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Fair Transition Spiral Using a Single Rational Quadratic Bézier Curve

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***Abstract*:** This paper describes spiral conditions for a single rational quadratic Bézier transition curve of G^2 contact in two fundamental cases; First, between a point and a circle, and second, between a line and a circle. Spiral segments are usable in the design of fair curves since they have several advantages of containing neither inflection points, singularities nor curvature extrema. They are important in CAD/CAM applications, such as highway, railway route, and satellite path design, trajectories of mobile robots and other similar applications. In this paper, we exploited the necessary conditions for the rational quadratic Bézier curve to possess monotonic increasing/decreasing curvatures which are based on the derivative of curvature functions. We have succeeded in obtaining an interval of the turning angle that assures the construction of a family of transition spiral between a point with a circle. As well as the construction of a family of transition spiral between a line with a circle which is characterized by the normal distance from the circle to the line. The proposed formulation can be used to construct S-shaped and C-shaped transition curve by joining the transition spiral with a straight line, along with right affine transformation procedure.

***Keywords*:** spiral, rational quadratic, fair curve.

1. Introduction

The transition curve can be characterized as a connecting curve between two separate points on primitive geometric objects, such as points, lines and circles. The type of the transition curve is normally identified as referring to the shape of the transition curve and the degree of geometric continuity of the transition curve generated by tangent vector and curvature at the contact points. The degree of G^1 continuity implies unit tangent vector continuity at the contact point, and G^2 if it also implies curvature continuity of the contact point. The shape of a transition curve joining two geometric figures is usually known as S-shaped and C-shaped curves.

The transition curve is useful for some Computer Graphics and CAD applications. It is important to meet the needs of the product design and also as an essential element to enhance the aesthetics in product design. Transition curves that have G^2 continuity at the contact points will give a visually smooth compare to G^1 continuity. Referring to the functional uses of transition curves, there were active used such as in the horizontal design of highways, railways and the trajectory of controlled automobile vehicles.

Transition curves with monotone curvature are considered to be an important factor to generate a shape preserving interpolation which gives “visually pleasing" curves. It is important for some CAD/CAM applications that we can strictly maintain the monotonous curvature along the curve segments. This is also known as the easement curve which are generally suitable to overcome the abrupt change in curvature. This good geometry property allows the curve segment contains no extraneous “bumps" or “wiggles", which makes it more readily acceptable to scientists and engineers [1]. Hence, the continuity of curvature profile is an essential indicator for the aesthetic value of an arbitrary curve. There have been some studies for the transition curve with monotone curvature when two circles have difference positions. The Clothoid or Cornu spiral is a curve whose curvature varies monotonically with arc-length has traditionally been used as the transition curves in highway design for many years [2, 3], but it cannot be represented exactly as a NURBS curve, so it is not used easily in standard graphics packages [4]. As an alternative, many researchers advocated the used of low degrees of Bézier curves such as quadratic, cubic, and quartic with monotone curvature for that purpose. Many considerable literature are available for the stated matter as discussed in [3, 5–13].

Additionally, the use of the Bezier rational representation can also give the same results as well as have more ability to control the shape of the curve. The rational Bézier curves are the special case of NURBS which have the adjustable weights that can be used to give closer approximations to arbitrary shapes. So they have become more popular due to its flexibility and its ability to maintain its shape which are important in the construction of curves and surfaces in CAD/CAM. Discussions on curvature analysis and the use of rational Bézier with low degrees can be refer in the following studies; Sapidis and Frey in [14] give a formula for finding the maximum curvature for the quadratic Bézier curve. Ahn and Kim [11] derive the characterization which is more complete in the sense that it tells the extrema as well as the monotonicity of curvature of the quadratic rational Bezier curves. In [15], Suenaga and Sakai proposed necessary and sufficient conditions for the rational quadratic Bézier curve to be a spiral or have local extrema by means of differential and Descartes' rule of signs. Furthermore, the analysis performed by Frey and Field [16] shows that if endpoints and weights are fixed then the curvature of the conic segment will always be monotonous if and only if the other control point lies inside well-defined regions bounded by circular arcs.

