Hyperbolic \( k \)-Fibonacci Quaternions

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Abstract: In this paper, hyperbolic \( k \)-Fibonacci quaternions are defined. Also, some algebraic properties of hyperbolic \( k \)-Fibonacci quaternions which are connected with hyperbolic numbers and \( k \)-Fibonacci numbers are investigated. Furthermore, d’Ocagne’s identity, the Honsberger identity, Binet’s formula, Cassini’s identity and Catalan’s identity for these quaternions are given.

Keywords: Fibonacci number, \( k \)-Fibonacci number, \( k \)-Fibonacci quaternion, \( k \)-Fibonacci dual quaternion, hyperbolic quaternion.

1. Introduction

The quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors.

Quaternions were first described by Irish mathematician Hamilton in 1843. Hamilton [1] introduced a set of quaternions which can be represented as

\[ H = \{ q = q_0 + iq_1 + jq_2 + kq_3 \mid q_i \in \mathbb{R}, i = 0, 1, 2, 3 \} \]

where

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \]
\[ ki = -ik = j. \quad (2) \]

Several authors worked on different quaternions and their generalizations ([2], [3], [4], [5], [6], [7]).

Horadam [8] defined complex Fibonacci and Lucas quaternions as follows

\[ Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \]
and

\[ K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \]

where

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \]
\[ ki = -ik = j. \]

In 2012, Halıcı [9] gave generating functions and Binet’s formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [10] defined complex Fibonacci quaternions as follows:

\[ H_{FC} = \{ R_n = C_n + c_1C_{n+1} + c_2C_{n+2} + c_3C_{n+3} \mid C_n = F_n + iF_{n+1}, i^2 = -1 \} \]

where

\[ e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1, \]
\[ e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1, \]
\[ e_3e_1 = -e_1e_3 = e_2, \quad n \geq 1. \]

In 2015, Ramirez [11] defined the \( k \)-Fibonacci and the \( k \)-Lucas quaternions as follows:

\[ D_{k,n} = \{ F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n \text{ th } k \text{-Fib. number} \}, \]
\[ F_{k,n} = \{ L_{k,n} + iL_{k,n+1} + jL_{k,n+2} + kL_{k,n+3} \mid L_{k,n}, n \text{ th } k \text{-Lucas number} \} \]

where \( i, j, k \) satisfy the multiplication rules (2).

In 2015, Polatlı Kızılateş and Kesim [12] defined split \( k \)-Fibonacci and split \( k \)-Lucas quaternions \((M_{k,n})\) and \((N_{k,n})\) respectively as follows:

\[ M_{k,n} = \{ F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n \text{ th } k \text{-Fibonacci number} \} \]

where \( i, j, k \) are split quaternionic units which satisfy the multiplication rules

\[ i^2 = -1, \quad j^2 = k^2 = ijk = 1, \]
\[ ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j. \]

In 2018, Aydın Torunbalci [13] defined \( k \)-Fibonacci dual quaternions as follows:

\[ DF_{k,n} = \{ DF_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n \text{ th } k \text{-Fibonacci number} \} \]

where \( i, j, k \) are dual quaternionic units which satisfy the multiplication rules

\[ i^2 = j^2 = k^2 = 0, \]
\[ ij = -ji = jk = -kj = ki = -ik = 0. \]

In 2018, Kösal [14] defined hyperbolic quaternions \((K)\) as follows:

\[ K = \{ q = a_0 + ia_1 + ja_2 + ka_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}, \]
\[ i, j, k \notin \mathbb{R} \} \]

where \( i, j, k \) are hyperbolic quaternionic units which satisfy the multiplication rules

\[ i^2 = j^2 = k^2 = 1, \]
\[ ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \]
In this paper, the hyperbolic $k$-Fibonacci quaternions and the hyperbolic $k$-Lucas quaternions will be defined respectively, as follows

$$H F_{k,n} = \{ q = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, \text{nth k-Fib. num.} \}$$

(3)

and

$$H L_{k,n} = \{ q = L_{k,n} + i L_{k,n+1} + j L_{k,n+2} + k L_{k,n+3} \mid L_{k,n}, \text{nth k-Lucas num.} \}$$

(4)

where

$$i^2 = j^2 = k^2 = 1,$$

$$ij = -ji, jk = i = -kj, ki = j = -ik.$$  

(5)

The aim of this work is to present in a unified manner a variety of algebraic properties of both the hyperbolic $k$-Fibonacci quaternions as well as the $k$-Fibonacci quaternions and hyperbolic quaternions. In accordance with these definitions, we give some algebraic properties and Binet’s formula for hyperbolic $k$-Fibonacci quaternions. Moreover, some sums formulas and some identities such as d’Ocagne’s, Honsberger, Cassini’s and Catalan’s identities for these quaternions are given.

