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Hyperbolic k-Fibonacci Quaternions

Fügen Torunbalci Aydin

Department of Mathematical Engineering, Faculty of Chemical and Metallurgical Engineering, Yildiz Technical University, Istanbul, TURKEY
faydin@yildiz.edu.tr

***Abstract*:** In this paper, hyperbolic k-Fibonacci quaternions are defined. Also, some algebraic properties of hyperbolic k-Fibonacci quaternions which are connected with hyperbolic numbers and k-Fibonacci numbers are investigated. Furthermore, d'Ocagne's identity, the Honsberger identity, Binet's formula, Cassini's identity and Catalan's identity for these quaternions are given.

***Keywords*:** Fibonacci number, k-Fibonacci number, k-Fibonacci quaternion, k-Fibonacci dual quaternion, hyperbolic quaternion.

1. Introduction

The quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors.

Quaternions were first described by Irish mathematician Hamilton in 1843. Hamilton [1] introduced a set of quaternions which can be represented as

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_i \in \mathbb{R}, i = 0,1,2,3\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i,$$
$$ki = -ik = j. \tag{2}$$

Several authors worked on different quaternions and their generalizations ([2], [3], [4], [5], [6], [7]).

Horadam [8] defined complex Fibonacci and Lucas quaternions as follows

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i,$$
$$ki = -ik = j.$$

In 2012, Halıcı [9] gave generating functions and Binet's formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [10] defined complex Fibonacci quaternions as follows:

$$H_{FC} = \{R_n = C_n + e_1C_{n+1} + e_2C_{n+2} + e_3C_{n+3} \mid C_n = F_n + iF_{n+1}, i^2 = -1\}$$

where

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = e_1e_2e_3 = -1,$$
$$e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1,$$
$$e_3e_1 = -e_1e_3 = e_2, \ n \geq 1.$$

In 2015, Ramirez [11] defined the k-Fibonacci and the k-Lucas quaternions as follows:

$$D_{k,n} = \{F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n-th\ k\text{-Fib. number}\},$$

$$P_{k,n} = \{L_{k,n} + iL_{k,n+1} + jL_{k,n+2} + kL_{k,n+3} \mid L_{k,n}, n-th\ k\text{-Lucas number}\}$$

where i, j, k satisfy the multiplication rules (2).

In 2015, Polatlı Kızılateş and Kesim [12] defined split k-Fibonacci and split k-Lucas quaternions $(M_{k,n})$ and $(N_{k,n})$ respectively as follows:

$$M_{k,n} = \{F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n-th\ k\text{-Fibonacci number}\}$$

where i, j, k are split quaternionic units which satisy the multiplication rules

$$i^2 = -1, j^2 = k^2 = ijk = 1,$$
$$ij = -ji = k, jk = -kj = -i, ki = -ik = j.$$

In 2018, Aydın Torunbalcı [13] defined k-Fibonacci dual quaternions as follows:

$$\mathbb{D}F_{k,n} = \{\mathbb{D}F_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3} \mid F_{k,n}, n-th\ k\text{-Fibonacci number}\}$$

where i, j, k are dual quaternionic units which satisy the multiplication rules

$$i^2 = j^2 = k^2 = 0,$$
$$ij = -ji = jk = -kj = ki = -ik = 0.$$

In 2018, Kösal [14] defined hyperbolic quaternions (K) as follows:

$$K = \{q = a_0 + ia_1 + ja_2 + ka_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}, i, j, k \notin \mathbb{R}\}$$

where i, j, k are hyperbolic quaternionic units which satisy the multiplication rules

$$i^2 = j^2 = k^2 = 1,$$
$$ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik.$$

In this paper, the hyperbolic k-Fibonacci quaternions and the hyperbolic k-Lucas quaternions will be defined respectively, as follows

$$\mathbb{H}F_{k,n} = \{q = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{k}F_{k,n+3} \mid F_{k,n},\, nth\, k\text{-Fib. num.}\} \tag{3}$$

and

$$\mathbb{H}L_{k,n} = \{q = L_{k,n} + \mathbf{i}L_{k,n+1} + \mathbf{j}L_{k,n+2} + \mathbf{k}L_{k,n+3} \mid L_{k,n},\, nth\, k\text{-Lucas num.}\} \tag{4}$$

where

$$\begin{aligned} &i^2 = j^2 = k^2 = 1, \\ &ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik. \end{aligned} \tag{5}$$

The aim of this work is to present in a unified manner a variety of algebraic properties of both the hyperbolic k-Fibonacci quaternions as well as the k-Fibonacci quaternions and hyperbolic quaternions. In accordance with these definitions, we given some algebraic properties and Binet's formula for hyperbolic k-Fibonacci quaternions. Moreover, some sums formulas and some identities such as d'Ocagne's, Honsberger, Cassini's and Catalan's identities for these quaternions are given.

