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Strongly Deferred Invariant Convergence and Deferred Invariant Statistical Convergence

Fatih Nuray

Deparment of Mathematics, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey
fnuray@aku.edu.tr

Abstract: In this paper, we introduce the concepts of deferred invariant convergence, strongly deferred invariant convergence and deferred invariant statistical convergence and investigate the relation between these concepts.

Keywords: statistical convergence, invariant convergence, strongly invariant convergence, deferred Cesaro mean, deferred invariant convergence, deferred invariant statistical convergence.

1. Introduction

A sequence $x=(x_k)$ is said to be strongly Cesaro summable to the number ℓ if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_k-\ell|=0.$$

Nearly all of the transformations used in the theory of summability have undesirable features. For example, the Cesaro transformation of any given positive order increases ultimate bounds and oscillations of certain sequences of functions, and does not always preserve uniform convergence, or continuous convergence, of sequences of functions. Deferred Cesaro means have useful properties not possessed by Cesaro's and other well known transformations. R.P. Agnew [1] defined the deferred Cesaro mean as a generalization of Cesaro mean of real (or complex) valued sequence $x=(x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x_k, \qquad n=1,2,3,\ldots,$$

where $p=\{p(n):n\in\mathbb{N}\}$ and $q=\{q(n):n\in\mathbb{N}\}$ are the sequences of nonnegative integers satisfying $p(n)<q(n)$ and $\lim_{n\to\infty}q(n)=\infty$. A sequence $x=(x_k)$ is said to be strongly $D_{p,q}$-convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-\ell|=0.$$

A sequence $x=(x_k)$ is said to be statistically convergent to the number $\ell\in\mathbb{R}$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: \quad |x_k-\ell|\ge\epsilon\}|=0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The idea of statistical convergence was introduced by Steinhaus in [2] and Fast in [3] independently and since then has been studied by other authors including [4–11] and [12]. The following relationship between statistical convergence and strong Cesaro summability is known [4]:

If a sequence $x=(x_k)$ is strongly Cesaro convergent to ℓ, then $x=(x_k)$ is statistically convergent to ℓ and the converse is true if $x=(x_k)$ is a bounded sequence.

The concept of deferred statistical convergence was introduced in [13] as:

A sequence $x=(x_k)$ is said to be deferred statistically convergent to the number $\ell\in\mathbb{R}$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{p(n)-q(n)}|\{p(n)<k\le q(n): \quad |x_k-\ell|\ge\epsilon\}|=0.$$

Recently, $\triangle^m$-deferred statistical convergence of order α was introduced in [14] and the concept of asymptotically deferred statistical equivalence of sequences was defined and studied in [15].

A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be a Banach limit if

(a) $\phi(x)\ge 0$ when the sequence $x=(x_n)$ has $x_n\ge 0$ for all n,

(b) $\phi(e)=1$ where $e=(1,1,1,\ldots)$ and

(c) $\phi(x_{n+1})=\phi(x_n)$ for all $x\in l_\infty$.

A sequence $x\in l_\infty$ is said to be almost convergent to the value ℓ if all of its Banach limits equal to ℓ. Lorentz [16] has given the following characterization.

A bounded sequence $\{x_n\}$ is almost convergent to L if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}x_{m+k}=\ell$$

uniformly in m.

Maddox [17] has defined strongly almost convergent sequence as follows:

A bounded sequence $\{x_n\}$ is said to be strongly almost convergent to ℓ if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_{m+k}-\ell|=0$$

uniformly in m.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if it satisfies conditions (a) and (b) stated above and

(d) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in l_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m*th* iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently $c \subset V_\sigma$. In the case σ is the translation mapping $\sigma(n) = n+1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \Big\{ x = (x_n) \in \ell_\infty : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{\sigma^k(m)} = \ell \quad \text{uniformly in } m \Big\},$$

where ℓ_∞ denotes the set of all bounded sequences.

Several authors including Raimi [18], Schaefer [19], Mursaleen [20, 21], Savaş [11, 22] and others have studied invariant convergent sequences.

The concept of strongly σ-convergence was defined by Mursaleen in [23]:

A bounded sequence $x = (x_k)$ is said to be strongly invariant convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{m} |x_{\sigma^k(m)} - \ell| = 0$$

uniformly in m.

In this case we will write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly invariant convergent sequences. In the case $\sigma(n) = n+1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$. The concept of strong s-invariant convergent sequence was defined by Savaş in [11]: A bounded sequence $x = (x_k)$ is said to be strongly s-invariant convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{\sigma^k(m)} - \ell|^s = 0$$

uniformly in m, where $0 < s < \infty$.