2. Hyperbolic $k$-Fibonacci quaternions

The $k$-Fibonacci sequence $\{F_{k,n}\}_{n \in \mathbb{N}}$ [11] is defined as

$$\begin{align*}
F_{k,0} &= 0, F_{k,1} = 1 \\
F_{k,n+1} &= k F_{k,n} + F_{k,n-1}, \quad n \geq 1 \\
\{F_{k,n}\}_{n \in \mathbb{N}} &= \{0,1,k,k^2+1,k^3+2k,k^4+3k^2+1, \ldots \}
\end{align*}$$

(6)

Here, $k$ is a positive real number. In this section, firstly hyperbolic $k$-Fibonacci quaternions will be defined. Hyperbolic $k$-Fibonacci quaternions are defined by using the $k$-Fibonacci numbers and hyperbolic quaternionic units as follows

$$H F_{k,n} = \{ q = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, \text{nth k-Fib. num.} \},$$

(7)

where

$$i^2 = j^2 = k^2 = 1,$$

$$ij = -ji, jk = i = -kj, ki = j = -ik.$$  

Let $H F_{k,n}$ and $H F_{k,m}$ be two hyperbolic $k$-Fibonacci quaternions such that

$$H F_{k,n} = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}$$

(8)

and

$$H F_{k,m} = F_{k,m} + i F_{k,m+1} + j F_{k,m+2} + k F_{k,m+3}$$

(9)

Then, the addition and subtraction of two hyperbolic $k$-Fibonacci quaternions are defined in the obvious way,

$$\begin{align*}
H F_{k,n} \pm H F_{k,m} &= (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) \\
&\pm (F_{k,m} + i F_{k,m+1} + j F_{k,m+2} + k F_{k,m+3}) \\
&= (F_{k,n} \pm F_{k,m}) + i (F_{k,n} \pm F_{k,m}) + j (F_{k,n} \pm F_{k,m}) + k (F_{k,n} \pm F_{k,m}).
\end{align*}$$

(10)

Multiplication of two hyperbolic $k$-Fibonacci quaternions is defined by

$$\begin{align*}
H F_{k,n} H F_{k,m} &= (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) \\
&\times (F_{k,m} + i F_{k,m+1} + j F_{k,m+2} + k F_{k,m+3}) \\
&= (F_{k,n} \cdot F_{k,m}) + i (F_{k,n} \cdot F_{k,m}) + j (F_{k,n} \cdot F_{k,m}) + k (F_{k,n} \cdot F_{k,m}).
\end{align*}$$

(11)

The scaler and the vector parts of hyperbolic $k$-Fibonacci quaternion $H F_{k,n}$ are denoted by

$$\begin{align*}
S_{H F_{k,n}} &= F_{k,n} \\
V_{H F_{k,n}} &= i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}.
\end{align*}$$

(12)

Thus, hyperbolic $k$-Fibonacci quaternion $H F_{k,n}$ is given by $H F_{k,n} = S_{H F_{k,n}} + V_{H F_{k,n}}$. The conjugate of hyperbolic $k$-Fibonacci quaternion $H F_{k,n}$ is denoted by $H F_{k,n}^*$ and it is

$$H F_{k,n}^* = F_{k,n} - i F_{k,n+1} - j F_{k,n+2} - k F_{k,n+3}.$$  

(13)

The norm of hyperbolic $k$-Fibonacci quaternion $H F_{k,n}$ is defined as follows

$$\| H F_{k,n} \|^2 = H F_{k,n} H F_{k,n}^* = F_{k,n}^2 - F_{k,n+1}^2 - F_{k,n+2}^2 - F_{k,n+3}^2.$$  

(14)

In the following theorem, some properties related to hyperbolic $k$-Fibonacci quaternions are given.

