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A Finite Power Prime Group and Some Applications for its Conjugacy Classes

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Abstract: Suppose that $\mathcal{G}$ be a non-abelian metacyclic 2-group of positive type and $\Delta_{\mathcal{G}}$ be its non-commuting graph. Using the number of conjugacy classes of $\mathcal{G}$, we investigate some graph properties of $\mathcal{G}$. Also we give explicit formulas for the number of edges, vertices, clique number and chromatic number of $\mathcal{G}$. It is shown that the graph $\mathcal{G}$ is weakly perfect.

Keywords: graph, positive type, prime power.

1. Introduction

Let $\mathcal{G}$ be a finite group and $Z(\mathcal{G})$ be its center. The non-commuting graph, $\Delta_{\mathcal{G}}$, of $\mathcal{G}$ is defined as the graph whose vertex set is $\mathcal{G}-Z(\mathcal{G})$ and two distinct vertices x and y are connected by an edge if and only if $x^y = xyx^{-1} \neq y$. A group $\mathcal{G}$ is called metacyclic if it contains a normal cyclic subgroup N such that the quotient group $\mathcal{G}/N$ is also cyclic. The $p-$group $\mathcal{G}$ is presented in [1] as follows:

$$\mathcal{G} = \langle a,b : a^{p^m} = 1, b^{p^n} = a^{p^k}, a^b = a^r\rangle,$$

for some $m,n \geq 0$ $r > 0, k \leq p^m, p^m \mid k(r-1)$ and $p^m \mid r^{p^n-1}$.

The following theorem gives a classification of 2-groups of positive type of class at least three $\mathcal{G}$, which is divided into split and non-split families of non-isomorphic 2-groups [2]. These groups are studied through their centralizers in [3].

Theorem 1 ([2]) If $\mathcal{G}$ is a $2-$group of nilpotency class at least three and positive type, then $\mathcal{G}$ is isomorphic to exactly one group in the following list:

(1) $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+) \simeq \langle a,b | a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, where α, β, $\gamma \in \mathbb{N}$, $1+\gamma < \alpha < 2\gamma$ and $\beta \geq \gamma$;

(2) $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+) \simeq \langle a,b | a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\epsilon}}, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, where α, β, γ, $\epsilon \in \mathbb{N}$, $1+\gamma < \alpha < 2\gamma$, $\gamma \leq \beta$ and $\alpha < \beta+\epsilon$.

These groups have nilpotency class of at least three since $\alpha < 2\gamma$. The case (1) is split since $\epsilon = 0$, and the group of case (2) is non-split since $\alpha < \beta+\epsilon$. The above two 2-groups of positive type are not isomorphic. For brevity, the 2-groups of positive type given in the above cases can be presented as the following definition.

Definition 1 ([2]) Let $\mathcal{G}$ be a 2-group for some positive integers α, β, γ and, ϵ . The group is called of positive type if $r = 2^{\alpha-\gamma}+1$. We shorten the notation to $\mathcal{G} = \mathcal{G}(2,+)$ for $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+)$ 2-groups of positive type of class at least three.

For the graph $\Delta_{\mathcal{G}}$, a clique is a set of vertices in $\Delta_{\mathcal{G}}$ such that any two vertices are adjacent. The clique number, $\omega(\Delta_{\mathcal{G}})$, of $\Delta_{\mathcal{G}}$ is the cardinality of a largest clique in $\Delta_{\mathcal{G}}$. The chromatic number $\chi(\Delta_{\mathcal{G}})$ of $\Delta_{\mathcal{G}}$ is the smallest number of colors needed to color the vertices of $\Delta_{\mathcal{G}}$ such that no two vertices get the same color. A graph $\Delta_{\mathcal{G}}$ is said to be weakly perfect if $\omega(\Delta_{\mathcal{G}}) = \chi(\Delta_{\mathcal{G}})$. The $\deg \Delta_{\mathcal{G}}(v)$ of a vertex v of a graph $\Delta_{\mathcal{G}}$ is the number of edges incident with v. Finally, Deg($\Delta_{\mathcal{G}}$) stands for the set of all degrees of vertices of the graph $\Delta_{\mathcal{G}}$.

