+1}} - Q_{F_n}^2 = (-1)^n 2(1 + i + 6\varepsilon + 6i\varepsilon). \tag{66}$$

Proof: Eq. (66): By using Eq. (44) we get

$$\begin{aligned}&Q_{P_{n-1}}Q_{P_{n+1}} - Q_{P_n}^2 = (P_{n-1}P_{n+1} - P_n^2) \\ &\quad + (P_{n+1}^2 - P_nP_{n+2}) - i(P_{n+1}P_n - P_{n+2}P_{n-1}) \\ &\quad + \varepsilon[-(P_{n+2}P_n - P_{n+3}P_{n-1}) - (P_nP_{n+2} - P_{n+1}P_{n+1}) \\ &\quad + (P_{n+1}P_{n+3} - P_{n+2}P_{n+2}) + (P_{n+3}P_{n+1} - P_{n+4}P_n)] \\ &\quad + i\varepsilon[-P_nP_{n+3} - P_{n+1}P_{n+2}) - (P_{n+3}P_n - P_{n+4}P_{n-1}) \\ &\quad - (P_{n+2}P_{n+1} - P_{n+3}P_n) - (P_{n+1}P_{n+2} - P_{n+2}P_{n+1})] \\ &= (-1)^n 2(0 + 0i + 6j + 3ij) \\ &= (-1)^n 2(1 + i + 6\varepsilon + 6i\varepsilon).\end{aligned}$$

where the identities of the Pell numbers $P_mP_{n+1} - P_{m+1}P_n = (-1)^n P_{m-n}$ is used [9]. □

Theorem 7 (Catalan's Identity): Let Q_{P_n} be the dual-complex Pell quaternion. For $n \geq 1$, Catalan's identity for Q_{P_n} is as follows:

$$Q_{P_n}^2 - Q_{P_{n+r}}Q_{P_{n-r}} = (-1)^{n-r}P_r^2 2(1 + i + 6\varepsilon + 6i\varepsilon). \tag{67}$$

Proof: Eq. (67): By using Eq. (44) we get

$$\begin{aligned}Q_{P_n}^2 - Q_{P_{n+r}}Q_{P_{n-r}} =& [(P_n^2 - P_{n+r}P_{n-r}) \\ & - (P_{n+1}^2 - P_{n+r+1}P_{n-r+1})] \\ & + i[(P_nP_{n+1} - P_{n+r}P_{n-r+1}) \\ & + (P_{n+1}P_n - P_{n+r+1}P_{n-r})] \\ & + \varepsilon[(P_{n+2}P_n - P_{n+r+2}P_{n-r}) \\ & + (P_nP_{n+2} - P_{n+r}P_{n-r+2}) \\ & - (P_{n+1}P_{n+3} - P_{n+r+1}P_{n-r+3}) \\ & - (P_{n+3}P_{n+1} - P_{n+r+3}P_{n-r+1})] \\ & + i\varepsilon[(P_nP_{n+3} - P_{n+r}P_{n-r+3}) \\ & + (P_{n+3}P_n - P_{n+r+3}P_{n-r}) \\ & + (P_{n+1}P_{n+2} - P_{n+r+1}P_{n-r+2}) \\ & + (P_{n+2}P_{n+1} - P_{n+r+2}P_{n-r+1})] \\ =& (-1)^{n-r})2P_r^2(1 + i + 6\varepsilon + 6i\varepsilon)\end{aligned}$$

where the identities of the Pell numbers

$$P_mP_n - P_{m+r}P_{n-r} = (-1)^{n-r}P_{m+r-n}P_r$$

and

$$P_nP_n - P_{n-r}P_{n+r} = (-1)^{n-r}P_r^2$$

are used [9]. □

4. Conclusion

In this study, a number of new results on dual-complex Pell quaternions were derived. Quaternions have great importance as they are used in quantum physics, applied mathematics, quantum mechanics, Lie groups, kinematics and differential equations.

This study fills the gap in the literature by providing the dual-complex Pell quaternion using definitions of the dual-complex number, dual-complex Pell number and Pell quaternion [12].