**Theorem 1:** Let $F_{k,n}$ and $H F_{k,n}$ be the $n$-th terms of $k$-Fibonacci sequence $(F_{k,n})$ and hyperbolic $k$-Fibonacci quaternion $(H F_{k,n})$, respectively. In this case, for $n \geq 1$ we can give the following relations:

$$H F_{k,n+2} = k H F_{k,n+1} + H F_{k,n}$$

(15)

$$H F_{k,n}^2 = 2 F_{k,n} \cdot H F_{k,n} - H F_{k,n} \cdot H F_{k,n}^*$$

(16)

$$H F_{k,n} - i H F_{k,n+1} - j H F_{k,n+2} - k H F_{k,n+3} = F_{k,n} - F_{k,n+2} - F_{k,n+4} - F_{k,n+6}$$

(17)
Proof: (15): By the equation (8) we get,

\[ HF_{k,n} + kHF_{k,n+1} = (F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3}) + k(F_{k,n+1} + iF_{k,n+2} + jF_{k,n+3} + kF_{k,n+4}) = (F_{k,n} + kF_{k,n+1}) + i(F_{k,n+1} + kF_{k,n+2}) + j(F_{k,n+2} + kF_{k,n+3}) + k(F_{k,n+3} + kF_{k,n+4}) = F_{k,n+2} + iF_{k,n+3} + jF_{k,n+4} + kF_{k,n+5} = \mathbb{H}F_{k,n+2}. \]

(16): By the equation (7) we get,

\[
\mathbb{H}F_{k,n} = (F_{k,n}^2 + F_{k,n+1}^2 + F_{k,n+2}^2 + F_{k,n+3}^2) + 2F_{k,n}(iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3}) = 2F_{k,n}HF_{k,n} - 2F_{k,n}^2 + (F_{k,n}^2 + F_{k,n+1}^2 + F_{k,n+2}^2 + F_{k,n+3}^2) = 2F_{k,n}HF_{k,n} - \mathbb{H}F_{k,n}. \]

(17): By the equation (7) we get,

\[
\mathbb{H}F_{k,n} - i\mathbb{H}F_{k,n+1} - j\mathbb{H}F_{k,n+2} - k\mathbb{H}F_{k,n+3} = F_{k,n} - F_{k,n+2} - F_{k,n+4} - F_{k,n+6}. \]

Theorem 2: For \( m \geq n+1 \) the d’Ocagne’s identity for hyperbolic \( k \)-Fibonacci quaternions \( \mathbb{H}F_{k,m} \) and \( \mathbb{H}F_{k,n} \) is given by

\[
\mathbb{H}F_{k,m} \mathbb{H}F_{k,n+1} - \mathbb{H}F_{k,m+1} \mathbb{H}F_{k,n} = (-1)^n \left[ 0 - 2iF_{k,m-n-1} + 2jF_{k,m-n-2} + k(L_{k,m-n} + (k^3 + 3k)F_{k,m-n}) \right].
\]

Proof: (18): By using (7)

\[
\mathbb{H}F_{k,m} \mathbb{H}F_{k,n+1} - \mathbb{H}F_{k,m+1} \mathbb{H}F_{k,n} = (F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n}) + (F_{k,m+1}F_{k,n+2} - F_{k,m+2}F_{k,n+1}) + (F_{k,m+2}F_{k,n+3} - F_{k,m+3}F_{k,n+2}) + (F_{k,m+3}F_{k,n+4} - F_{k,m+4}F_{k,n+3}) + i(F_{k,m}F_{k,n+2} - F_{k,m+1}F_{k,n+1}) + j(F_{k,m}F_{k,n+3} - F_{k,m+1}F_{k,n+2}) + k(F_{k,m}F_{k,n+4} - F_{k,m+1}F_{k,n+3}) \]

\[ = (-1)^n \left[ 0 - 2iF_{k,m-n-1} + 2jF_{k,m-n-2} + k(L_{k,m-n} + (k^3 + 3k)F_{k,m-n}) \right]. \]

Here, d’Ocagne’s identity of \( k \)-Fibonacci number

\[ F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n} \text{ in Falcon and Plaza [15]} \] was used.

