Journal of Computer Science & Computational Mathematics, Volume 8, Issue 2, June 2018
DOI: 10.20967/jcscm.2018.02.001

Topological Properties like Separation Axioms Satisfying Properly Hereditary Property

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Abstract: In this paper we shall improve the definition of "properly hereditary property" which raised previously [1] and we shall prove that the topological properties: Functionally Hausdorff, $\boldsymbol{T}_{1\frac{2}{3}}$, completely regular, normal, perfectly normal, S_2, $\boldsymbol{S}_\infty$, locally connected, locally strongly connected, totally disconnected and Riesz separation axiom: T_R are properly hereditary properties.

Keywords: hereditary property, proper subspace, completely regular, functionally Hausdorff, Riesz's separation, strongly connected.

1. Introduction

In 1996, Arenas [1] introduced a definition of properly hereditary property as: A property P of a space X is *properly hereditary property* if every proper subspace A of X has the property P, then the whole space X has the property P. The property P is *properly* (*closed*, *open*, *etc*. *respectively*) *hereditary property* if every (closed, open, etc. respectively) proper subspace has the property P, hence the whole space X has the property P. If the property P is hereditary and properly hereditary property, then it is called a *strongly properly hereditary property*.

All proper subspaces of the trivial topology on $X=\{a, b\}$ are T_i and the whole space is not T_i for i=0, $\frac{1}{4}$, $\frac{1}{2}$, 1, $1\frac{1}{3}$, $1\frac{2}{3}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, 5, 6. Also, Al-Bsoul [2, Example 2.1] introduced a topological space where every proper subspace is $T_{\frac{1}{4}}$ and the whole space is not $T_{\frac{1}{4}}$. So, we shall rewrite the definition of proper hereditary property as the next definition.

Definition 1.1. A property P is *properly hereditary property* denoted by *PHP* if there exists a cardinal number μ such that given any topological space X with $|X| \geq \mu$ and all proper subspaces have the property P, then the whole space has the property P.

In this paper, $|A|$ denotes the cardinal number of the set A and I denotes the topological space $([0,1],\tau_u)$.

2. Properties Looks like Separation Axioms

Arenas [1] proved that T_i for all i=0, 1, 2, 3 are properly hereditary properties. In 2003, Al-Bsoul [2] showed that some of non-familiar separation axioms: $T_{\frac{1}{4}}$, $T_{\frac{1}{2}}$, $TT_{1\frac{1}{3}}$, $T_{2\frac{1}{2}}$ are properly hereditary properties. In this section, we shall show that the topological properties: Functionally Hausdorff, $T_{1\frac{2}{3}}$, completely regular, normal, perfectly normal and R_0 are PHPs.

Definition 2.1. [3] A topological space X is *functionally Hausdorff* if for any two distinct points a and b in X, there is a continuous function $f: X \to I$ such that $f(a) = 0$ and $f(b) = 1$.

Theorem 2.1. Functionally Hausdorff is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and all proper subspaces are functionally Hausdorff. Since, all proper subspaces are functionally Hausdorff, then all proper subspaces are T_1, so X is T_1. Assume that x and y are distinct points in X, since $|X| \geq 3$, so there is $z \in X - \{x, y\}$. Hence, X-$\{z\}$ and X-$\{y\}$ are functionally Hausdorff proper subspaces, then there are continuous functions $f_y: X - \{y\} \to I$ and $f_z: X - \{z\} \to I$ such that $f_y(x) = f_z(x) = 1$ and $f_y(z) = f_z(y) = 0$.

Define $f: X \to I$ by $f(t) = f_y(t) f_z(t)$ for all $t \notin \{y, z\}$ and $f(t) = 0$ otherwise. Since f is a multiplication of two continuous functions on $X - \{y, z\}$, so f is a continuous function on $X - \{y, z\}$, hence we shall show that f is a continuous function on $\{y, z\}$. Let $\varepsilon > 0$.

Case 1: $t = y$, so there is an open set U in $X - \{z\}$ containing y such that $f_z(U) \subseteq [0, \varepsilon)$, since X is T_1, then $\{z\}$ is a closed set in X, moreover $X - \{z\}$ is an open set in X, so U is open in X, also $f(u) = f_y(u) f_z(u) < f_z(u) < \varepsilon$ for any $u \neq y$ and $f(y) = 0 < \varepsilon$, then $f(U) \subseteq [0, \varepsilon)$.