In this work, we obtain the number of conjugacy classes of 2-groups of positive type $\mathcal{G}$, denoted by $k(\mathcal{G})$. We also show that if $\mathcal{H}$ is a group such that $\omega(\Delta_{\mathcal{G}}) \simeq \chi(\Delta_{\mathcal{G}})$, then $\mathcal{G}$ and $\mathcal{H}$ have the same orders.

Proposition 1 ([4]) Let $|E(\Delta_{\mathcal{G}})|$, be the number of edges in $\Delta_{\mathcal{G}}$. Then $|E(\Delta_{\mathcal{G}})| = \frac{1}{2}(|\mathcal{G}|^2 - k(\mathcal{G})|\mathcal{G}|)$.

2. Some properties of 2-group $\mathcal{G}$

In the following lemma some general properties of elements in the group $\mathcal{G}(2,+)$ are listed.

Lemma 1 ([5]) Let $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+)$ be a 2-group of positive type. If $x,y \in \mathcal{G}$ with $x = a^i b^j$ and $y = a^s b^t$, then the following hold in $\mathcal{G}$.

(1) $b^j a^i = a^{ir^j} b^j$

(2) $xy = a^{i+sr^j} b^{j+t}$;

(3) $x^y = a^{s(1-r^j)+ir^t} b^j$;

(4) $[x,y] = a^{i(1-r^t)+s(r^j-1)}$.

The following lemma collects basic information on the order and order of the center for the 2-groups of positive type. This information will be used in the main results.

Theorem 2: Let $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+)$ be a 2-group of nilpotency class at least three. Then

(1) $|\mathcal{G}(2,+)| = 2^{\alpha+\beta}$;

(2) $Z(\mathcal{G}(2,+)) = \langle a^{2^\gamma}, b^{2^\gamma}\rangle$;

(3) $|Z(\mathcal{G}(2,+))| = 2^{\alpha+\beta-2\gamma}$.

Proof: (1) $G = \langle a\rangle\langle b\rangle$ and $\langle a\rangle \cap \langle b\rangle = \langle a^{2^{\alpha-\varepsilon}}\rangle$ has order 2^ε, then the order of G is $2^{\alpha-\beta}$. Parts (ii) is a straightforward consequence of Lemma 2.1 in [5]. ■

Our next goal is to compute the number of conjugacy classes of groups of type $\mathcal{G}(2,+)$, in split and non-split cases. The same calculations can be applied for both types. We will observe that $k(\mathcal{G})$ for both cases are the same.

Theorem 3: Let $\mathcal{G}(2,+)$ be a split 2-group of nilpotency class at least three. Then

$$k(\mathcal{G}) = |\mathcal{G}(2,+)|(\frac{3}{2^{\gamma+1}} - \frac{1}{2^{2\gamma+1}}).$$

Proof: Suppose that $\mathcal{G}_\gamma = \mathcal{G}_\gamma(2,+) = \mathcal{G}(2,\alpha,\beta,\gamma,+)$. If $z = a^{2^{\alpha-1}}$, then z is a central element of order 2 and we define the group $\mathcal{G}_{\gamma-1} = \mathcal{G}_\gamma/\langle z\rangle$. If we let $\bar{a} = a\langle z\rangle$ and $\bar{b} = b\langle z\rangle$, then $|\bar{a}| = 2^{\alpha-1}$ and $|\bar{b}| = 2^\beta$. Also we have $[\bar{b},\bar{a}] = a^{2^{\alpha-\gamma}}\langle z\rangle = (a\langle z\rangle)^{2^{\alpha-\gamma}} = \bar{a}^{2^{\alpha-\gamma}} = \bar{a}^{2^{(\alpha-1)-(\gamma-1)}}$. Hence

$$\begin{aligned}\mathcal{G}_{\gamma-1} =&\langle \bar{a},\bar{b}|\bar{a}^{2^{\alpha-1}} = \bar{b}^{2^\beta} = 1, [\bar{b},\bar{a}] = \bar{a}^{2^{(\alpha-1)-(\gamma-1)}}\rangle \\ \simeq&\mathcal{G}_{\gamma-1}(2,\alpha-1,\beta,\gamma-1,+).\end{aligned}$$