2. Hyperbolic k-Fibonacci quaternions

The k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ [11] is defined as

$$\begin{cases} F_{k,0} = 0, F_{k,1} = 1 \\ F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1 \\ or \\ \{F_{k,n}\}_{n\in\mathbb{N}} = \{0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, \ldots\} \end{cases} \tag{6}$$

Here, k is a positive real number. In this section, firstly hyperbolic k-Fibonacci quaternions will be defined. Hyperbolic k-Fibonacci quaternions are defined by using the k-Fibonacci numbers and hyperbolic quaternionic units as follows

$$\mathbb{H}F_{k,n} = \{q = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{k}F_{k,n+3} \mid F_{k,n},\, n-th\ k\text{-Fib. num.}\}, \tag{7}$$

where

$$\begin{aligned} &i^2 = j^2 = k^2 = 1, \\ &ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik. \end{aligned}$$

Let $\mathbb{H}F_{k,n}$ and $\mathbb{H}F_{k,m}$ be two hyperbolic k-Fibonacci quaternions such that

$$\mathbb{H}F_{k,n} = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{k,n+3} \tag{8}$$

and

$$\mathbb{H}F_{k,m} = F_{k,m} + \mathbf{i}F_{k,m+1} + \mathbf{j}F_{k,m+2} + \mathbf{k}F_{k,m+3} \tag{9}$$

Then, the addition and subtraction of two hyperbolic k-Fibonacci quaternions are defined in the obvious way,

$$\begin{aligned} &\mathbb{H}F_{k,n} \pm \mathbb{H}F_{k,m} = (F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{k}F_{k,n+3}) \\ &\quad \pm (F_{k,m} + \mathbf{i}F_{k,m+1} + \mathbf{j}F_{k,m+2} + \mathbf{k}F_{k,m+3}) \\ &= (F_{k,n} \pm F_{k,m}) + \mathbf{i}(F_{k,n+1} \pm F_{k,m+1}) \\ &\quad + \mathbf{j}(F_{k,n+2} \pm F_{k,m+2}) + \mathbf{k}(F_{k,n+3} \pm F_{k,m+3}). \end{aligned} \tag{10}$$

Multiplication of two hyperbolic k-Fibonacci quaternions is defined by

$$\begin{aligned} &\mathbb{H}F_{k,n}\mathbb{H}F_{k,m} = (F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3}) \\ &\quad (F_{k,m} + iF_{k,m+1} + jF_{k,m+2} + kF_{k,m+3}) \\ &= (F_{k,n}F_{k,m} + F_{k,n+1}F_{k,m+1} + F_{k,n+2}F_{k,m+2} \\ &\quad + F_{k,n+3}F_{k,m+3}) \\ &\quad + i(F_{k,n}F_{k,m+1} + F_{k,n+1}F_{k,m} + F_{k,n+2}F_{k,m+3} \\ &\quad - F_{k,n+3}F_{k,m+2}) \\ &\quad + j(F_{k,n}F_{k,m+2} + F_{k,n+2}F_{k,m} - F_{k,n+1}F_{k,m+3} \\ &\quad + F_{k,n+3}F_{k,m+1}) \\ &\quad + k(F_{k,n}F_{k,m+3} + F_{k,n+3}F_{k,m} + F_{k,n+1}F_{k,m+2} \\ &\quad - F_{k,n+2}F_{k,m+1}) \\ &\neq \mathbb{H}F_{k,m}\mathbb{H}F_{k,n}. \end{aligned} \tag{11}$$

The scaler and the vector parts of hyperbolic k-Fibonacci quaternion $\mathbb{H}F_{k,n}$ are denoted by

$$\begin{aligned} &S_{\mathbb{H}F_{k,n}} = F_{k,n} \\ &V_{\mathbb{H}F_{k,n}} = iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3}. \end{aligned} \tag{12}$$

Thus, hyperbolic k-Fibonacci quaternion $\mathbb{H}F_{k,n}$ is given by $\mathbb{H}F_{k,n} = S_{\mathbb{H}F_{k,n}} + V_{\mathbb{D}F_{k,n}}$. The conjugate of hyperbolic k-Fibonacci quaternion $\mathbb{H}F_{k,n}$ is denoted by $\overline{\mathbb{H}F}_{k,n}$ and it is

$$\overline{\mathbb{H}F}_{k,n} = F_{k,n} - iF_{k,n+1} - jF_{k,n+2} - kF_{k,n+3}. \tag{13}$$

The norm of hyperbolic k-Fibonacci quaternion $\mathbb{H}F_{k,n}$ is defined as follows

$$\begin{aligned} \|\mathbb{H}F_{k,n}\|^2 &= \mathbb{H}F_{k,n}\overline{\mathbb{H}F}_{k,n} \\ &= F_{k,n}^2 - F_{k,n+1}^2 - F_{k,n+2}^2 - F_{k,n+3}^2. \end{aligned} \tag{14}$$

In the following theorem, some properties related to hyperbolic k-Fibonacci quaternions are given.

