On the Commutator Subgroups of Groups of Order $8q$

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Abstract: Let G be a group. For any $a, b \in G$, the element $a^{-1}b^{-1}ab$ is called the commutator of a and b. The commutator $a^{-1}b^{-1}ab$ is denoted by $[a, b]$. If A and B are subsets of G, then $[A,B]$ denotes the subgroup of G generated by $\{[a, b] \mid a \in A, b \in B\}$. The subgroup of G generated by all the commutators in G (that is, the smallest subgroup of G containing all the commutators) is called the derived subgroup, or the commutator subgroup, of G and denoted by $[G,G]$. In this paper, we determine the commutator subgroup G' for groups of order $8q$, where q is an odd prime.

Keywords: Groups of order $8q$, commutator subgroup.

1. Introduction

In 1898, Miller [3] introduced the derived subgroup G' of a group G as the subgroup generated by K(G) = $\{[a, b] \mid a \in A, b \in B\}$ the set of commutators of G. According to Miller, commutators $[a, b]$ were introduced by Dedekind a few years earlier. Commutators can act as a tool in all of group theory. For example, commutators can be used to compute the Schur multiplier, Schur multiplier of a pair and nonabelian tensor squares of groups.

In 1899, Western [4] obtained the classication of groups of order $8q$, where q is an odd prime. Western proved that there are 16 types of groups of order $8q$. Miah [2] showed that a nonabelian group of order $8q$ is isomorphic to exactly one group of the following types.

Theorem 1.1. Let G be a nonabelian group of order $8q$, where q is an odd prime. Then G is isomorphic to exactly one group of the following types.

(2.10.1)

$$\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, ac=ca, bc=cb\rangle.$$

(2.10.2)

$$\langle a,b,c \mid a^4=b^4=c^q=1, b^2=a^2, b^{-1}ab=a^{-1}, ac=ca, bc=cb\rangle$$

(2.10.3) $\langle a,b \mid a^8=b^q=1, a^{-1}ba=b^{-1}\rangle$.

(2.10.4)

$$\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, ac=ca, bcb=c^{-1}\rangle.$$

(2.10.5)

$$\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, a^{-1}c=c^{-1}, bc=cb\rangle.$$

(2.10.6)

$$\langle a,b,c,d \mid a^2=b^2=c^2=d^q=1, ab=ba, ac=ca, bc=cb, ad=da, bc=cb, cdc=d^{-1}\rangle.$$

(2.10.7)

$$\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, ac=ca, bcb=c^{-1}\rangle.$$

(2.10.8)

$$\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, a^{-1}ca=c^{-1}, bc=cb\rangle,$$

$q \equiv 1 \pmod 2$.

(2.10.9)

$$\langle a,b,c \mid a^4=b^4=c^q=1, b^2=a^2, b^{-1}ab=a^{-1}, ac=ca, b^{-1}cb=c^{-1}\rangle$$

(2.10.10)

$$\langle a,b \mid a^8=b^q=1, a^{-1}ba=c^m\rangle,$$

where m is any primitive root of

$$m^4 \equiv 1 \pmod q \text{ and } q \equiv 1 \pmod 4.$$

(2.10.11)

$$\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, a^{-1}ca=c^m, bc=cb\rangle,$$

where m is any

primitive root of $m^4 \equiv 1 \pmod q$ and $q \equiv 1 \pmod 4$.

(2.10.12)

$$\left\langle a,b \mid a^8 = b^q = 1, a^{-1}ba = b^m \right\rangle,$$

where m is any primitive root of

$$m^8 \equiv 1 \pmod q \text{ and } q \equiv 1 \pmod 8.$$

(2.10.13)

$$\langle a,b,c,d \mid a^2 = b^2 = c^2 = d^q = 1, ab = ba,$$
$$ac = ca, bc = cb, ad = da, d^{-1}bd = c, d^{-1}cd = bc\rangle\rangle$$

(2.10.14)

$$\langle a,b,c \mid a^4 = b^4 = c^3 = 1, a^2 = b^2, b^{-1}ab = a^{-1},$$
$$c^{-1}ac = b, c^{-1}bc = ab\rangle$$

(2.10.15)

$$\langle a,b,c \mid a^4 = b^4 = c^3 = 1, bab = a^{-1}, c^{-1}a^2b = b,$$
$$c^{-1}bc = a^2b, a^{-1}ca = c^2a^2b\rangle$$

(2.10.16)

$$\langle a,b,c,d \mid a^2 = b^2 = c^2 = d^7 = 1, ab = ba,$$
$$ac = ca, bc = cb, d^{-1}ad = b,$$
$$d^{-1}bd = c, d^{-1}cd = ab\rangle.\rangle$$

The commutator subgroup of order 8q is computed in the next theorem.

Main Theorem: Let G be a nonabelian group of order 8q, where q is an odd prime. Then for the commutator subgroup of G exactly one of the following holds:

$$G' \cong \begin{cases} \mathbb{Z}_2; G \text{ is of type (1.1.1) and (1.1.2)}, \\ \mathbb{Z}_q; G \text{ is of type (1.1.3)-(1.1.6)} \\ \quad \text{and (1.1.10)-(1.1.12)}, \\ \mathbb{Z}_{2q}; G \text{ is of type (1.1.7)-(1.1.9)}, \\ (\mathbb{Z}_2)^2; G \text{ is of type (1.1.13)}, \\ Q_2; G \text{ is of type (1.1.14)}, \\ A_4; G \text{ is of type (1.1.15)}, \\ (\mathbb{Z}_2)^3; G \text{ is of type (1.1.16)}. \end{cases}$$

2. Basic Definition and Theorems

This section includes some definitions and results on the derived subgroups of nonabelian groups.

