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A New Version of the Accelerated FRFD Method for Solving Time-Fractional Diffusion Equations

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Abstract: Over the last few years, iterative schemes, derived from the rotated finite difference approximation, have been proven to work well in solving standard diffusion equations. However, its application on time fractional diffusion counterpart still needs to be investigated. In this paper, we have formulated new preconditioned fractional rotated finite difference (PFRFD) method for solving 2D time-fractional diffusion equation. Numerical experiments are conducted to examine the effectiveness of the proposed PFRFD method. The computed results indicate that the new scheme achieves the effect of PFRFD for accelerating the convergence speed.

Keywords: preconditioned fractional rotated finite difference method, time-factional diffusion equation.

1. Introduction

In the last decades, differential equations involving fractional derivatives and integrals have been studied by many researchers. Due to their ability to model more adequately some phenomena, fractional partial differential equations (FPDE's) have been used in numerous areas such as viscoelasticity, finance, hydrology and porous media, engineering, control systems, etc. Furthermore, Fractional Partial Differential Equations (FPDEs) can be seen as a generalization of the classical partial differential equations (FPDEs) in the sense that it takes into account the memory and hereditary properties of the physical phenomena [1, 2]. As it was in the classical FPDEs, there is no general method that can be used in solving FPDEs. Numerical solution of FPDEs has received great progress in the recent years [3, 4]. Fast computational methods for solving partial differential equations using finite difference schemes derived from skewed (rotated) difference operators have been extensively investigated over the years. These iterative methods based on the rotated finite difference approximations have been shown to be much faster than the methods based on the standard five-point formula in solving the partial differential equations which is due to the formers' overall lower computational complexities [5-7]. In Saeed's study [8], the preconditioned iterative method based on rotated finite difference method for solving fractional elliptic partial differential equations was formulated where the preconditioned scheme was shown to have a better rate of convergence compared to its unpreconditioned counterpart.

This work involves an investigation on the utilization of the new Preconditioned Fractional Rotated Finite Difference (PFRFD) method for solving 2D Time-Fractional Diffusion Equations. The paper is organized in five sections: Section 2 describes the formulation of the Fractional Standard Point Iterative Method for solving 2D Time-Fractional Diffusion Equations. In Section 3, the proposed accelerated version of fractional rotated five point's approximation method will be derived. In Section 4, the numerical results are presented in order to show the efficiency of the new proposed method. Finally, the conclusion is given in Section 5.

2. Formulation of Fractional Standard Finite Difference Method

We consider the following time fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \tag{1}$$

where α is the order of the time fractional derivative in Caputo sense which is defined as the following [9]:.

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^\alpha}, \; 0<\alpha<1. \tag{2}$$

Suppose that the domains are constant for both x and y, while the grid dimensions in relation to space and time for the positive integers n and l are respectively represented by $h=\frac{1}{n}$ and $\tau=\frac{T}{l}$. The grid points in the space interval [0,1] are denoted $x_i = ih,\, x_j = jh,\, \{i,j=0,1,\ldots,n\}$ and the grid points for time are designated $t_k = k\tau,\, k=0,1,\ldots,l.$ Discretization with regard to time fractional with utilization of Crank-Nicolson finite difference approximations at $(x_i, y_j, t_{n+1/2})$ is realized through the formula displayed below [10]

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} = \{w_1 u^k + \sum_{s=1}^{k-1}[w_{k-s+1} - w_{k-s}]u_{i,j}^s - w_k u_{i,j}^0 + \sigma\frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}}\} + 0(\tau^{2-\alpha}), \tag{3}$$

where

$$\sigma = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}, \quad w_s = \sigma\{(\frac{s+1}{2})^{1-\alpha} - (\frac{s-1}{2})^{1-\alpha}\}$$

utilization of the standard second order Crank-Nicolson difference scheme with the formula (3) for finite difference discretization of (1) will result in the standard Crank-Nicolson formula portrayed below