Theorem 1: Let $F_{k,n}$ and $\mathbb{H}F_{k,n}$ be the n-th terms of k-Fibonacci sequence $(F_{k,n})$ and hyperbolic k-Fibonacci quaternion $(\mathbb{H}F_{k,n})$, respectively. In this case, for $n \geq 1$ we can give the following relations:

$$\mathbb{H}F_{k,n+2} = k\mathbb{H}F_{k,n+1} + \mathbb{H}F_{k,n} \tag{15}$$

$$\mathbb{H}F_{k,n}^2 = 2F_{k,n}\mathbb{H}F_{k,n} - \mathbb{H}F_{k,n}\overline{\mathbb{H}F}_{k,n} \tag{16}$$

$$\begin{aligned} &\mathbb{H}F_{k,n} - i\mathbb{H}F_{k,n+1} - j\mathbb{H}F_{k,n+2} - k\mathbb{H}F_{k,n+3} \\ &= F_{k,n} - F_{k,n+2} - F_{k,n+4} - F_{k,n+6} \end{aligned} \tag{17}$$

Proof: (15): By the equation (8) we get,

$$\begin{aligned}&\mathbb{H}F_{k,n}+k\mathbb{H}F_{k,n+1}\\=&(F_{k,n}+iF_{k,n+1}+jF_{k,n+2}+kF_{k,n+3})\\&+k(F_{k,n+1}+iF_{k,n+2}+jF_{k,n+3}+kF_{k,n+4})\\=&(F_{k,n}+kF_{k,n+1})+i(F_{k,n+1}+kF_{k,n+2})\\&+j(F_{k,n+2}+kF_{k,n+3})+k(F_{k,n+3}+kF_{k,n+4})\\=&F_{k,n+2}+iF_{k,n+3}+jF_{k,n+4}+kF_{k,n+5}\\=&\mathbb{H}F_{k,n+2}.\end{aligned}$$

(16): By the equation (7) we get,

$$\begin{aligned}\mathbb{H}F_{k,n}^2=&(F_{k,n}^2+F_{k,n+1}^2+F_{k,n+2}^2+F_{k,n+3}^2)\\&+2F_{k,n}(iF_{k,n+1}+jF_{k,n+2}+kF_{k,n+3})\\=&2F_{k,n}\mathbb{H}F_{k,n}-2F_{k,n}^2\\&+(F_{k,n}^2+F_{k,n+1}^2+F_{k,n+2}^2+F_{k,n+3}^2)\\=&2F_{k,n}\mathbb{H}F_{k,n}-\mathbb{H}F_{k,n}\overline{\mathbb{H}F}_{k,n}.\end{aligned}$$

(17): By the equation (7) we get,

$$\begin{aligned}&\mathbb{H}F_{k,n}-i\mathbb{H}F_{k,n+1}-j\mathbb{H}F_{k,n+2}-k\mathbb{H}F_{k,n+3}\\&=F_{k,n}-F_{k,n+2}-F_{k,n+4}-F_{k,n+6}.\end{aligned}$$ □

Theorem 2: For $m\geq n+1$ the d'Ocagne's identity for hyperbolic k-Fibonacci quaternions $\mathbb{H}F_{k,m}$ and $\mathbb{H}F_{k,n}$ is given by

$$\begin{aligned}&\mathbb{H}F_{k,m}\,\mathbb{H}F_{k,n+1}-\mathbb{H}F_{k,m+1}\,\mathbb{H}F_{k,n}\\&=(-1)^n\,[0-2\,\mathbf{i}\,F_{k,m-n-1}+2\,\mathbf{j}\,F_{k,m-n-2}\\&+\mathbf{k}\,(L_{k,m-n}+(k^3+3k)\,F_{k,m-n})].\end{aligned}\tag{18}$$