Definition 2.1: [2] (**Subgroup of G generated by a set**)

Let G be a group and X a subset of G . Let $\{ H_i \mid i \in I \}$ be the family of all subgroups of G which contains X . Then $\bigcap_{i\in I} H_i$ is called the **subgroup of G generated by the set X,** and is denoted by $<X>$.

Theorem 2.2: [2] Let G be a group and X a non empty subset of G .Then the subgroup $< X >$ generated by X consists of all finite product $a_1^{n_1} a_1^{n_2} \cdots a_t^{n_t}$ ($a_i \in X$, $n_i \in Z$). In particular for every $a \in G$, $< a > = \{ a^n \mid n \in Z \}$.

Definition 2.3: [2] (**Commutator subgroup**)
Let G be a group. The subgroup of G generated by the set $\{ x^{-1}y^{-1}xy \mid x , y \in G \}$ is called the **commutator subgroup** of G and denoted by G'.

Let G be a group and let $G^{(1)}$ be G'. Then for $i \geq 1$,define $G^{(i)} = G^{(i-1)'}$. The notation $G^{(i)}$ is called the ith derived subgroup of G. This gives a sequence of subgroups of G, each normal in preceding one: $G > G^{(1)} > G^{(2)} > \cdots$. Actually each $G^{(i)}$ is a normal subgroup of G .

3. The Proof of Main Theorem

Let G be a nonabelian group of order 8q, where q is an odd prime. By Theorem 1.1 there are 16 types of these groups in which groups include 2, 3 or 4-generators. Since computing the commutator subgroup for all 2-generator groups is similar, G' is computed for exactly one type in this family in details. This method is the proceeded for 3, 4-generators, that is, G' is computed for groups of types (1.1.1), (1.1.3) and (1.1.6) including 3, 2 and 4-generators, respectively.

For computing the commutator subgroup for groups of type (1.1.1) by the relations $bab = a^{-1}$; $ac = ca$ and $bc = cb$, every element of G can be uniquely written in the form $a^i b^j c^k$ where $0 \leq i \leq 3$, $0 \leq j \leq 1$ and $0 \leq k \leq q - 1$. Assume that $g = a^i b^j c^k$ and $g' = a^{i'} b^{j'} c^{k'}$ be two arbitrary elements of G. Then

Case 1: $j = j' = 0$.

In this case by the relations, it is clear that $[g, g'] = 1$.

Case 2: $j = 1, j' = 0$.

By the relations $ac = ca$ and $bc = cb$,

$[g, g'] = c^{-k}ba^{-i}c^{-k'}a^{-i'}a^{i}bc^{k}a^{i'}c^{k'} = a^{2i'}$

Case 3: $j = 0$, $j' = 1$.

By similar way, $[g, g'] = a^{2i'}$.
Case 4: $j = j' = 1$.

In this case,
$[g, g'] = c^{-k}ba^{-i}c^{-k'}a^{-i'}a^{i}bc^{k}a^{i'}c^{k'} = a^{2(i-i')}$,

These computations show that the commutator subgroup is isomorphic to a cyclic group of order 2.

For computing the commutator subgroup for groups of type (1.1.3) using the relations $a^{-1}ba = b^{-1}$, it is clear that every element of G can be uniquely written in the form $a^i b^j$, where $0 \leq i \leq 7$, $0 \leq j \leq q-1$. Let $g = a^i b^j$ and $g' = a^{i'} b^{j'}$ be two arbitrary elements of G. Then

$$[g, g'] = b^{j'} a^{i'} b^{-j'} a^{-i'} a^i b^j a^{i'} b^{j'} = b^{-2j'}$$

This computatiom shows that the commutator subgroup is isomorphic to a cyclic group of order q.

For computing the commutator subgroup for groups of type (1.1.6), note that every element of G can be uniquely written in the form $g = a^i b^j c^k d^s$ where $0 \leq i, j, k \leq 1$, $0 \leq s \leq q-1$. Let $g = a^i b^j c^k d^s$ and $g' = a^{i'} b^{j'} c^{k'} d^{s'}$ be two arbitrary elements of G. Then

$$[g,g'] = d^{-s} c^{-k} b^{-j} a^{-i} d^{-s'} c^{-k'} b^{-j'} a^{-i'} a^i b^j c^k d^s a^{i'} b^{j'} c^{k'} d^{s'}$$

The following cases are considered:

Case 1: $k = k' = 0$.

In this case by the relations, it is clear that $[g, g'] = 1$.

Case 2: $k = 1$, $k' = 0$.

By the relations, $[g, g'] = = d^{2s'}$.

Case 3: $k = 0$; $k' = 1$.

By the similar way, $[g, g'] = = d^{2s'}$.

Case 4: $k = k' = 1$.

In this case, $[g, g'] = d^{2(s-s')}$. This computatiom shows that the commutator subgroup is isomorphic to a cyclic group of order q.

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