The concept of σ-statistically convergent sequence was introduced by Nuray and Savaş in [24] as follows:

A sequence $x = (x_k)$ is σ-statistically convergent to ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \quad |x_{\sigma^k(m)} - \ell| \ge \epsilon\}| = 0$$

uniformly in m.

In this case we write $S_\sigma - \lim x = \ell$ or $x_k \to \ell(S_\sigma)$ and we define

$$S_\sigma := \{x = (x_k) : \quad S_\sigma - \lim x = \ell, \quad \text{for some } \ell\}.$$

By a lacunary sequence [6] we mean an increasing integer sequence $\theta = \{k_n\}$ such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$. The intervals determined by θ will be denoted by $I_n = (k_{n-1}, k_n]$.

A bounded sequence $x = (x_k)$ is said to be strongly lacunary $\sigma-$ convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} |x_{\sigma^k(m)} - \ell| = 0$$

uniformly in m.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞, and $\lambda_{n+1} - \lambda_n \le 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean[25] is defined by

$$\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} x_k.$$

A sequence $x = (x_k)$ is said to be strongly $\lambda-$ invariant convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} |x_{\sigma^k(m)} - \ell| = 0$$

uniformly in m.

$x = (x_k)$ is said to be $\lambda-$ statistically invariant convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \le k \le n : \quad |x_{\sigma^k(m)} - \ell| \ge \epsilon\}| = 0$$

uniformly in m.

Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ be a sequence. Let E and F be two nonempty subset of the space w of complex numbers. We write $Ax = (A_n x)$ if $A_n(x) = \sum_{k=1}^{\infty} a_{kn} x_k$ converges for each n. If $x = (x_k) \in E$ implies $Ax \in F$, we say that A defines a matrix transformation from E to F. A matrix A is said to be regular if A transforms every convergent sequence to convergent sequence by preserving the limit.

Following conditions are, by the Silverman-Toeplitz Theorem, necessary and sufficient conditions for regularity of $A = (a_{nk})$

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$

(ii) $\lim_{n\to\infty} a_{nk} = 0$ for each $k \in \mathbb{N}$

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

2. Deferred Invariant and Deferred Invariant Statistical Convergence

Definition 1: A sequence $x = (x_k)$ is said to be deferred invariant convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_{\sigma^k(m)} = \ell$$

uniformly in m.

Definition 2: A sequence $x=(x_k)$ is said to be strongly deferred invariant convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{\sigma^k(m)}-\ell|=0$$

uniformly in m. In this case we write $x_n\to\ell(D_\sigma[p,q])$.
Definition 3: A sequence $x=(x_k)$ is said to be strongly s-deferred invariant convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{\sigma^k(m)}-\ell|^s=0$$

uniformly in m where $0<s<\infty$. In this case we write $x_n\to\ell(D_\sigma^s[p,q])$.
It is clear that;

- If $q(n)=n$ and $p(n)=0$, then Definition 2 coincides with the definition of strong invariant convergence,
- Let $\theta=(k_n)$ be a lacunary sequence. If we consider $q(n)=k_n$ and $p(n)=k_{n-1}$, then Definition 2 coincides with the strong lacunary invariant convergence,
- If $q(n)=n$ and $p(n)=n-\lambda_n$, then Definition 2 coincides with the strong λ-invariant convergence.

Definition 4: A sequence $x=(x_k)$ is said to be deferred invariant statistically convergent to the number $\ell\in\mathbb{R}$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{\sigma^k(m)}-\ell|\ge\epsilon\}|=0$$

uniformly in m. In this case we write $x_n\to\ell(DS_\sigma[p,q])$.
It is clear that;

- If $q(n)=n$ and $p(n)=0$, then Definition 4 coincides with the definition of invariant statistical convergence,
- Let $\theta=(k_n)$ be a lacunary sequence.If we consider $q(n)=k_n$ and $p(n)=k_{n-1}$, then Definition 4 coincides with the lacunary invariant statistical convergence,
- If $q(n)=n$ and $p(n)=n-\lambda_n$, then Definition 4 coincides with the invariant λ-statistical convergence of sequences.

3. Inclusion Relations

Theorem 1: Let $\{p(n)\},\{q(n)\},\{p'(n)\}$ and $\{q'(n)\}$ be sequences of of non-negative integers satisfying $p(n)\le p'(n)<q'(n)\le q(n)$ for all $n\in\mathbb{N}$ and

$$\limsup_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}<\infty,$$

then $x_n\to\ell(DS_\sigma[p,q])$ implies $x_n\to\ell(DS_\sigma[p',q'])$.