Proof: (18): By using (7)

$$\begin{aligned}&\mathbb{H}F_{k,m}\,\mathbb{H}F_{k,n+1}-\mathbb{H}F_{k,m+1}\,\mathbb{H}F_{k,n}\\&=(F_{k,m}\,F_{k,n+1}-F_{k,m+1}\,F_{k,n})\\&+(F_{k,m+1}\,F_{k,n+2}-F_{k,m+2}\,F_{k,n+1})\\&+(F_{k,m+2}\,F_{k,n+3}-F_{k,m+3}\,F_{k,n+2})\\&+(F_{k,m+3}\,F_{k,n+4}-F_{k,m+4}\,F_{k,n+3})\\&+\mathbf{i}\,[(F_{k,m}\,F_{k,n+2}-F_{k,m+1}\,F_{k,n+1})\\&+(F_{k,m+1}\,F_{k,n+1}-F_{k,m+2}\,F_{k,n})\\&+(F_{k,m+2}\,F_{k,n+4}-F_{k,m+3}\,F_{k,n+3})\\&-(F_{k,m+3}\,F_{k,n+3}-F_{k,m+4}\,F_{k,n+2})]\\&+\mathbf{j}\,[(F_{k,m}\,F_{k,n+3}-F_{k,m+1}\,F_{k,n+2})\\&-(F_{k,m+1}\,F_{k,n+4}-F_{k,m+2}\,F_{k,n+3})\\&+(F_{k,m+2}\,F_{k,n+1}-F_{k,m+3}\,F_{k,n})\\&+(F_{k,m+3}\,F_{k,n+2}-F_{k,m+4}\,F_{k,n+1})]\\&+\mathbf{k}\,[(F_{k,m}\,F_{k,n+4}-F_{k,m+1}\,F_{k,n+3})\\&+(F_{k,m+1}\,F_{k,n+3}-F_{k,m+2}\,F_{k,n+2})\\&-(F_{k,m+2}\,F_{k,n+2}-F_{k,m+3}\,F_{k,n+1})\\&+(F_{k,m+3}\,F_{k,n+1}-F_{k,m+4}\,F_{k,n})]\\&=(-1)^n\,[0-2\,\mathbf{i}\,F_{k,m-n-1}+2\,\mathbf{j}\,F_{k,m-n-2}\\&+\mathbf{k}\,(L_{k,m-n}+(k^3+3k)\,F_{k,m-n})].\end{aligned}$$

Here, d'Ocagne's identity of k-Fibonacci number $F_{k,m}F_{k,n+1}-F_{k,m+1}F_{k,n}=(-1)^n\,F_{k,m-n}$ in Falcon and Plaza [15] was used. □

Theorem 3: For $n,m\geq 0$ the Honsberger identity for hyperbolic k-Fibonacci quaternions $\mathbb{H}F_{k,n}$ and $\mathbb{H}F_{k,m}$ is given by

$$\begin{aligned}&\mathbb{H}F_{k,n+1}\mathbb{H}F_{k,m}+\mathbb{H}F_{k,n}\mathbb{H}F_{k,m-1}\\&=2\mathbb{H}F_{k,n+m}+kF_{k,n+m+1}+L_{k,n+m+5}.\end{aligned}\tag{19}$$

Proof: (19) By using (11)

$$\begin{aligned}\mathbb{H}F_{k,n+1}\mathbb{H}F_{k,m}=&(F_{k,n+1}F_{k,m}+F_{k,n+2}F_{k,m+1}\\&+F_{k,n+3}F_{k,m+2}+F_{k,n+4}F_{k,m+3})\\&+\mathbf{i}(F_{k,n+1}F_{k,m+1}+F_{k,n+2}F_{k,m}\\&+F_{k,n+3}F_{k,m+3}-F_{k,n+4}F_{k,m+2})\\&+\mathbf{j}(F_{k,n+1}F_{k,m+2}-F_{k,n+2}F_{k,m+3}\\&+F_{k,n+3}F_{k,m}+F_{k,n+4}F_{k,m+1})\\&+\mathbf{k}(F_{k,n+1}F_{k,m+3}+F_{k,n+2}F_{k,m+2}\\&-F_{k,n+3}F_{k,m+1}+F_{k,n+4}F_{k,m}),\\\mathbb{H}F_{k,n}\mathbb{H}F_{k,m-1}=&(F_{k,n}F_{k,m-1}+F_{k,n+1}F_{k,m}\\&+F_{k,n+2}F_{k,m+1}+F_{k,n+3}F_{k,m+2})\\&+\mathbf{i}(F_{k,n}F_{k,m}+F_{k,n+1}F_{k,m-1}\\&+F_{k,n+2}F_{k,m+2}-F_{k,n+3}F_{k,m+1})\\&+\mathbf{j}(F_{k,n}F_{k,m+1}-F_{k,n+1}F_{k,m+2}\\&+F_{k,n+2}F_{k,m-1}+F_{k,n+3}F_{k,m})\\&+\mathbf{k}(F_{k,n}F_{k,m+2}+F_{k,n+1}F_{k,m+1}\\&-F_{k,n+2}F_{k,m}+F_{k,n+3}F_{k,m-1}).\end{aligned}$$