An interactive method of finding a fair curve matching given G^2 Hermite data with the monotonous curvature of the rational quadratic Bézier curve has been discussed by [17]. They succeeded in providing several theorems which clarify the position of end control points that guarantee the accomplishment of a monotone (decreasing or increasing) curve. In a later paper, Ahmad and Gobithaasan [18] analyzed the range of middle weight of the standard quadratic rational Bézier curve to be strictly monotone and provide the necessary conditions of monotone curvature of the rational quadratic Bézier curve.

The main focus in this paper is to develop a simple algorithm of using the monotone curvature condition of the quadratic rational Bézier curves for direct construction of single transition segment between one point and one circle. We also formulated a single transition segment between one line and one circle. This algorithm will enable us to produce an S-shaped and C-shaped transition curves between two circles by applying some affine transformation procedures.

2. General Notations And Conventions

All points and vectors in the plane are represented by italic capital letters. The notation $X(t)$ represent the vector valued function. We write the dot product of two vectors, A and B as $A \bullet B$, while $A \wedge B$ is used to represent the outer product of two plane vectors A and B. Note that the dot product and outer products are scalar which are written as $A \bullet B = \|A\|\|B\|\cos\omega$ and $A \wedge B = \|A\|\|B\|\sin\omega$, respectively, where ω is the turning angle from A to B. Positive angle is measured anti clockwise and $\|A\|$ is the Euclidean norm or length of vector A.

3. Rational Quadratic Bézier Curve

Let $Z(t)$ be a planar rational quadratic Bézier curve with control point $V_i \in R$ and it expressed as

$$Z(t) = \frac{\sum_{i=0}^{2} w_i B_i(t) V_i}{\sum_{i=0}^{2} w_i B_i(t)}, \quad 0 \le t \le 1, \tag{1}$$

where $B_0(t) = (1-t)^2$, $B_1(t) = 2(1-t)t$ and $B_2(t) = t^2$ are the Bernstein polynomial basis of the rational quadratic Bézier curve and t is a local parameter [19]. Parameter w_i represents the weight, which in practice $w_i > 0$, $i = 0,1,2$ to ensure that weighted basis functions, $B_i(t)w_i$ are non-negative and therefore $Z(t)$ is well defined. The standard form can be obtained when we assign $w_0 = w_2 = 1$ which make the shape of $Z(t)$ is unchanged. The quadratic rational Bézier in standard form is well known as conic segments and they can be classified into the following form [20] as,

- The curve is a segment of ellipse if $w_1 < 1$.
- The curve is a segment of parabola if $w_1 = 1$.
- The curve is a segment of hyperbola if $w_1 > 1$.

The curve is a circular arc if $w_1 = \cos(\theta/2)$ when the control polygon $V_0V_1V_2$ forms an isosceles triangle with V_0V_2 is the base, where $\theta \in [0, \pi)$ is turning angle and its determined by

$$\theta = \cos^{-1} \frac{(V_1 - V_0) \bullet (V_2 - V_1)}{\|V_1 - V_0\|\|V_2 - V_1\|}. \tag{2}$$

Finally, the rational quadratic Bézier curve is a spiral if $Z(t)$ satisfies the following theorem.

Theorem 1: Let us consider the standard form of rational quadratic Bézier curves as stated in Eq. (1). Suppose that we denoted the length between two control points as $a = \|V_1 - V_0\|$, $b = \|V_2 - V_1\|$ and ratio $m = b/a$. Let $w_L = \frac{\sqrt{1+m\cos\theta}}{\sqrt{2}}$ and $w_U = \frac{\sqrt{m+\cos\theta}}{\sqrt{2}\sqrt{m}}$ with $\theta \in \left[0, \frac{\pi}{2}\right]$. If $w_L < w_1 < w_U$ and $0 < m < 1$, then the curvature of the standard form of rational quadratic Bézier curves is monotonically increasing. And if $w_U < w_1 < w_L$ and $m > 1$, then the curvature of the rational quadratic Bézier curves is monotonically decreasing.