Using the above consideration we have,

$$|\mathcal{G}_1(2,+)/Z(\mathcal{G}_1(2,+))| = 2^{\alpha+\beta-\gamma+1}/2^{\alpha+\beta-\gamma-1} = 2^2.$$

That is, the central factor group $\frac{\mathcal{G}_1(2,+)}{Z(\mathcal{G}_1(2,+))} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Using this correspondence together with, Lemma 1 and Theorem 2, it can concluded that $k(\mathcal{G}_\gamma) = 2^{\alpha+\beta}(\frac{1}{2^\gamma} + \frac{1}{2^{\gamma+1}} - \frac{1}{2^{2\gamma+1}}) = |\mathcal{G}(2,+)|(\frac{3}{2^{\gamma+1}} - \frac{1}{2^{2\gamma+1}})$, and the proof is complete. ■

Using similar method as in Theorem 4, it is proved that the non-split 2-groups of positive type have the same number of conjugacy classes of split case.

3. Some applications

In the following theorem we calculate the set of edges and vertices in the graph $\Delta_\mathcal{G}$.

Theorem 4: Let $\Delta_\mathcal{G}$ be the non-commuting graph of 2-group $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+)$. Then $E(\Delta_\mathcal{G}) = 4^{\alpha+\beta}(1 - \frac{3}{2^{\gamma+1}} + \frac{1}{2^{2\gamma+1}})$ and $V(\Delta_\mathcal{G}) = 2^{\alpha+\beta}(1 - \frac{1}{2^{2\gamma}})$.

Theorem 5: By Proposition 5 in [4], the number of edges of graph $\Delta_\mathcal{G}$ is $2|E(\Delta_\mathcal{G})| = \big(|\mathcal{G}|^2 - k(\mathcal{G})|\mathcal{G}|\big)$. So, using Theorem 4 and definition of the non-commuting graph the results hold.

In Theorem 6 it will be shown that if $\mathcal{H}$ is a group such that the non-commuting graph $\Delta_\mathcal{G}$ be isomorphic to $\Delta_\mathcal{H}$, then the orders of $\mathcal{G}$ and $\mathcal{H}$ are identical.

Theorem 6: Let $\mathcal{G}(2,+)$ be a 2-group of nilpotency class at least three. If $\mathcal{H}$ is a group such that $\Delta_\mathcal{G} \simeq \Delta_\mathcal{H}$, then $|\mathcal{G}| = |\mathcal{H}|$.

Theorem 7: Suppose $\sigma : \Delta_\mathcal{G} \longrightarrow \Delta_\mathcal{H}$ is an isomorphism and $g_1, g_2 \in \mathcal{G} - Z(\mathcal{G})$ such that $\deg(g_2) = \min\deg(\Delta_\mathcal{G}) - \deg(g_1)$. For every $h_1 \in \sigma(g_1)$ and $h_2 \in \sigma(g_2)$ we have $\deg(g_i) = \deg(h_i)$, where $i = 1,2$. Recall that for every $x \in V(\Delta_\mathcal{G})$, $\deg(x) = |\mathcal{G}| - |C_\mathcal{G}(x)|$. Clearly, $\deg(g_i) = \deg(h_i)$. So, using the above results the proof is completed.

Theorem 8: Let $\Delta_\mathcal{G}$ be the non-commuting graph of 2-group $\mathcal{G}(2,\alpha,\beta,\epsilon,\gamma,+)$. Then $\Delta_\mathcal{G}$ is weakly perfect.