$$u_{i,j}^{k+1} = \frac{1}{1+2r}\{\frac{r}{2}[(u_{i+1,j}^{k+1} + u_{i-1,j}^{k+1} + u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1}) + (u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k})] + (1 - 2^{1-\alpha} w_1^* - 2r)u_{i,j}^k + 2^{1-\alpha} \sum_{s=1}^{k-1}[(w_{k-s}^* - w_{k-s+1}^*)u_{i,j}^s + 2^{1-\alpha}\ w_k^* u_{i,j}^0 + m_0\ f_{i,j}^{\frac{k+1}{2}}\}, \quad (4)$$

where

$$m_0 = \tau^\alpha \Gamma(2-\alpha)*2^{1-\alpha},\quad r = \frac{m_0}{h^2},\ w_s^* = [(\frac{s+1}{2})^{1-\alpha} - (\frac{s-1}{2})^{1-\alpha}].$$

3. The proposed Accelerated FRFD Method

The rotated finite difference approximation (achieved through 45^0 degree clockwise rotation of the x-y axis) for equation (1) can be utilized to unveil the following:

$$u_{i,j}^{k+1} = \frac{1}{1+r}\{\frac{r}{4}[(u_{i+1,j+1}^{k+1} + u_{i-1,j-1}^{k+1} + u_{i-1,j+1}^{k+1} + u_{i+1,j-1}^{k+1}) + (u_{i+1,j+1}^{k} + u_{i-1,j-1}^{k} + u_{i-1,j+1}^{k} + u_{i+1,j-1}^{k})] + (1 - 2^{1-\alpha} w_1^* - r)u_{i,j}^k + 2^{1-\alpha} \sum_{s=1}^{k-1}(w_{k-s}^* - w_{k-s+1}^*)u_{i,j}^s + 2^{1-\alpha}\ w_k^* u_{i,j}^0 + m_0\ f_{i,j}^{\frac{k+1}{2}}, \quad (5)$$

where m_0, r and w_s^* are the same as mentioned before in (4).

Usually the resulted system from equation (5) is large and the coefficient matrix A is sparse. Therefore, matrix A can be written as

$$A = D - L - U \quad (6)$$

where D is diagonal matrix A, $-L$ is strictly lower triangular parts of A and $-U$ is strictly upper triangular parts of A.

A preconditioner $(I + PU)$ where $0 \le P < 1.5$ is used to modify the original system into new system that is equivalent in the sense that it has the same solution, but with more favourable spectral properties.

4. Numerical Results and Discussion

In this section we give numerical results for the proposed method applied to two particular examples. The first problem is as the following [11],

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [\Gamma(2+\alpha)t - 2t^{1+\alpha}]e^{x+y} \quad (7)$$

where the solution domain is $\Omega = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 1\}$, with Dirichlet boundary requisites which comply with the exact solution $u(x,y,t) = e^{x+y}\, t^{1+\alpha}$. Ultimately, a Gauss-Sidel method holding a relaxation value equivalent to 1 was applied on a variety of grid dimensions (4, 8,16, 20 and 24) with varying time steps $(\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{20}, \frac{1}{24})$ for $0 < t < 1$. Preconditioned FRFD method were deemed efficient through investigations which revealed their superiority in the context of execution time (measured in seconds), number of iterations (Ite) and maximum absolute error (Max) with tolerance $\varepsilon = 10^{-6}$.

The computer processing unit is Intel(R) Core(TM) i5 with memory of 4Gb and the software used to implement and generate the results was Developer C++ Version 4.9.9.2. The attainment of a solution to an additional test problem (2) [12] substantiates the efficiency of these procedures:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 2t^2)[\sin(x)\sin(y)], \quad 0 < \alpha < 1, \quad (8)$$

with initial and boundary conditions:

$u(x,y,0) = 0,\ \ u(0,y,t) = 0,\ u(x,0,t) = 0$

$u(1,y,t) = t^2 \sin(1)\sin(y),\ \ u(x,1,t) = t^2 \sin(x)\sin(1),$

$0 < t < 1, \qquad 0 < x, y < 1.$

The exact solution is $u(x,y,t) = t^2 \sin(x)\sin(y)$. Numerical data of the original FRFD and the preconditioned systems are summarized in Tables 1, 2, 3, and 4 for problems (1) and (2) with $\alpha = 0.25$ and $\alpha = 0.75$ respectively.