The proof of this theorem can be refer in [18].

To construct the smooth transition curve, first, let us concentrate on the curvature of a quadratic rational Bézier curve. The curvature is one of the most important shape interrogation tool of curves and surfaces which was established from differential geometry of R^j, $j = 2,3$. It is widely used in determining the quality of the approximated curves or surfaces, and to construct fair interpolation of curves and surfaces. In particular, it is used as a measure of how much a curve or surface 'bends' and to describe the shape of a curve or surface in the vicinity of a point on that curve or surface. For a planar parametric curve, the signed curvature of $Z(t)$ is defined in Eq. (3). The curvature of a curve is the reciprocal of radius of osculating circle and the curvature of a straight line is zero. The curvature can be computed from the first and second derivatives of rational Bézier curve. The signed curvature is defined as positive when the curve turns left and negative when it turns right as we travel along the curve.

$$\kappa(t) = \frac{Z'(t) \wedge Z''(t)}{\|Z'(t)\|^3}. \tag{3}$$

The curvature at the end point of a general rational Bézier curve is given as in [21],

$$\kappa(0) = \frac{w_0 w_2}{w_1^2} \frac{n-1}{n} \frac{h}{\ell^2}, \tag{4}$$

where $\ell = \|V_1 - V_0\|$ and h is the perpendicular distance from V_2 to the first leg of control polygon $V_0V_1V_2 \cdots V_n$. The curvature at end points of the standard rational quadratic Bézier with control points V_0, V_1, V_2 can be obtained directly from Eq. (4) as follows:

$$\kappa(0) = \frac{m\sin\theta}{2w_1^2 a}, \tag{5}$$

and

$$\kappa(1) = \frac{\sin\theta}{2w_1^2 m^2 a}. \tag{6}$$

Hence, we get $m^3 = \kappa(0)/\kappa(1)$. The next section discuss the technique used to produce a circle connecting a given point and a circle where the point is located outside the circle.

4. Constructing Transition Curves between One Point And One Circle

In this section, we describe the steps for construct transition curves between one point to one circle with G^2 contact. Thus, the problem we address can be expressed in following problem statement;

Given a planar circle with specified center and radius, and one point that is outside the circle. Find a single rational quadratic Bézier curve joining the point and the circle with monotone curvature.

Firstly, let us consider a standard form of rational quadratic Bézier as defined in Eq. (1). Since our goal is to construct a transition curve with G^2 contacts on the given circle, so we need to assign that $\kappa(0) = \frac{1}{r}$, where r is the radius. In other words we set the first control point of the Bézier curve on the circumference of the circle and the third control point is at the given point. Indeed from Eq. (5) it is clear that $w_1 = \sqrt{\frac{mr\sin\theta}{2a}}$. Thus, w_1 is controlled by a, b, r and θ. Therefore, we prefer to use a canonical position to determine w_1, this leads to the similar transition curve by using proper transformation procedures.

We considered the canonical position as follows. Let us assign the coordinate of the point as $S_1 = \{0,0\}$, the center of the circle is on x-axis as $C_1 = \{1,0\}$, therefore $\|S_1 - C_1\| = 1$. We also denoted the radius of the circle as j with $0 < j < 1$. And, we consider the construction of a positive-oriented transition curve, which refers to its curvature. Next, the control polygon $V_0V_1V_2$ is used with V_0 is assigned on the circle and $V_2 = S_1$.

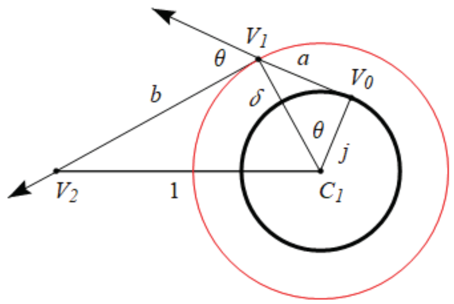


Figure 1. The canonical position of a point and a circle

Figure 1 illustrates the canonical position as described above. Here are some other important entities that need to be named before the analysis is made. First, let θ be a turning angle given by Eq. (2) which implies $\angle V_0C_1V_1 = \theta$. Second, we denoted acute angle formed between vector $V_0 - C_1$ and positive x-axis as ζ, so

$$\zeta = \cos^{-1}\frac{(C_1 - V_2)\bullet(V_0 - C_1)}{\|C_1 - V_2\|\|V_0 - C_1\|}. \tag{7}$$

And the normal distance from V_1 to the center of the circle is denoted by $j + \delta = \|V_1 - C_1\|$.