Case 2: $t = z$, so similarly for $t = y$.

Thus, f is a continuous function satisfy $f(x) = 1$ and $f(y) = 0$, therefore X is functionally Hausdorff. □

Definition 2.2. [4] A space X is R_0-*space* if any open set is a union of closed sets.

It is easy to see that X is R_0-*space* if for each open set U in X and a in U, there is a closed set A such that $a \in A \subseteq U$.

Theorem 2.2. R_0-*space* is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and all proper subspaces are R_0. Given an open set U in X.

Case 1: $U = X$, we are done.

Case 2: $U \subset X$, then there is $x \notin U$. Let $p \in U$. Since, $|X| \geq 3$, then there is $y \in X - \{x, p\}$

1. $y \in U$. Since, $X - \{y\}$ is a proper R_0-subspace, hence there is a closed set F_y in $X - \{y\}$ such that $p \in F_y \subseteq U - \{y\}$, so there is a closed set $F \subseteq X$ such

that $F_y = F \cap X - \{y\}$, then $p \in F \subseteq F_y \cup \{y\} \subseteq (U - \{y\}) \cup \{y\} \subseteq U$.

2. $y \notin U$, since $X - \{x\}$ and $X - \{y\}$ are proper subspaces, then there are closed sets F_x and F_y in $X - \{x\}$ and $X - \{y\}$, respectively, such that $p \in F_x \subseteq U$ and $p \in F_y \subseteq U$. Moreover, there are closed sets F_1 and F_2 in X such that $F_x = F_1 \cap X - \{x\}$ and $F_y = F_2 \cap X - \{y\}$. Define $F = F_1 \cap F_2$, so $p \in F = F_1 \cap F_2 \subseteq (F_x \cup \{x\}) \cap (F_y \cup \{y\}) = F_x \cap F_y \subseteq U$.

Therefore, the whole space X is R_0-*space*. □

Definition 2.3. A space X is $T_{1\frac{2}{3}}$ if every compact subset of X is closed.

Definition 2.4. [2] A space X is $T_{1\frac{1}{3}}$ if every convergent sequence converges to unique limit point.

It is easy to show that every $T_{1\frac{2}{3}}$ is $T_{1\frac{1}{3}}$. Last idea will help us to show that $T_{1\frac{2}{3}}$ is properly hereditary property.

Theorem 2.3. $T_{1\frac{2}{3}}$ is PHP.

Proof. Let X be a topological space with $|X| \geq 3$. Assume that all proper subspaces of X are $T_{1\frac{2}{3}}$, then each proper subspaces are $T_{1\frac{1}{3}}$, hence X is $T_{1\frac{1}{3}}$. Suppose that F is a nonclosed compact set in X, then there exists $p \notin F$ and $p \in Bd(F)$, so there is a net (x_λ) in F such that $x_\lambda \to p$, since (x_λ) is a net in compact subspace F, then (x_λ) has a convergent subnet (x_{λ_β}) in F to $q \in F$, also X is $T_{1\frac{1}{3}}$, then $p = q$, implies F is closed, therefore X is $T_{1\frac{2}{3}}$. □

Theorem 2.4. The completely regular is PHP.

Proof. Let X be a topological space with $|X| \geq 3$ such that all proper subspaces are completely regular. Assume that A be a closed set in X and $a \notin A$

Case 1: $A^o \neq \emptyset$, then there is $p \in A$ such that p has a neighborhood $U_o \subseteq A$. Since, $X - \{p\}$ is completely regular, hence there is a continuous function $g: X - \{p\} \to I$ such that $g(A - \{p\}) = 0$ and $g(a) = 1$. Define $f: X \to I$ by $f(t) = g(t)$ at $t \neq p$ and $f(p) = 0$. For each $\varepsilon > 0$, then $f^{-1}((\varepsilon, 1]) = g^{-1}((\varepsilon, 1])$ and $f^{-1}([0, \varepsilon)) = g^{-1}([0, \varepsilon)) \cup \{p\}$ are open sets in X.

