A Finite Power Prime Group and Some Applications for its Conjugacy Classes

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Abstract: Suppose that $G$ be a non-abelian metacyclic 2-group of positive type and $\Delta G$ be its non-commuting graph. Using the number of conjugacy classes of $G$, we investigate some graph properties of $G$. Also we give explicit formulas for the number of edges, vertices, clique number and chromatic number of $G$. It is shown that the graph $G$ is weakly perfect.

Keywords: graph, positive type, prime power.

1. Introduction

Let $G$ be a finite group and $Z(G)$ be its center. The non-commuting graph, $\Delta G$, of $G$ is defined as the graph whose vertex set is $G\setminus Z(G)$ and two distinct vertices $x$ and $y$ are connected by an edge if and only if $xyx^{-1}y^{-1}\neq 1$. A group $G$ is called metacyclic if it contains a normal cyclic subgroup $N$ such that the quotient group $G/N$ is also cyclic. The $p-$group $G$ is presented in [1] as follows:

$G = \langle a, b : a^{p^m} = 1, b^p = a^k, b = a^t \rangle$,

for some $m,n \geq 0$ and $k, l \in \mathbb{Z}$.

The following theorem gives a classification of 2-groups of positive type of class at least three $G$, which is divided into split and non-split families of non-isomorphic 2-groups [2]. These groups are studied through their centralizers in [3].

Theorem 1 ([2]) If $G$ is a 2-group of nilpotency class at least three and positive type, then $G$ is isomorphic to exactly one group in the following list:

$(1) \ G(2, \alpha, \beta, \epsilon, \gamma, +) \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}, 1 + \gamma < \alpha < 2\gamma$ and $\beta \geq \gamma$;

$(2) \ G(2, \alpha, \beta, \epsilon, \gamma, +) \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\epsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \epsilon, \gamma \in \mathbb{N}, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta$ and $\alpha < \beta + \epsilon$.

These groups have nilpotency class of at least three since $\alpha < 2\gamma$. The case (1) is split since $\epsilon = 0$, and the group of case (2) is non-split since $\alpha < \beta + \epsilon$. The above two 2-groups of positive type are not isomorphic. For brevity, the 2-groups of positive type given in the above cases can be presented as the following definition.

Definition 1 ([2]) Let $G$ be a 2-group for some positive integers $\alpha, \beta, \gamma$ and $\epsilon$. The group is called of positive type if $r = 2^{\alpha-\gamma} - 1$. We shorten the notation to $G = G(2, +)$ for $G(2, \alpha, \beta, \epsilon, \gamma, +)$ 2-groups of positive type of class at least three.

For the graph $\Delta G$, a clique is a set of vertices in $\Delta G$ such that any two vertices are adjacent. The clique number, $\omega(\Delta G)$, of $\Delta G$ is the cardinality of a largest clique in $\Delta G$. The chromatic number $\chi(\Delta G)$ of $\Delta G$ is the smallest number of colors needed to color the vertices of $\Delta G$ such that no two vertices get the same color. A graph $\Delta G$ is said to be weakly perfect if $\omega(\Delta G) = \chi(\Delta G)$. The $\deg(\Delta G)(v)$ of a vertex $v$ of a graph $\Delta G$ is the number of edges incident with $v$. Finally, $\Deg(\Delta G)$ stands for the set of all degrees of vertices of the graph $\Delta G$.

In this work, we obtain the number of conjugacy classes of 2-groups of positive type $G$, denoted by $k(G)$. We also show that if $H$ is a group such that $\omega(\Delta G) = \chi(\Delta G)$, then $G$ and $H$ have the same orders.

Proposition 1 ([4]) Let $|E(\Delta G)|$, be the number of edges in $\Delta G$. Then $|E(\Delta G)| = \frac{1}{2}(|G|^2 - k(G)|G|)$.

2. Some properties of 2-group $G$

In the following lemma some general properties of elements in the group $G(2, +)$ are listed.