Finally, adding by two sides to the side, we obtain

$$\begin{aligned}&\mathbb{H}F_{k,n+1}\mathbb{H}F_{k,m}+\mathbb{H}F_{k,n}\mathbb{H}F_{k,m-1}\\=&(F_{k,n+m}+F_{k,n+m+2}\\&+F_{k,n+m+4}+F_{k,n+m+6})\\&+2\mathbf{i}F_{k,n+m+1}+2\mathbf{j}F_{k,n+m+2}\\&+2\mathbf{k}F_{k,n+m+3}\\=&2(F_{k,n+m}+\mathbf{i}F_{k,n+m+1}+\mathbf{j}F_{k,n+m+2}\\&+\mathbf{k}F_{k,n+m+3})-F_{k,n+m}+F_{k,n+m+2}\\&+L_{k,n+m+5})\\=&2\mathbb{H}F_{k,n+m}+kF_{k,n+m+1}+L_{k,n+m+5}.\end{aligned}$$

Here, the Honsberger identity of k-Fibonacci number $F_{k,n+1}F_{k,m}+F_{k,n}F_{k,m-1}=F_{k,n+m}$ in Falcon and Plaza [16] and $F_{k,n+1}+F_{k,n-1}=L_{k,n}$ [11] was used. □

Theorem 4: Let $\mathbb{H}F_{k,n}$ and $\mathbb{H}L_{k,n}$ be n-th terms of hyperbolic k-Fibonacci quaternion $(\mathbb{H}F_{k,n})$ and hyperbolic k-Lucas quaternion $(\mathbb{H}L_{k,n})$, respectively. The following relation is satisfied

$$\mathbb{H}F_{k,n+1}+\mathbb{H}F_{k,n-1}=\mathbb{H}L_{k,n}.\tag{20}$$

and

$$\mathbb{H}F_{k,n+2}-\mathbb{H}F_{k,n-2}=k\mathbb{H}L_{k,n}.\tag{21}$$

Proof: (20) From equation (8) and identity $F_{k,n+1}+F_{k,n-1}=L_{k,n},\ n\geq 1$ Ramirez [11] between k-Fibonacci number and k-Lucas number, it follows that

$$\begin{aligned}&\mathbb{H}F_{k,n+1}+\mathbb{H}F_{k,n-1}\\=&(F_{k,n+1}+F_{k,n-1})+\mathbf{i}(F_{k,n+2}+F_{k,n})\\&+\mathbf{j}(F_{k,n+3}+F_{k,n+1})+\mathbf{k}(F_{k,n+4}+F_{k,n+2})\\=&(L_{k,n}+\mathbf{i}L_{k,n+1}+\mathbf{j}L_{k,n+2}+\mathbf{k}L_{k,n+3})\\=&\mathbb{H}L_{k,n}.\end{aligned}$$

(21) From equation (7) and identity $F_{k,n+2}-F_{k,n-2}=kL_{k,n},\ n\geq 1$ between k-Fibonacci number and k-Lucas number, it follows that

$$\begin{aligned}&\mathbb{H}F_{k,n+2}-\mathbb{H}F_{k,n-2}\\=&(F_{k,n+2}-F_{k,n-2})+\mathbf{i}(F_{k,n+3}-F_{k,n-1})\\&+\mathbf{j}(F_{k,n+4}-F_{k,n})+\mathbf{k}(F_{k,n+5}-F_{k,n+1})\\=&(kL_{k,n}+\mathbf{i}kL_{k,n+1}+\mathbf{j}kL_{k,n+2}+\mathbf{k}kL_{k,n+3})\\=&k\mathbb{H}L_{k,n}.\end{aligned}$$

□

Theorem 5: Let $\overline{\mathbb{H}F}_{k,n}$ be conjugation of hyperbolic k-Fibonacci quaternion ($\mathbb{H}F_{k,n}$). In this case, we can give the following relations between these quaternions:

$$\mathbb{H}F_{k,n}+\overline{\mathbb{H}F}_{k,n}=2F_{k,n} \tag{22}$$

$$\begin{aligned}&\mathbb{H}F_{k,n}\overline{\mathbb{H}F}_{k,n}+\mathbb{H}F_{k,n-1}\overline{\mathbb{H}F}_{k,n-1}\\&=F_{k,2n-1}-F_{k,2n+1}-F_{k,2n+3}-F_{k,2n+5}\end{aligned} \tag{23}$$