Theorem 3: For \( n, m \geq 0 \) the Honsberger identity for hyperbolic \( k \)-Fibonacci quaternions \( \mathbb{H}F_{k,n} \) and \( \mathbb{H}F_{k,m} \) is given by

\[
\mathbb{H}F_{k,n+1} \mathbb{H}F_{k,m} + \mathbb{H}F_{k,n} \mathbb{H}F_{k,m-1} = \mathbb{H}F_{k,n+m+1} \mathbb{H}F_{k,m-1} + \mathbb{H}F_{k,n} \mathbb{H}F_{k,m+1} = 2\mathbb{H}F_{k,n+m} + kF_{k,n+m+1} + L_{k,n+m+5}. \]

Proof: (19) By using (11)

\[
\mathbb{H}F_{k,n+1} \mathbb{H}F_{k,m} = (F_{k,n+1}F_{k,m} + F_{k,n+2}F_{k,m+1} + F_{k,n+3}F_{k,m+2} + F_{k,n+4}F_{k,m+3}) + i(F_{k,n+1}F_{k,m+1} + F_{k,n+2}F_{k,m}) + j(F_{k,n+2}F_{k,m+2} + F_{k,n+3}F_{k,m+3}) + k(F_{k,n+3}F_{k,m+3} + F_{k,n+4}F_{k,m+4}) \]

\[ = \mathbb{H}F_{k,n} \mathbb{H}F_{k,m-1} = (F_{k,n+1}F_{k,m-1} + F_{k,n+2}F_{k,m} + F_{k,n+3}F_{k,m+1} + F_{k,n+4}F_{k,m+2}) + i(F_{k,n+1}F_{k,m+1} + F_{k,n+2}F_{k,m}) + j(F_{k,n+2}F_{k,m+2} + F_{k,n+3}F_{k,m+3}) + k(F_{k,n+3}F_{k,m+3} + F_{k,n+4}F_{k,m+4}) \]

Finally, adding by two sides to the side, we obtain

\[
\mathbb{H}F_{k,n+1} \mathbb{H}F_{k,m} + \mathbb{H}F_{k,n} \mathbb{H}F_{k,m-1} = (F_{k,n+m} + F_{k,n+m+2} + F_{k,n+m+4} + F_{k,n+m+6}) + 2iF_{k,n+m+1} + 2jF_{k,n+m+2} + 2kF_{k,n+m+3} + 2\mathbb{H}F_{k,n+m} + kF_{k,n+m+1} + L_{k,n+m+5} \]

Here, the Honsberger identity of \( k \)-Fibonacci number

\[ F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1} = F_{k,n+m} \] in Falcon and Plaza [16] and \( F_{k,n+1} + F_{k,n-1} = L_{k,n} \) [11] was used.

Theorem 4: Let \( \mathbb{H}F_{k,n} \) and \( \mathbb{H}F_{k,n} \) be \( n \)th terms of hyperbolic \( k \)-Fibonacci quaternion \( \mathbb{H}F_{k,n} \) and hyperbolic \( k \)-Lucas quaternion \( \mathbb{H}L_{k,n} \), respectively. The following relation is satisfied

\[
\mathbb{H}F_{k,n+1} + \mathbb{H}F_{k,n-1} = \mathbb{H}L_{k,n}, \quad (20)
\]

and

\[
\mathbb{H}F_{k,n+2} + \mathbb{H}F_{k,n} = k\mathbb{H}L_{k,n} \quad (21)
\]

Proof: (20) From equation (8) and identity \( F_{k,n+1} + F_{k,n-1} = L_{k,n} \), \( n \geq 1 \) Ramirez [11] between \( k \)-Fibonacci number and \( k \)-Lucas number, it follows...
that
\[ HF_{k,n+1} + HF_{k,n-1} = (F_{k,n+1} + F_{k,n-1}) + i(F_{k,n+2} + F_{k,n}) + j(F_{k,n+4} + F_{k,n+2}) + k(F_{k,n+5} - F_{k,n+3}) = \mathbb{H}L_k, n. \]