By applying prior notation, $V0$ and V_1 can be written as follow

$$V_0 = C_1 + j(\cos\zeta, \sin\zeta), \tag{8}$$
$$V_1 = C_1 + (j+\delta)(\cos(\theta+\zeta), \sin(\theta+\zeta)). \tag{9}$$

From Eq. (8) and Eq. (9), the norms a, b and ratio m can be rewritten as

$$a = \sqrt{\delta(2j+\delta)}, \quad b = \sqrt{1+(j+\delta)^2}, \quad m = \frac{b}{a}. \tag{10}$$

Next, by applying the some vector operations;

$$(V_0 - C_1)\bullet(V_1 - V_0) = 0 \text{ and } (V_1 - C_1)\bullet(V_2 - V_1) = 0 \tag{11}$$

we obtained

$$\delta = j(-1+\cos\theta) \text{ and } \delta = -j + \cos(\theta+\zeta). \tag{12}$$

Hence,

$$\zeta = \cos^{-1}(-j\cos\theta) - \theta. \tag{13}$$

For maintaining curvature on V_0 as $\frac{1}{j}$, we applied Eq. (12) to obtained w_1 in the terms of θ and j as

$$w_1 = \sqrt{\frac{\cos\theta\cot\theta\sqrt{1-j^2\sec^2\theta}}{2j}}. \tag{14}$$

Is clear from Eq. (14) that the interior weight is controlled by turning angle only whenever the radius of the circle is fixed or vice versa. Next we need to specify the domain of θ such that condition $w_U < w_1 < w_L$ is satisfied after radius is fixed. Angle θ is obtain by solving equation $w_1 = w_U$, we will get the following nonlinear equation

$$1 - 3j^2 + (1+j^2)\cos 2\theta - 2j\sqrt{1-j^2}\tan\theta = 0. \tag{15}$$

With the specific value of j we can solve Eq. (15) by using numerical methods such as the Newton Raphson method, with the initial value selected from $\theta \in \left(0, \sec^{-1}\sqrt{1/j^2}\right)$. If we denoted the solution of θ from Eq. (15) as Θ, therefore we need to choose $\theta < \Theta$ hence w_1 will always meets the spiral requirement.

As a summary, there are several steps in constructing a transition spiral between one point and one circle in a canonical position. First, by using the radius j of a given circle can we get Θ from Eq. (15). By choosing $\theta \in (0, \Theta]$, we obtained $\delta, \zeta, w_1, V_0, V_1$, along with $V_2 = S_1$ from Eq. (12), Eq. (13), Eq. (14), Eq. (8), and Eq. (9). Finally, the transition spiral can be constructed using Eq. (1).

4.1 Numerical example of transition spirals between one point and one circle

An example of a transition spiral construction that connect one point to one circle. By using radius $j = 0.3$, we obtained $\Theta = 0.9314$ from the solution of Eq. (15). Figure 2(a) shows a transition spiral with G^2 continuity at the circle by using $\theta = 0.9$. Which give us $\delta = 0.183$, $\zeta = 1.174$ and control points $V_0 = (1.1158, 0.2767)$ and $V_1 = (0.671, 0.4227)$. Whereas Figure 2(b) shows several transition spirals resulting from multiple values of $\theta \in (0, \Theta]$.