Proof: By Lemma 1, we have $Z(\mathcal{G}(2,+)) = \langle a^{2^\gamma}, b^{2^\gamma}\rangle$. Thus $[(\mathcal{G}(2,+)) : Z((\mathcal{G}(2,+)))] = 2^{\alpha+\beta-1}$. Let

$$\tau = \{a^ib^j : 0 \le i,j \le 2^\gamma - 1\}$$

be a set of left transversal of $Z(\mathcal{G}(2,+))$ in $\mathcal{G}(2,+)$ which includes $2^\gamma - 1$ elements of the form, $a^i, i \neq 0$ and $2^\gamma - 1$ elements of the form b^j . Also, this set contains $(2^\gamma - 1)(2^\gamma - 1)$ elements of the form a^ib^j in which i and j are not equal to zero at the same time. Moreover, the vertices a^ib^j and a^rb^s in the partition $\mathcal{G}/Z(\mathcal{G})$ are adjacent, for $0 \le i,j,r,s \le 2^\gamma - 1$. Clearly, the vertices a^i and b^j are adjacent with a^ib^j, but two vertices a^i and a^r, also two vertices b^j and b^s are not adjacent. Now, the following set:

$$\Upsilon_{u,v} = \{a^u, b^v, a^ib^j : 0 \le i,j \le 2^\gamma - 1\},$$

in which, $0 \le u,v \le 2^\gamma$ is a clique in the graph with maximum size. Thus, $|\Upsilon_{u,v}| = (2^\gamma - 1)(2^\gamma - 1) + 2 = \Delta_\mathcal{G}$. These results can be extended to obtain the chromatic number in the graph $\Delta_\mathcal{G}$, $\chi(\Delta_\mathcal{G}) = (2^\gamma - 1)^2 + 2$ which is shown that $\Delta_\mathcal{G}$ is weakly perfect. ■

As application, using the exact number of conjugacy classes of quasi dihedral group, $QD_{2^{\alpha+1}}$, we find the clique number and chromatic number of the related non-commuting graph.

Example 1: A class of 2-groups of positive type is the quasi dihedral group, $QD_{2^{\alpha+1}}$, which is obtained by taking $\beta = \gamma = 1$ in Theorem 1 Thus $\mathcal{G}(2,\alpha,1,0,1,+) \simeq \langle a,b|a^{2^\alpha} = b^2 = 1, [b,a] = a^{2^{\alpha-1}}\rangle = QD_{2^{\alpha+1}}$. Using Theorem 4 it is seen that the number of conjugacy classes of $QD_{2^{\alpha+1}}$ is equal to: $k(QD_{2^{\alpha+1}}) = 2^\alpha\left(\frac{3}{2} - \frac{1}{4}\right) = 2^\alpha + 2^{\alpha-2}$. Consider the non-commuting graph of $\Delta_{QD_{2^{\alpha+1}}}$, the number of edges of this graph is $E(\Delta_{QD_{2^{\alpha+1}}}) = 3\cdot 2^{2\alpha+5}$ and the number of vertices of the graph is $V(\Delta_{QD_{2^{\alpha+1}}}) = 3\cdot 2^{\alpha-1}$. Therefore, $\chi(\Delta_{QD_{2^{\alpha+1}}}) = \omega(\Delta_{QD_{2^{\alpha+1}}}) = 3$.

4. Conclusion

Some formulas for the number of edges and vertices of non-commuting graph of finite non-abelian 2-groups $\mathcal{G}$ have been given. It have been shown that the clique number and chromatic number of the graph are the same.

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References

[1] B. W. King, "Presentations of metacyclic groups," *Bulletin of the Australian Mathematical Society*, vol. 10, no. 2, pp. 103–131, 1973.

[2] J. R. Beuerle, "An elementary classification of finite metacyclic $p-$groups of class at least three," *Algebra Colloq*, vol. 12, no. 04, pp. 553–562, 2005.

[3] K. Moradipour, "Conjugacy class sizes and $n-$th commutativity degrees of some finite groups," *Compt. Rend. Acad. Bulg. Sci.*, vol. 71, no. 4, pp. 453–459, 2018.

[4] W. Z. A. R. Moghaddamfar, W. J. Shi and A. Zokayi, "On non-commuting graph associated with a finite group," *Siberian Math. J.*, vol. 46, no. 2, pp. 325–332, 2005.

[5] N. S. K. Moradipour and A. Erfanian, “Conjugacy classes and commutativity degree of metacyclic 2-groups,” *Cr. Acad. Bulg. Sci.*, vol. 66, no. 10, pp. 1363–1372, 2013.