(21) From equation (7) and identity \( F_{k,n+2} - F_{k,n-2} = kL_k, n, n \geq 1 \) between \( k \)-Fibonacci number and \( k \)-Lucas number, it follows that
\[ HF_{k,n+2} - HF_{k,n-2} = (F_{k,n+2} - F_{k,n-2}) + i(F_{k,n+3} - F_{k,n-1}) + j(F_{k,n+4} - F_{k,n+2}) + k(F_{k,n+5} - F_{k,n+3}) = \mathbb{H}L_k, n. \]

\[ HF_{k,n} + \overline{HF}_{k,n} = 2F_{k,n} \]  
\[ HF_{k,n} + HF_{k,n+1} + HF_{k,n-1} + HF_{k,n+3} = F_{k,2n+1} - F_{k,2n+3} - F_{k,2n+5} \]  
\[ HF_{k,n} + iHF_{k,n} + jHF_{k,n} + kHF_{k,n} = 2F_{k,n}. \]

**Theorem 5:** Let \( HF_{k,n} \) be conjugation of hyperbolic \( k \)-Fibonacci quaternion \( \overline{HF}_{k,n} \). In this case, we can give the following relations between these quaternions:
\[ HF_{k,n} + HF_{k,n} = 2F_{k,n} \]

\[ HF_{k,n} + HF_{k,n+1} + HF_{k,n-1} + HF_{k,n+3} = F_{k,2n+1} - F_{k,2n+3} - F_{k,2n+5} \]

**Proof:** (24): Since \( \sum_{i=1}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1) \)

Falcon and Plaza [16], we get
\[ \sum_{s=1}^{n} HF_{k,s} = \sum_{s=1}^{n} F_{k,s} + i\sum_{s=1}^{n} F_{k,s+1} + j\sum_{s=1}^{n} F_{k,s+2} + k\sum_{s=1}^{n} F_{k,s+3} = \frac{1}{k}(HF_{k,2n+1} - HF_{k,1}). \]

\[ HF_{k,n} + \overline{HF}_{k,n} = \mathbb{H}F_{k,n}. \]

**Theorem 6:** Let \( HF_{k,n} \) be hyperbolic \( k \)-Fibonacci quaternion. Then, we have the following identities
\[ \sum_{s=1}^{n} HF_{k,s} = \frac{1}{k}(HF_{k,n+1} + HF_{k,n} - HF_{k,1} - HF_{k,0}). \]
\[ \sum_{s=1}^{n} HF_{k,2s-1} = \frac{1}{k}(HF_{k,2n} - HF_{k,0}). \]
\[ \sum_{s=1}^{n} HF_{k,2s} = \frac{1}{k}(HF_{k,2n+1} - HF_{k,1}). \]  

**Proof:** (24): Since \( \sum_{i=1}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1) \)

Falcon and Plaza [16], we get
\[ \sum_{s=1}^{n} HF_{k,s} = \sum_{s=1}^{n} F_{k,s} + i\sum_{s=1}^{n} F_{k,s+1} + j\sum_{s=1}^{n} F_{k,s+2} + k\sum_{s=1}^{n} F_{k,s+3} = \frac{1}{k}(HF_{k,2n+1} - HF_{k,1}). \]

\[ HF_{k,n} + \overline{HF}_{k,n} = \mathbb{H}F_{k,n}. \]
Theorem 7 (Binet’s Formula): Let $\mathbb{H}F_{k,n}$ be hyperbolic $k$-Fibonacci quaternion. For $n \geq 1$, Binet’s formula for these quaternions is as follows Falcon and Plaza [16]:

$$\mathbb{H}F_{k,n} = \frac{1}{\sqrt{k^2 + 4}} \left( \hat{\alpha} \alpha^n - \hat{\beta} \beta^n \right) \quad (27)$$

where

$$\hat{\alpha} = 1 + i[(k - \beta)] + j[(k^2 + 1) - k\beta] + k[(k^3 + 2k) - (k^2 + 1)\beta],$$

and

$$\hat{\beta} = -1 + i(\alpha - k) + j[\alpha k - (k^2 + 1)] + k[(k^2 + 1)\alpha - (k^3 + 2k)].$$

Proof: The characteristic equation of recurrence relation $\mathbb{H}F_{k,n+2} = k\mathbb{H}F_{k,n+1} + \mathbb{H}F_{k,n}$ is

$$t^2 - kt - 1 = 0.$$ 

The roots of this equation are

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{k - \sqrt{k^2 + 4}}{2}$$

where $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4}$, $\alpha \beta = -1$.