Case 2: $A^o = \emptyset$, since X is regular, there are two disjoint open sets U and V such that $a \in U$ and $A \subseteq V$, thus U^c is closed, $a \neq U^c$ and $U^{c\,o} \neq \emptyset$, then apply Case 1, there is a continuous function $f: X \to I$ such that $f(U^c) = 0$ and $f(a) = 1$, hence $f(A) = 0$ and $f(a) = 1$, therefore X is completely regular.

The last result shows that $T_{3\frac{1}{2}}$ and all the equivalent properties to $T_{3\frac{1}{2}}$ like gauge space [5] are PHP.

Based on Al-Bsoul [2] T_D is PHP without proof. Next, we will show that T_D is PHP.

A set A is locally closed if $\overline{A} - A$ is closed. A topological space X is T_D if each singleton point is locally closed. □

Theorem 2.5. T_D is PHP.

Proof. Let X be a topological space with $|X| \geq 3$ such that all proper subspaces are T_D. Assume that $x \in X$.

Case 1: If $\overline{\{x\}} = X$, since $|X| \geq 3$, so there are $a, b \in \overline{\{x\}} - \{x\}$. So, $X - \{a\}$ and $X - \{b\}$ are proper T_D subspaces, then $\overline{\{x\}} - \{a, x\}$ and $\overline{\{x\}} - \{b, x\}$ are closed sets in $\overline{\{x\}} - \{a\}$ and $\overline{\{x\}} - \{b\}$ respectively. Hence, there are closed sets of F_a and F_b in X such that $X - \{x, a\} = F_a \cap (X - \{a\})$ and $X - \{x, b\} = F_b \cap X - \{b\}$

1. If $a \in F_a$ or $b \in F_b$, then $F_a = X - \{x\} = \overline{\{x\}} - \{x\}$ or $F_b = X - \{x\} = \overline{\{x\}} - \{x\}$, so $\overline{\{x\}} - \{x\}$ is closed in X.
2. If $a \notin F_a$ and $b \notin F_b$, then $F_a = X - \{a, x\}$ and $F_b = X - \{b, x\}$, thus $F_a \cup F_b = X - \{a, x\} \cup X - \{b, x\} = X - (\{a, x\} \cap \{b, x\}) = X - \{x\}$, so $\overline{\{x\}} - \{x\}$ is closed in X.

Case 2: If $\overline{\{x\}} \neq X$, then $\overline{\{x\}}$ is a proper subspace, so $\overline{\{x\}} - \{x\}$ is closed in $\overline{\{x\}}$, thus it is closed in X. □

Definition 2.5. [3] A space X is *perfectly normal* if for each pair of disjoint closed sets A and B in X there is a continuous function $f: X \to I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. A space X is T_6 if X is a T_1 perfectly normal space.

Al-Bsoul proved that T_4, T_5 and T_6 are properly hereditary properties. Next Theorems show that normal, completely normal and perfectly normal are properly hereditary properties.

Theorem 2.6. Normality is PHP.

Proof. Let X be a topological space with $|X| \geq 4$ and all proper subspaces are normal space. Assume that A and B are disjoint closed sets in X

Case 1: $X = A \cup B$, thus A and B are clopen sets, hence X is normal.

Case 2: There are at least two points p and q in $X - (A \cup B)$, since $X - \{p\}$ is a normal proper subspace, so there are two disjoint open sets U_A^p and U_B^p in $X - \{p\}$, thus there are two open sets U_A and U_B in X such that $U_A^p = U_A \cup X - \{p\}$ and $U_B^p = U_B \cap X - \{p\}$ containing A and B, respectively, furthermore $U_A \cap U_B \subseteq (U_A^p \cap U_B^p) \cup \{p\} \subseteq \{p\}$. Similarly, there are two open sets V_A and V_B in X such that $V_A \cap V_B \subseteq (V_A^q \cap V_B^q) \cup \{q\} \subseteq \{q\}$. Now, $H = U_A \cap V_A$ and $G = U_B \cap V_B$ are disjoint open sets containing A and B, respectively.