Proof: (22): By using (14), we get

$$\begin{aligned}&\mathbb{H}F_{k,n}+\overline{\mathbb{H}F}_{k,n}\\=&(F_{k,n}+\mathbf{i}F_{k,n+1}+\mathbf{j}F_{k,n+2}+\mathbf{k}F_{k,n+3})\\&+(F_{k,n}-\mathbf{i}F_{k,n+1}-\mathbf{j}F_{k,n+2}-\mathbf{k}F_{k,n+3})\\=&2F_{k,n}.\end{aligned}$$

(23): By using (15), we get

$$\begin{aligned}&\mathbb{H}F_{k,n}\overline{\mathbb{H}F}_{k,n}+\mathbb{H}F_{k,n-1}\overline{\mathbb{H}F}_{k,n-1}\\=&(F_{k,n}^2+F_{k,n-1}^2)-(F_{k,n+1}^2+F_{k,n}^2)\\&-(F_{k,n+2}^2+F_{k,n+1}^2)-(F_{k,n+3}^2+F_{k,n+2}^2)\\=&F_{k,2n-1}-F_{k,2n+1}-F_{k,2n+3}-F_{k,2n+5}.\end{aligned}$$

where the identity of k-Fibonacci number $F_{k,n}^2+F_{k,n+1}^2=F_{k,2n+1},\ n\geq 0$ Ramirez [11] was used. □

Theorem 6: Let $\mathbb{H}F_{k,n}$ be hyperbolic k-Fibonacci quaternion. Then, we have the following identities

$$\sum_{s=1}^{n}\mathbb{H}F_{k,s}=\frac{1}{k}(\mathbb{H}F_{k,n+1}+\mathbb{H}F_{k,n}-\mathbb{H}F_{k,1}-\mathbb{H}F_{k,0}), \tag{24}$$

$$\sum_{s=1}^{n}\mathbb{H}F_{k,2s-1}=\frac{1}{k}(\mathbb{H}F_{k,2n}-\mathbb{H}F_{k,0}), \tag{25}$$

$$\sum_{s=1}^{n}\mathbb{H}F_{k,2s}=\frac{1}{k}(\mathbb{H}F_{k,2n+1}-\mathbb{H}F_{k,1}). \tag{26}$$

Proof: (24): Since $\sum_{i=1}^{n}F_{k,i}=\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)$ Falcon and Plaza [16], we get

$$\begin{aligned}\sum_{s=1}^{n}\mathbb{H}F_{k,s}=&\sum_{s=1}^{n}F_{k,s}+\mathbf{i}\sum_{s=1}^{n}F_{k,s+1}+\mathbf{j}\sum_{s=1}^{n}F_{k,s+2}\\&+\mathbf{k}\sum_{s=1}^{n}F_{k,s+3}\\=&\frac{1}{k}\{(F_{k,n+1}+F_{k,n}-1)\\&+\mathbf{i}(F_{k,n+2}+F_{k,n+1}-k-1)\\&+\mathbf{j}[F_{k,n+3}+F_{k,n+2}-(k^2+1)-k]\\&+\mathbf{k}[F_{k,n+4}+F_{k,n+3}-(k^3+2k)-(k^2+1)]\}\\=&\frac{1}{k}\{(F_{k,n+1}+\mathbf{i}F_{k,n+2}+\mathbf{j}F_{k,n+3}+\mathbf{k}F_{k,n+4})\\&+(F_{k,n}+\mathbf{i}F_{k,n+1}+\mathbf{j}F_{k,n+2}+\mathbf{k}F_{k,n+3})\\&-[1+\mathbf{i}(k+1)+\mathbf{j}(k^2+k+1)+\mathbf{k}(k^3+k^2+2k+1)]\}\\=&\frac{1}{k}\{\mathbb{H}F_{k,n+1}+\mathbb{H}F_{k,n}\\&-[F_{k,1}+\mathbf{i}(F_{k,2}+F_{k,1})+\mathbf{j}(F_{k,3}+F_{k,2})\\&+\mathbf{k}(F_{k,4}+F_{k,3})]\}\\=&\frac{1}{k}(\mathbb{H}F_{k,n+1}+\mathbb{H}F_{k,n}-\mathbb{H}F_{k,1}-\mathbb{H}F_{k,0}).\end{aligned}$$