Lemma 1 ([5]) Let $G(2, \alpha, \beta, \epsilon, \gamma, +)$ be a 2-group of positive type. If $x, y \in G$ with $x = a^tb^j$ and $y = a^tb^l$, then the following hold in $G$:

$(1) \ b^ja^t = a^t + b^j$

$(2) \ xy = a^{j+t}b^{j+t}$;

$(3) \ x^y = a^{(1-r)^{t}} + a^{r^{t}}b^{j}$;

$(4) \ [x, y] = a^{(1-r)^{t} + s(r^{t} - 1)}$.

The following lemma collects basic information on the order and order of the center for the 2-groups of positive type. This information will be used in the main results.

Theorem 2: Let $G(2, \alpha, \beta, \epsilon, \gamma, +)$ be a 2-group of nilpotency class at least three. Then

$(1) \ |G(2, +)| = 2^{\alpha + \beta}$;

$(2) \ Z(G(2, +)) = \langle a^{2^\epsilon}, b^{2^\gamma} \rangle$;

$(3) \ |Z(G(2, +))| = 2^{2^\gamma - 2^\epsilon}$.

Proof: (1) $G = \langle a \rangle b$ and $\langle a \rangle \cap \langle b \rangle = \langle a^{2^{\alpha-\epsilon}} \rangle$ has order $2^\epsilon$, hence the order of $G$ is $2^{\alpha + \beta}$. Parts (ii) is a straightforward consequence of Lemma 2.1 in [5].
Our next goal is to compute the number of conjugacy classes of groups of type \( G(2,+) \), in split and non-split cases. The same calculations can be applied for both types. We will observe that \( k(G) \) for both cases are the same.

**Theorem 3:** Let \( G(2,+) \) be a split 2-group of nilpotency class at least three. Then

\[
k(G) = |G(2,+)| \left( \frac{3}{2\gamma+1} - \frac{1}{2\gamma+3} \right).
\]

**Proof:** Suppose that \( G\gamma = G_\gamma(2,+,+) = G(2,\alpha,\beta,\gamma,+) \).

If \( z = a^{2\alpha-1} \), then \( z \) is a central element of order 2 and we define the group \( G\gamma = G_\gamma(2,+,+) \). If we let \( a = a(z) \) and \( b = b(z) \), then \( |a| = 2^{\alpha-1} \) and \( |b| = 2^\beta \). Also we have \( [b,a] = a^{\alpha-\gamma}(z) = (a(z))^{2^{\alpha-\gamma}} = a^2^{(\alpha-1)-(\gamma-1)} \). Hence

\[
G\gamma_1 = \langle \bar{a},b \rangle a^{2\alpha-1} = b^2 = 1, [\bar{a},a] = a^{2(\alpha-1)-(\gamma-1)}.
\]

Thus

\[
G\gamma_1(2,\alpha-1,\beta,\gamma-1,+) \geq G\gamma_1(2,+,+) = 2^2.
\]

Using the above consideration we have,

\[
|G_1(2,+)/Z(G_1(2,+) | = 2^{2\alpha-\gamma-1}/2^{2\alpha+\gamma-1} = 2^2.
\]

That is, the central factor group \( \frac{G_1(2,+)}{Z(G_1(2,+))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Using this correspondence together with Lemma 1 and Theorem 2, it can be concluded that \( k(G_\gamma) = 2^\alpha+\beta(1 - \frac{1}{2\gamma+3} + \frac{1}{2\gamma+1}) \) and \( V(G_\gamma) = 2^\alpha+\beta(1 - \frac{1}{2\gamma+3}) \).

**Theorem 5:** By Proposition 5 in [4], the number of edges of graph \( \Delta_G \) is \( 2|E(G_\gamma)| = \langle |G|^2 - k(G)/|G| \rangle \). So, using Theorem 4 and definition of the non-commuting graph the results hold.

In Theorem 6 it will be shown that if \( \mathcal{H} \) is a group such that the non-commuting graph \( \Delta_{G_\gamma} \) be isomorphic to \( \Delta_{G_\gamma} \), then the orders of \( \mathcal{H} \) and \( \mathcal{H} \) are identical.