Table 1. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.25$

Problem(1)					
Δt	n	Method	Time	Ite	Max Error
$\frac{1}{4}$	4	FRFD	0.0148	8	7.36E-3
		PFRFD	0.0139	7	7.35E-3
$\frac{1}{8}$	8	FRFD	0.5910	26	2.13E-3
		PFRFD	0.5781	20	2.13E-3
$\frac{1}{16}$	16	FRFD	4.8145	46	8.04E-3
		PFRFD	4.7324	38	8.21E-3
$\frac{1}{20}$	20	FRFD	224.014	72	7.36E-4
		PFRFD	223.832	58	6.22E-4
$\frac{1}{24}$	24	FRFD	244.492	134	2.08E-4
		PFRFD	202.012	104	1.28E-4

Table 2. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.75$

Problem(1)					
Δt	n	Method	Time	Ite	Max Error
$\frac{1}{4}$	4	FRFD	0.0136	8	3.36E-3
		PFRFD	0.0136	7	3.33E-3
$\frac{1}{8}$	8	FRFD	0. 5237	22	1.41E-3
		PFRFD	0.4301	18	1.48E-3
$\frac{1}{16}$	16	FRFD	3.8211	34	5.22E-4
		PFRFD	3.1731	29	4.71E-4
$\frac{1}{20}$	20	FRFD	128.231	72	3.76E-4
		PFRFD	102.038	51	2.61E-4
$\frac{1}{24}$	24	FRFD	142.091	84	3.15E-3
		PFRFD	123.512	67	2.12E-3

It can be observed that the proposed PFRFD method requires less time and iteration number when compared to FRFD with the same levels of precision.

5. Conclusion and Future Work

In this work, we have formulated new preconditioned iterative method based on FRFD method for solving 2D time-fractional diffusion equation. From observation of all experimental results, it can be concluded that the proposed scheme may be a good alternative to solve this type of equations and many other numerical problems. The convergence analysis of the present iterative method regarding solutions for 2D time-fractional diffusion equation is currently under study. Furthermore, the idea of this proposed method can be extended to group iterative solver which will be reported separately in the future.

Table 3. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.25$

Problem(2)					
Δt	n	Method	Time	Ite	Max Error
$\frac{1}{4}$	4	FRFD	0.0132	6	4.81E-4
		PFRFD	0.0132	6	4.81E-4
$\frac{1}{8}$	8	FRFD	0.3682	21	1.22E-4
		PFRFD	0.3221	16	1.22E-4
$\frac{1}{16}$	16	FRFD	3.1051	37	2.09E-5
		PFRFD	2.8626	23	2.38E-5
$\frac{1}{20}$	20	FRFD	98.015	70	2.67E-5
		PFRFD	82.341	50	2.39E-5
$\frac{1}{24}$	24	FRFD	143.018	90	5.01E-5
		PFRFD	96.878	65	4.83E-5

Table 4. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.75$

Problem(2)					
Δt	n	Method	Time	Ite	Max Error
$\frac{1}{4}$	4	FRFD	0.0135	7	1.98E-4
		PFRFD	0.0135	6	1.98E-4
$\frac{1}{8}$	8	FRFD	0.3613	18	1.26E-4
		PFRFD	0.2851	14	1.14E-4
$\frac{1}{16}$	4	FRFD	2.5064	33	1.05E-4
		PFRFD	2.0051	25	1.03E-4
$\frac{1}{20}$	4	FRFD	87.411	52	1.31E-4
		PFRFD	66.725	38	1.28E-4
$\frac{1}{24}$	4	FRFD	117.318	74	1.82E-4
		PFRFD	98.702	58	1.61E-4

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