(25): Using $\sum_{i=1}^{n}F_{k,2i-1}=\frac{1}{k}F_{k,2n}$, Falcon and Plaza [16], we get

$$\begin{aligned}&\sum_{s=1}^{n}\mathbb{H}F_{k,2s-1}\\=&\frac{1}{k}\{(F_{k,2n})+\mathbf{i}(F_{k,2n+1}-1)+\mathbf{j}(F_{k,2n+2}-k)\\&+\mathbf{k}[F_{k,2n+3}-(k^2+1)]\}\\=&\frac{1}{k}\{[F_{k,2n}+\mathbf{i}F_{k,2n+1}+\mathbf{j}F_{k,2n+2}+\mathbf{k}F_{k,2n+3}]\\&-[\mathbf{i}+k\mathbf{j}+(k^2+1)\mathbf{k}]\}\\=&\frac{1}{k}\{\mathbb{H}F_{k,2n}-(F_{k,0}+\mathbf{i}F_{k,1}+\mathbf{j}F_{k,2}+\mathbf{k}F_{k,3})\}\\=&\frac{1}{k}(\mathbb{H}F_{k,2n}-\mathbb{H}F_{k,0}).\end{aligned}$$

(26): Using $\sum_{i=1}^{n}F_{k,2i}=\frac{1}{k}(F_{k,2n+1}-1)$ Falcon and Plaza [16], we obtain

$$\begin{aligned}&\sum_{s=1}^{n}\mathbb{H}F_{k,2s}\\=&\frac{1}{k}\{(F_{k,2n+1}-1)+\mathbf{i}[F_{k,2n+2}-k]\\&+\mathbf{j}[F_{2n+3}-(k^2+1)]+\mathbf{k}[F_{k,2n+4}-(k^3+2k)]\}\\=&\frac{1}{k}\{(F_{k,2n+1}+\mathbf{i}F_{k,2n+2}+\mathbf{j}F_{k,2n+3}+\mathbf{k}F_{k,2n+4})\\&-[1+k\mathbf{i}+(k^2+1)\mathbf{j}+(k^3+2k)\mathbf{k}]\}\\=&\frac{1}{k}(\mathbb{H}F_{k,2n+1}-\mathbb{H}F_{k,1}).\end{aligned}$$

□

Theorem 7 (Binet's Formula): Let $\mathbb{H}F_{k,n}$ be hyperbolic k-Fibonacci quaternion. For $n \geq 1$, Binet's formula for these quaternions is as follows Falcon and Plaza [16]:

$$\mathbb{H}F_{k,n} = \frac{1}{\sqrt{k^2+4}}\left(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n\right) \tag{27}$$

where

$$\hat{\alpha} = 1 + \mathbf{i}[(k-\beta)] + \mathbf{j}[(k^2+1) - k\beta] + \mathbf{k}[(k^3+2k) - (k^2+1)\beta],$$

and

$$\hat{\beta} = -1 + \mathbf{i}(\alpha - k) + \mathbf{j}[k\alpha - (k^2+1)] + \mathbf{k}[(k^2+1)\alpha - (k^3+2k)].$$

Proof: The characteristic equation of recurrence relation $\mathbb{H}F_{k,n+2} = k\mathbb{H}F_{k,n+1} + \mathbb{H}F_{k,n}$ is

$$t^2 - kt - 1 = 0.$$

The roots of this equation are

$$\alpha = \frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta = \frac{k-\sqrt{k^2+4}}{2}$$

where $\alpha+\beta = k$, $\alpha-\beta = \sqrt{k^2+4}$, $\alpha\beta = -1$.

Using recurrence relation and initial values $\mathbb{H}F_{k,0} = (0,1,k,k^2+1)$,
$\mathbb{H}F_{k,1} = (1,k,k^2+1,k^3+2k)$, the Binet formula for $\mathbb{H}F_{k,n}$ is

$$\mathbb{H}F_{k,n} = A\alpha^n + B\beta^n = \frac{1}{\sqrt{k^2+4}}\left[\hat{\alpha}\alpha^n - \hat{\beta}\beta^n\right],$$

where $A = \dfrac{\mathbb{H}F_{k,1} - \beta\mathbb{H}F_{k,0}}{\alpha-\beta}$, $B = \dfrac{\alpha\mathbb{H}F_{k,0} - \mathbb{H}F_{k,1}}{\alpha-\beta}$ and $\hat{\alpha} = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$, $\hat{\beta} = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$. □

Theorem 8 (Cassini's Identity): Let $\mathbb{H}F_{k,n}$ be hyperbolic k-Fibonacci quaternion. For $n \geq 1$, Cassini's identity for $\mathbb{H}F_{k,n}$ is as follows:

$$\mathbb{H}F_{k,n-1}\mathbb{H}F_{k,n+1} - (\mathbb{H}F_{k,n})^{\mathbf{2}} = (-1)^n[2k\mathbf{i} + 2(k^2+1)\mathbf{j} + (k^3+2k)\mathbf{k}]. \tag{28}$$