**Theorem 7:** Suppose \( \sigma : \Delta_G \rightarrow \Delta_{G_\gamma} \) is an isomorphism and \( g_1,g_2 \in G \setminus Z(G) \) such that \( \deg(g_2) = \min \deg(G_\gamma) - \deg(g_1) \). For every \( h_1 \in \sigma(g_1) \) and \( h_{2} \in \sigma(g_2) \) we have \( g_i = g(h_i) \), where \( i = 1,2 \). Recall that for every \( x \in V(G_\gamma) \), \( \deg(x) = |G| - |C_G(x)| \). Clearly, \( \deg(g_i) = \deg(h_i) \). So, using the above results the proof is completed.

**Theorem 8:** Let \( G_\gamma \) be the non-commuting graph of 2-group \( G(2,\alpha,\beta,\epsilon,\gamma,+) \). Then \( \Delta_G \) is weakly perfect.

**Proof:** By Lemma 1, we have \( Z(G(2,+)) = \langle a^{2r}, b^{2s} \rangle \).

Thus \( \langle G(2,+)) : Z(G(2,+)) \rangle = 2^{\alpha+\beta-1} \). Let

\[
\tau = \{ab^j : 0 \leq i,j \leq 2^\gamma - 1 \}
\]

be a set of left transversal of \( Z(G(2,+,+)) \) in \( G(2,+) \) which includes \( 2^\gamma - 1 \) elements of the form, \( a^i, i \neq 0 \) and \( 2^\gamma - 1 \) elements of the form \( b^j \). Also, this set contains \( (2^\gamma - 1)(2^\gamma - 1) \) elements of the form \( ab^j \) in which \( i,j \neq 0 \) and are not equal to zero at the same time. Moreover, the vertices \( ab^i \) and \( ab^j \) are adjacent in the partition \( G/Z(G) \) are adjacent, for \( 0 \leq i,j \leq 2^\gamma - 1 \). Clearly, the vertices \( a^i \) and \( b^j \) are adjacent with \( a^i b^j \), but two vertices \( a^i \) and \( a^j \), also two vertices \( b^i \) and \( b^j \) are not adjacent. Now, the following set:

\[
T_{u,v} = \{a^n,b^n,a^ib^j : 0 \leq i,j \leq 2^\gamma - 1 \},
\]

in which, \( 0 \leq u,v \leq 2^\gamma \) is a clique in the graph with maximum size. Thus, \( |T_{u,v}| = (2^\gamma - 1)(2^\gamma - 1) + 2 = \Delta_G \). These results can be extended to obtain the chromatic number in the graph \( \Delta_G \). \( \chi(\Delta_G) = (2^\gamma - 1)^2 + 2 \) which is shown that \( \Delta_G \) is weakly perfect.

As application, using the exact number of conjugacy classes of quasi dihedral group, \( QD_{2n+1} \), we find the clique number and chromatic number of the related non-commuting graph.

**Example 1:** A class of 2-groups of positive type is the quasi dihedral group, \( QD_{2n+1} \), which is obtained by taking \( \beta = \gamma = 1 \) in Theorem 1. Thus \( G(2,\alpha,1,1,+) \cong \langle a,b | a^{2\alpha} = b^2 = 1, [b,a] = a^{2\alpha-1} = QD_{2n+1} \rangle \). Using Theorem 4 it is seen that the number of conjugacy classes of \( QD_{2n+1} \) equals to: \( k(QD_{2n+1}) = 2^\alpha(\frac{3}{2} - \frac{1}{4}) = 2^\alpha + 2^{\alpha-2} \). Consider the non-commuting graph of \( G(QD_{2n+1}) \), the number of edges of this graph is \( E(\Delta_{QD_{2n+1}}) = 3 \cdot 2^{\alpha+5} \) and the number of vertices of the graph is \( V(\Delta_{QD_{2n+1}}) = 3 \cdot 2^{\alpha-1} \). Therefore, \( \chi(\Delta_{QD_{2n+1}}) = \omega(\Delta_{QD_{2n+1}}) = 3 \).

4. Conclusion

Some formulas for the number of edges and vertices of non-commuting graph of finite non-abelian 2-groups \( G \) have been given. It have been shown that the clique number and chromatic number of the graph are the same.

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