Proof: From the inclusion

$$\{p'(n)<k\le q'(n):\ |x_{\sigma^k(m)}-\ell|\ge\epsilon\}$$
$$\subset\{p(n)<k\le q(n):\ |x_{\sigma^k(m)}-\ell|\ge\epsilon\}$$

we can write

$$\frac{1}{q'(n)-p'(n)}|\{p'(n)<k\le q'(n):\quad |x_{\sigma^k(m)}-\ell|\ge\epsilon\}|$$
$$\le\frac{q(n)-p(n)}{q'(n)-p'(n)}\frac{1}{q(n)-p(n)}$$
$$\times|\{p(n)<k\le q(n):|x_{\sigma^k(m)}-\ell|\ge\epsilon\}|.$$

After taking limit when $n\to\infty$ desired result is obtained. □

Theorem 2: Let $\{p(n)\},\{q(n)\},\{p'(n)\}$ and $\{q'(n)\}$ be sequences of of non-negative integers satisfying $p(n)\le p'(n)<q'(n)\le q(n)$ for all $n\in\mathbb{N}$ and

$$\limsup_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}<\infty,$$

then $x_n\to\ell(D_\sigma[p,q])$ implies $x_n\to\ell(D_\sigma[p',q'])$.
Since the proof is similar to the proof of Theorem 1, we omit it.
Theorem 3: If (x_n) is strongly deferred invariant convergent to ℓ, then (x_n) is deferred invariant statistically convergent to ℓ, that is, if $x_n\to\ell(D_\sigma[p,q])$, then $x_n\to\ell(DS_\sigma[p,q])$.
Proof: Let $x_n\to\ell(D_\sigma[p,q])$. For an arbitrary $\epsilon>0$, we get

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{\sigma^k(m)}-\ell|$$
$$=\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{\sigma^k(m)}-\ell|\ge\epsilon}}^{q(n)}|x_{\sigma^k(m)}-\ell|$$
$$+\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{\sigma^k(m)}-\ell|<\epsilon}}^{q(n)}|x_{\sigma^k(m)}-\ell|$$
$$\ge\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{\sigma^k(m)}-\ell|\ge\epsilon}}^{q(n)}|x_{\sigma^k(m)}-\ell|$$
$$\ge\frac{\epsilon}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{\sigma^k(m)}-\ell|\ge\epsilon\}|$$

for each m. Hence, we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{\sigma^k(m)}-\ell|\ge\epsilon\}|=0$$

uniformly in m, that is $x_n\to\ell(DS_\sigma[p,q])$. □
Theorem 4: If (x_n) is bounded and deferred invariant statistically convergent to ℓ, then (x_n) is strongly deferred invariant convergent to ℓ, that is,if x bounded and $x_n\to\ell(DS_\sigma[p,q])$, then $x_n\to\ell(D_\sigma[p,q])$.
Proof: Suppose that $x_n\to\ell(DS_\sigma[p,q])$ and (x_n) is bounded,

say $|x_{\sigma^k(m)}-\ell|\leq M$ for all k and m. Given $\epsilon>0$, we get

$$\begin{aligned}&\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{\sigma^k(m)}-\ell|\\ =&\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\ |x_{\sigma^k(m)}-\ell|\geq\epsilon}}^{q(n)}|x_{\sigma^k(m)}-\ell|\\ &+\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\ |x_{\sigma^k(m)}-\ell|<\epsilon}}^{q(n)}|x_{\sigma^k(m)}-\ell|\\ \leq&\frac{M}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\ |x_{\sigma^k(m)}-\ell|\geq\epsilon}}^{q(n)}1\\ &+\frac{\epsilon}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\ |x_{\sigma^k(m)}-\ell|<\epsilon}}^{q(n)}1\\ \leq&\frac{M}{q(n)-p(n)}|\{p(n)<k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\\ &+\frac{\epsilon}{q(n)-p(n)}|\{p(n)<k\leq q(n):|x_{\sigma^k(m)}-\ell|<\epsilon\}|\end{aligned}$$

for each m, hence we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{\sigma^k(m)}-\ell|=0$$

uniformly in m. □

Theorem 5: If the sequence $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded, then $x_n\to\ell(S_\sigma)$ implies $x_n\to\ell(DS_\sigma[p,q])$.

Proof: Let $x_n\to\ell(S_\sigma)$ then for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\}|=0$$

uniformly in m. Hence, for every $\epsilon>0$

$$\lim_{n\to\infty}\frac{1}{q(n)}|\{k\leq q(n):\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\}|=0$$

uniformly in m. From the inclusion

$$\begin{aligned}&\left\{p(n)<k\leq q(n):\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ &\subseteq\left\{k\leq q(n):\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\end{aligned}$$

and the inequality

$$\begin{aligned}&|\{p(n)<k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\\ \leq&|\{k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{q(n)-p(n)}|\{p(n)<k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\\ =&\frac{q(n)-p(n)+p(n)}{q(n)-p(n)}\frac{1}{q(n)}\\ &\times|\{p(n)<k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\\ \leq&\left(1+\frac{p(n)}{q(n)-p(n)}\right)\frac{1}{q(n)}\\ &\times|\{k\leq q(n):|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\end{aligned}$$

for each m and we obtain $x_n\to\ell(DS_\sigma[p,q])$. □

Theorem 6: Let $q(n)=n$ for all $n\in\mathbb{N}$. Then, $x_n\to\ell(DS_\sigma[p,n])$ if and only if $x_n\to\ell(S_\sigma)$.