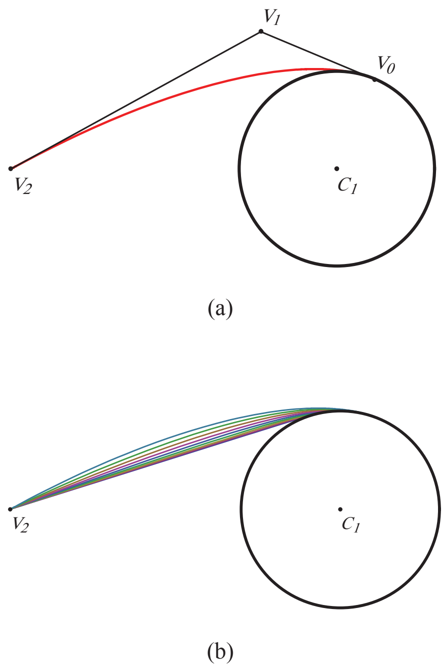


Figure 2. (a) A single transition spiral, and (b) several transition spirals with G^2 continuity between one point and one circles

5. Constructing Transition Curves between One Line And One Circle

Now, we will describe a technique for constructing transition curves between one straight line and one circle with G^1G^2 contact. Thus, the problem we address can be summarized in following problem statement;

Given a circle with specified center and radius, and a line which does not intersect or touch the circle. Find a single rational quadratic Bézier of monotone curvature that joining the line and the circle with G^1 and G^2 contact, respectively.

To solve this problem, we used similar approach as in the prior problem. We applied a canonical position to determine the range of middle weight w_1 which then give the appropriate location of the control points. In this approach the weights are also invariant under affine transformation. First, let us consider the canonical position used as illustrated in Figure 3 below. We denoted L as a line that parallel to x-axis and the center of the circle as C_1 is placed on origin with radius $j = 1$. The control polygon $V_0V_1V_2$ is used with V_0 is on the circle, V_1 and V_2 are on line L, also V_1 is the intersection point between line L and y-axis. The main idea of this design is to generate the rational quadratic Bézier curve which have the end vector that parallel to the straight line and having the other end point controllable curvature as equal to the reciprocal of radius of the given circle. Here we write $V_1 = \{0, 1+\delta\}$ and $V_2 = \{-b, 1+\delta\}$, where $\delta > 0$ since L is not intersect the circle. Once again we denoted $a = |V_1 - V_0|$, $b = |V_2 - V_1|$ and $m = b/a$. Without loss of generality, we apply positive curvature orientation which means the turning angle is in the left side of line V_0V_1.

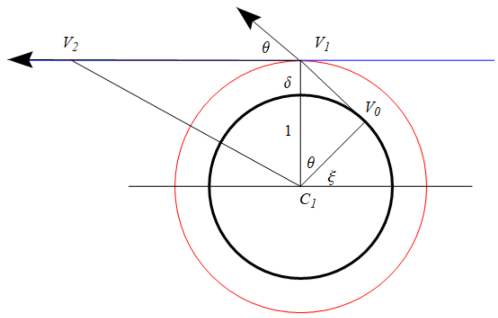


Figure 3. Canonical position for constructing transition curves between a line and a circle

Clearly from a canonical position, we write V_0 as

$$V_0 = C_1 + (\cos\zeta, \sin\zeta), \tag{16}$$

where ζ represents the acute angel between segment C_1V_0 and positive x-axis. By replacing the assumptions and conventions given above, we can rewrite a and m as

$$a = \sqrt{\delta(2+\delta)}, \qquad m = \frac{b}{\sqrt{\delta(2+\delta)}}. \tag{17}$$