Using recurrence relation and initial values $\mathbb{H}F_{k,0} = (0, 1, k, k^2 + 1)$, $\mathbb{H}F_{k,1} = (1, k, k^2 + 1, k^3 + 2k)$, the Binet formula for $\mathbb{H}F_{k,n}$ is

$$\mathbb{H}F_{k,n} = A\alpha^n + B\beta^n = \frac{1}{\sqrt{k^2 + 4}} \left[ \hat{\alpha} \alpha^n - \hat{\beta} \beta^n \right],$$

where $A = \frac{\mathbb{H}F_{k,1} - \beta\mathbb{H}F_{k,0}}{\alpha - \beta}$, $B = \frac{\alpha \mathbb{H}F_{k,0} - \mathbb{H}F_{k,1}}{\alpha - \beta}$ and

$$\hat{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3, \quad \hat{\beta} = 1 + i\beta + j\beta^2 + k\beta^3.$$

Theorem 8 (Cassini’s Identity): Let $\mathbb{H}F_{k,n}$ be hyperbolic $k$-Fibonacci quaternion. For $n \geq 1$, Cassini’s identity for $\mathbb{H}F_{k,n}$ is as follows:

$$\mathbb{H}F_{k,n-1}\mathbb{H}F_{k,n+1} - (\mathbb{H}F_{k,n})^2 = (-1)^n[2ki + (k^2 + 1)j + (k^3 + 2k)k]. \quad (28)$$

Proof: (28): By using (7) and (11), we get

$$\mathbb{H}F_{k,n-1}\mathbb{H}F_{k,n+1} - (\mathbb{H}F_{k,n})^2 \ni \left[ (F_{k,n-1}F_{k,n+1} - F_{k,n}^2) + (F_{k,n}F_{k,n+2} - F_{k,n+1}^2) + (F_{k,n+1}F_{k,n+3} - F_{k,n+2}^2) + (F_{k,n+2}F_{k,n+4} - F_{k,n+3}^2) \right]$$

$$+ j \left[ (F_{k,n-1}F_{k,n+2} - F_{k,n+1}F_{k,n+3}) + (F_{k,n}F_{k,n+3} - F_{k,n+1}F_{k,n+2}) + (F_{k,n+2}F_{k,n+4} - F_{k,n+3}F_{k,n+3}) \right]$$

$$= (-1)^n[2ki + (k^2 + 1)j + (k^3 + 2k)k].$$

Here, the identity of the $k$-Fibonacci number $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$, Falcon and Plaza [16] was used.

Theorem 9 (Catalan’s Identity): Let $\mathbb{H}F_{k,n}$ be hyperbolic $k$-Fibonacci quaternion. For $n \geq 1$, Catalan’s identity for $\mathbb{H}F_{k,n}$ is as follows:

$$\mathbb{H}F_{k,n+r-1}\mathbb{H}F_{k,n+r+1} - \mathbb{H}F_{k,n+r}^2 = (-1)^{n+r}[0 + 2ki + (k^2 + 1)j + (k^3 + 2k)k]. \quad (29)$$

Proof: (29): By using (7) and (11), we get

$$\mathbb{H}F_{k,n+r-1}\mathbb{H}F_{k,n+r+1} - \mathbb{H}F_{k,n+r}^2 \ni \left[ (F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2) + (F_{k,n+r}F_{k,n+r+2} - F_{k,n+r+1}^2) + (F_{k,n+r+1}F_{k,n+r+3} - F_{k,n+r+2}^2) + (F_{k,n+r+2}F_{k,n+r+4} - F_{k,n+r+3}^2) \right]$$

$$+ j \left[ (F_{k,n+r-1}F_{k,n+r+2} - F_{k,n+r+1}F_{k,n+r+3}) + (F_{k,n+r}F_{k,n+r+3} - F_{k,n+r+1}F_{k,n+r+2}) + (F_{k,n+r+1}F_{k,n+r+4} - F_{k,n+r+2}F_{k,n+r+3}) \right]$$

$$= (-1)^{n+r}[0 + 2ki + (k^2 + 1)j + (k^3 + 2k)k].$$

Here, the identity of the $k$-Fibonacci number $F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2 = (-1)^{n+r}$, Falcon and Plaza [16] was used.
3. Conclusion

In this study, a number of new results on hyperbolic $k$-Fibonacci quaternions are derived. I hope that these results will be important in applied mathematics, quantum physics and kinematics.

References