Proof: Assume that $x_n\to\ell(DS_\sigma[p,n])$. By using the technique which was used by Agnew in [1], for each $n\in\mathbb{N}$, letting $p(n)=n^{(1)}$, $p(n^{(1)})=n^{(2)}$, $p(n^{(2)})=n^{(3)},\ldots$, we may write

$$\begin{aligned}&\left\{k\leq n:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ =&\left\{k\leq n^{(1)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ &\cup\left\{n^{(1)}<k\leq n:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\},\\ &\left\{k\leq n^{(1)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ =&\left\{k\leq n^{(2)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ &\cup\left\{n^{(2)}<k\leq n^{(1)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\end{aligned}$$

and

$$\begin{aligned}&\left\{k\leq n^{(2)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ =&\left\{k\leq n^{(3)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ &\cup\left\{n^{(3)}<k\leq n^{(2)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\end{aligned}$$

for each m. This process may be continued until for some positive integer h depending on n, we obtain

$$\begin{aligned}&\left\{k\leq n^{(h-1)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ =&\left\{k\leq n^{(h)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\\ &\cup\left\{n^{(h)}<k\leq n^{(h-1)}:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\right\}\end{aligned}$$

for each m where $n^{(h)}\geq1$ and $n^{(h+1)}=0$. Therefore, we can write

$$\frac{1}{n}|\{k\leq n:\quad |x_{\sigma^k(m)}-\ell|\geq\epsilon\}|=\sum_{r=0}^{h}\frac{n^{(r)}-n^{(r+1)}}{n}t_{rm}$$

for every n and and m where

$$t_{rm}=\frac{1}{n^{(r)}-n^{(r+1)}}|\{n^{(r+1)}<k\leq n^{(r)}:|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|.$$

If we consider a matrix $A=(a_{nr})$ as

$$a_{nr}=\begin{cases}\frac{n^{(r)}-n^{(r+1)}}{n}, & r=0,1,2,\ldots,h\\ 0, & \text{otherwise}\end{cases}$$

where $n^{(0)}=n$, then the sequence

$$\left\{\frac{1}{n}|\{k\leq n:|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\right\}$$

is the (a_{nr}) transformation of the sequence (t_{rm}). Since $n^{(r)}>n^{(r+1)}, r=1,2,\ldots,h$, and $n^{(h+1)}=0$, this transformation evidently satisfies (i) and (iii); and for fixed k,

$\frac{n^{(k)}-n^{(k+1)}}{n}$ is either zero or a fraction of which the denominator is n and the numerator is $\leq k$ so that (ii) holds. Therefore, the matrix (a_{nr}) is a regular and since the sequence

$$\left\{\frac{1}{n^{(r)}-n^{(r+1)}}|\{n^{(r+1)}<k\leq n^{(r)}:|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|\right\}$$

is convergent to zero for each m, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:|x_{\sigma^k(m)}-\ell|\geq\epsilon\}|=0$$

uniformly in m.

Conversely, since $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded for $q(n)=n$, by Theorem 5, we have $x_n\to\ell(S_\sigma)$ implies $x_n\to\ell(DS_\sigma[p,q])$. □

When $\sigma(m)=m+1$, from Definitions 1, 2, 3 and 4 we have the following definitions of deferred almost convergence, deferred strongly almost convergence, strongly s-deferred almost convergence and deferred almost statistically convergence for a sequence $x=(x_k)$.

Definition 5: A sequence $x=(x_k)$ is said to be deferred almost convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}x_{k+m}=\ell$$

uniformly in m.

Definition 6: A sequence $x=(x_k)$ is said to be strongly deferred almost convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{k+m}-\ell|=0$$

uniformly in m.

Definition 7: A sequence $x=(x_k)$ is said to be strongly s-deferred almost convergent to $\ell\in\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{k+m}-\ell|^s=0$$

uniformly in m where $0<s<\infty$.

Definition 8: A sequence $x=(x_k)$ is said to be deferred almost statistically convergent to the number $\ell\in\mathbb{R}$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}|\{p(n)<k\leq q(n):|x_{k+m}-\ell|\geq\epsilon\}|=0$$

uniformly in m.

Similar inclusions to Theorems 3, 4, 5 and 6 hold between strongly deferred almost convergent sequences and deferred almost statistical convergent sequences, which have not appeared anywhere by this time.

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