Using the vector operation; $(V_0 - C_1) \bullet (V_1 - V_0) = 0$, we obtained

$$\zeta = \sin^{-1}\frac{1}{1+\delta}. \tag{18}$$

To maintain the curvature on V_0 as 1 we require the middle weight as follows

$$w_1 = \sqrt{\frac{b}{2(1+\delta)\sqrt{\delta(2+\delta)}}}. \tag{19}$$

Clearly from Eq. (19), the value of w_1 can be determined by δ and b. Next, to ensure the curvature of the curve is monotonic in the domain $w_U < w_1 < w_L$, we have to get the minimum value of δ by assigning $w_1 = w_U$. Thus, $w_1 = w_U$ is the minimum weight for a rational quadratic Bézier curve to have a monotone decreasing curvature. The results is obtained directly after solving the following quadratic equations in the terms of b,

$$b^2 - (1+\delta)\sqrt{\delta(2+\delta)}\,b - \delta(2+\delta) = 0. \tag{20}$$

Hence, if we denoted the positive root of Eq. (20) as Υ so we can write

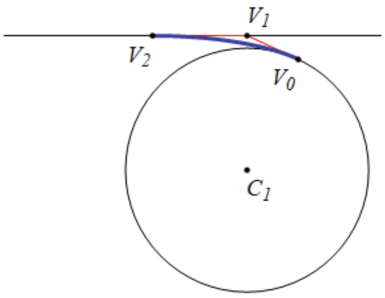
$$\Upsilon = \frac{1}{2}\left((1+\delta)\sqrt{\delta(2+\delta)} + \sqrt{\delta(10+9\delta+4\delta^2+\delta^3)}\right). \tag{21}$$

As the results, Z is a spiral with decreasing curvature after the distance between the line and the circle, δ is determined and follow by choosing $b > \Upsilon$. It is clear that Z has only one degree of freedom.

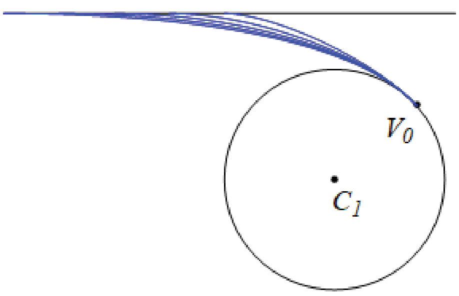
The following steps explain how the transition curve with decreasing curvature can be constructed between one line and one circle in canonical position. When the distance, δ from one line and one circle is given. Υ can be obtained from Eq. (21) then we selected the desirable value of $b > \Upsilon$. We can easily obtain a, ζ, w_1, from Eq. (17), Eq. (18) and Eq. (19), respectively. Next, we can find V_0, V_1, V_2 and θ, from $V_0 = C_1 + (\cos\zeta, \sin\zeta)$, and $V_1 = \{0, 1+\delta\}$, $V_2 = \{-b, 1+\delta\}$. Finally, we obtained a spiral with decreasing curvature from Eq. (1). A brief numerical example is given in the next subsection to explain the use of the procedure in constructing transition spirals.

5.1 Numerical example of transition spirals between one line and one circle

Suppose that we choose the distance between the line and the circle as $\delta = 0.1$. From Eq. (21), we directly obtain $\Upsilon = 0.7750$. If we choose $b = 0.7750$, we will get $V_0 = (0.4166, 0.9091)$, $V_1 = (0, 1.1)$ and $V_2 = (0, -0.7750)$. Figure 4(a) shows a positive oriented transition spiral Z joining a line with G^1 and a circle with G^2 continuity at the circle. While Figure 4(b) shows several transition spirals resulting from multiple values of $b > \Upsilon$.



(a)



(b)

Figure 4. (a) A single transition spiral, and (b) several transition spirals with G^2 continuity between one line and one circles

6. Conclusion

This paper successfully describes techniques for constructing monotonous curvature transition curves by using a single rational quadratic Bézier curve. First, the transition curve connects one point and one circle, and secondly, the transition curve that connects a line and a circle with the G^1 and G^2 continuity. A transition curve between a point and a circle can be controlled by the middle weight which generated from selected turning angle. Meanwhile, the control of the variability of the transition curves between a straight line and a circle is determined by the normal distance between the line and the circle. As a result, we managed to determine the limits of the middle weight to obtain a constantly monotonous transition curve of the both cases. Indeed, with this knowledge it can help us continue our efforts to design a visually pleasing curve such as S-shaped and C-shaped through the combination of transition curves and line segment, along with the use of affine transformation procedures as commonly practiced in the geometric modelling.

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