Proof: (28): By using (7) and (11), we get

$$\begin{aligned}
&\mathbb{H}F_{k,n-1}\mathbb{H}F_{k,n+1} - (\mathbb{H}F_{k,n})^2\\
=&[(F_{k,n-1}F_{k,n+1} - F_{k,n}^2)\\
&+(F_{k,n}F_{k,n+2} - F_{k,n+1}^2)\\
&+(F_{k,n+1}F_{k,n+3} - F_{k,n+2}^2)\\
&+(F_{k,n+2}F_{k,n+4} - F_{k,n+3}^2)]\\
&+\mathbf{i}[(F_{k,n-1}F_{k,n+2} - F_{k,n}F_{k,n+1})\\
&-(F_{k,n+1}F_{k,n+4} - F_{k,n+2}F_{k,n+3})]\\
&+\mathbf{j}[(F_{k,n-1}F_{k,n+3} - F_{k,n}F_{k,n+2})\\
&-(F_{k,n}F_{k,n+4} - F_{k,n+1}F_{k,n+3})\\
&+(F_{k,n+1}F_{k,n+1} - F_{k,n+2}F_{k,n})\\
&+(F_{k,n+2}F_{k,n+2} - F_{k,n+3}F_{k,n+1})]\\
&+\mathbf{k}[(F_{k,n-1}F_{k,n+4} - F_{k,n}F_{k,n+3})\\
&+(F_{k,n}F_{k,n+3} - F_{k,n+1}F_{k,n+2})\\
&+(F_{k,n+2}F_{k,n+1} - F_{k,n+3}F_{k,n})]\\
=&(-1)^n[2k\mathbf{i} + 2(k^2+1)\mathbf{j} + (k^3+2k)\mathbf{k}].
\end{aligned}$$

Here, the identity of the k-Fibonacci number $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$, Falcon and Plaza [16] was used. □

Theorem 9 (Catalan's Identity): Let $\mathbb{H}F_{k,n}$ be hyperbolic k-Fibonacci quaternion. For $n \geq 1$, Catalan's identity for $\mathbb{H}F_{k,n}$ is as follows:

$$\mathbb{H}F_{k,n+r-1}\mathbb{H}F_{k,n+r+1} - \mathbb{H}F_{k,n+r}^2 = (-1)^{n+r}[0 + 2k\mathbf{i} + 2(k^2+1)\mathbf{j} + (k^3+2k)\mathbf{k}]. \tag{29}$$

Proof: (29): By using (7) and (11), we get

$$\begin{aligned}
&\mathbb{H}F_{k,n+r-1}\mathbb{H}F_{k,n+r+1} - \mathbb{H}F_{k,n+r}^2\\
=&[(F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2)\\
&+(F_{k,n+r}F_{k,n+r+2} - F_{k,n+r+1}^2)\\
&+(F_{k,n+r+1}F_{k,n+r+3} - F_{k,n+r+2}^2)\\
&+(F_{k,n+r+2}F_{k,n+r+4} - F_{k,n+r+3}^2)]\\
&+i[(F_{k,n+r-1}F_{k,n+r+2} - F_{k,n+r}F_{k,n+r+1})\\
&+(F_{k,n+r+1}F_{k,n+r+4} - F_{k,n+r+2}F_{k,n+r+3})]\\
&+j[(F_{k,n+r-1}F_{k,n+r+3} - F_{k,n+r}F_{k,n+r+2})\\
&+(F_{k,n+r+1}F_{k,n+r+1} - F_{k,n+r+2}F_{k,n+r})\\
&-(F_{k,n+r}F_{k,n+r+4} - F_{k,n+r+1}F_{k,n+r+3})\\
&+(F_{k,n+r+2}F_{k,n+r+2} - F_{k,n+r+3}F_{k,n+r+1})]\\
&+k[(F_{k,n+r-1}F_{k,n+r+4} - F_{k,n+r}F_{k,n+r+3})\\
&+(F_{k,n+r+2}F_{k,n+r+1} - F_{k,n+r+3}F_{k,n+r})\\
&+(F_{k,n+r}F_{k,n+r+3} - F_{k,n+r+1}F_{k,n+r+2})]\\
=&(-1)^{n+r}[0 + 2k\mathbf{i} + 2(k^2+1)\mathbf{j} + (k^3+2k)\mathbf{k}].
\end{aligned}$$

Here, the identity of the k-Fibonacci number $F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2 = (-1)^{n+r}$, Falcon and Plaza [16] was used. □

3. Conclusion

In this study, a number of new results on hyperbolic k-Fibonacci quaternions are derived. I hope that these results will be important in applied mathematics, quantum physics and kinematics.

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