References

[1] W. R. Hamilton, *Elements of quaternions*. Longmans, Green, & Company, 1866.
[2] A. F. Horadam, "Complex fibonacci numbers and fibonacci quaternions," *The American Mathematical Monthly*, vol. 70, no. 3, pp. 289–291, 1963.
[3] S. Halici, "On complex fibonacci quaternions," *Advances in applied Clifford algebras*, vol. 23, no. 1, pp. 105–112, 2013.
[4] S. Halici, "On fibonacci quaternions," *Adv. Appl. Clifford Algebras*, vol. 22, no. 2, pp. 321–327, 2012.
[5] S. K. Nurkan and İ. A. Güven, "Dual fibonacci quaternions," *Advances in Applied Clifford Algebras*, vol. 25, no. 2, pp. 403–414, 2015.
[6] M. Akyiğit, H. H. Kösal, and M. Tosun, "Split fibonacci quaternions," *Advances in applied Clifford algebras*, vol. 23, no. 3, pp. 535–545, 2013.
[7] V. Majernik, "Quaternion formulation of the galilean space-time transformation," *Acta physica Slovaca*, vol. 56, no. 1, pp. 9–14, 2006.
[8] S. Yüce and F. T. Aydın, "A new aspect of dual fibonacci quaternions," *Advances in applied Clifford algebras*, vol. 26, no. 2, pp. 873–884, 2016.
[9] A. Horadam, "Pell identities," *Fibonacci Quart*, vol. 9, no. 3, pp. 245–252, 1971.
[10] A. Horadam, J. Mahon, *et al.*, "Pell and pell-lucas polynomials," *The Fibonacci Quarterly*, vol. 23, no. 1, pp. 7–20, 1985.
[11] A. Horadam, "Quaternion recurrence relations," *Ulam Quarterly*, vol. 2, no. 2, pp. 23–33, 1993.
[12] C. B. Cimen and A. İpek, "On pell quaternions and pell-lucas quaternions," *Advances in Applied Clifford Algebras*, vol. 26, no. 1, pp. 39–51, 2016.
[13] A. Szynal-Liana and I. Włoch, "The pell quaternions and the pell octonions," *Advances in Applied Clifford Algebras*, vol. 26, no. 1, pp. 435–440, 2016.
[14] F. Torunbalcı Aydın and S. Yüce, "Dual pell quaternions," *Journal of Ultra Scientist of Physical Sciences*, vol. 28, no. 7, pp. 328–339, 2016.
[15] F. T. Aydın, K. Köklü, and S. Yüce, "Generalized dual pell quaternions," *Notes on Number Theory and Discrete Mathematics*, vol. 23, no. 4, pp. 66–84, 2017.
[16] F. T. Aydin and K. Köklü, "On generalizations of the pell sequence," *arXiv preprint arXiv:1711.06260*, 2017.
[17] Ü. Tokeşer, Z. Ünal, and G. Bilgici, "Split pell and pell–lucas quaternions," *Advances in Applied Clifford Algebras*, vol. 27, no. 2, pp. 1881–1893, 2017.
[18] P. Catarino and P. Vasco, "On dual k-pell quaternions and octonions," *Mediterranean Journal of Mathematics*, vol. 14, no. 2, p. 75, 2017.
[19] F. T. Aydin, "On bicomplex pell and pell-lucas numbers," *Communications in Advanced Mathematical Sciences*, vol. 1, no. 2, pp. 142–155, 2018.
[20] W. Clifford, "Preliminary sketch of biquaters," *Proc. London Math.Soc.London.*, vol. 64, no. 4, pp. 381–395, 1873.
[21] A. Kotelnikov, "Screw calculus and some applications to geometry and mechanics," *Annals of the Imperial University of Kazan*, vol. 24, 1895.
[22] E. Study, "Geometrie der dynamen," 1903.
[23] V. Majernik, "Nulticohponent number systems," *Acta Physica. Polonica A.*, vol. 90, no. 3, pp. 491–498, 2016.
[24] F. Messelmi, "Dual-complex numbers and their holomorphic functions," 2015.
[25] M. A. Güngör and A. Z. Azak, "Investigation of dual-complex fibonacci, dual-complex lucas numbers and their properties," *Advances in Applied Clifford Algebras*, vol. 27, no. 4, pp. 3083–3096, 2017.
[26] M. Bicknell, "Primer on pell sequence and related sequences," *Fibonacci Quarterly*, vol. 13, no. 4, p. 345, 1975.
[27] R. Melham, "Sums involving fibonacci and pell numbers," *Portugaliae Mathematica*, vol. 56, no. 3, pp. 309–318, 1999.