Case 3: There is exactly one point p in $X - (A \cup B)$. Since, $|X| \geq 4$, then one of the two sets has at least two distinct points x and y. Without loss of generality, assume that $\{x, y\} \subseteq A$. Since, $X - \{x\}$ is a normal proper subspace, thus there are two disjoint open sets U_A^x and U_B^x in $X - \{x\}$ containing $A - \{x\}$ and B, respectively. Moreover, there exist U_A and U_B open in X such that $U_A^x = U_A \cap X - \{x\}$ and $U_B = U_B^x \cap X - \{x\}$, also $U_A \cap U_B \subseteq (U_A^x \cap U_B^x) \cup \{x\} \subseteq \{x\}$. Similarly, there are two disjoint open sets V_A and V_B in $X -$

$\{y\}$ such that $V_A^y = V_A \cap X - \{y\}$ and $V_B^y = V_B \cap X - \{y\}$, so $V_A \cap V_B \subseteq (V_A^y \cap V_B^y) \cup \{y\} \subseteq \{y\}$. Now, $H = U_A \cup V_A$ and $G = U_B \cap V_B$ are disjoint open sets in X containing A and B, respectively. Therefore, X is normal. □

Corollary 2.7. Completely normal is PHP.

Theorem 2.8. Perfectly normal is PHP.
Proof. Let X be a topological space with $|X| \geq 4$ and let all proper subspaces are perfectly normal. Assume that A and B are disjoint closed sets in X.

Case 1: $A^o \neq \emptyset$ or $B^o \neq \emptyset$. Without loss of generality, assume that $A^o \neq \emptyset$, then there is $p \in A$ has a neighborhood $U_0 \subseteq A$. Since, $X - \{p\}$ is perfectly normal, hence there is a continuous function $g: X - \{p\} \to I$ such that $g^{-1}(0) = A - \{p\}$ and $g^{-1}(1) = B$. Define $f: X \to I$ by $f(t) = g(t)$ for all $x \neq p$ and $f(p) = 0$. For each $\varepsilon > 0$, then $f^{-1}((\varepsilon, 1]) = g^{-1}([\varepsilon, 1))$ and $f^{-1}([0, \varepsilon)) = g^{-1}([0, \varepsilon)) \cup \{p\}$ are open sets in X.

Case 2: $A^o = \emptyset$ and $B^o = \emptyset$, since X is normal, there are two disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$, thus U^c is closed with $U^{co} \neq \emptyset$ and $A \cap U^c = \emptyset$, then apply Case 1, there are continuous functions $f_1, f_2: X \to I$ such that $f_1^{-1}(0) = U^c$, $f_1^{-1}(1) = A$, $f_2^{-1}(0) = B$ and $f_2^{-1}(1) = V^c$, hence there is a continuous function $f: X \to I$ defined by $f(t) = (f_1(t) + f_2(t)/2)$ such that $f^{-1}(1) = A$ and $f^{-1}(0) = B$, therefore X is perfectly normal. □

3. Riesz's Separation Axiom and Related Axioms

Császár [6] introduced Riesz separation axiom and some related Axioms. In this section, we shall show that these separation axioms are properly hereditary properties.

Definition 3.1. [6]
A topological space X is said to be

1. T_R if it is a T_1-space and for a subset A of X, with p and q different elements in A', there is a subset B of A such that $p \in B'$ and $q \notin B'$.
2. S_1 if $p \notin \overline{q}$ implies $q \notin \overline{p}$ for all p, q in X.
3. S_2 if $p \notin \overline{\{q\}}$ implies that p and q have disjoint neighborhoods for all p, q in X.
4. P_R if given any two distinct points p and q satisfy $p \notin \overline{\{q\}}$ and $p \in \overline{A}$ for any subset $A \subseteq X$, there is a subset $B \subseteq A$ such that $p \in \overline{B}$ and $q \notin \overline{B}$.
5. Q_R if given any two distinct points p and q satisfy $q \notin \overline{\{p\}}$ and $p \in \overline{A}$ for any subset $A \subseteq X$, there is a subset $B \subseteq A$ such that $p \in \overline{B}$ and $q \notin \overline{B}$.
6. S_∞ if for any $p \in \overline{A}$ where A is a subset of X and $\overline{A}$ is not compact there is a subset $B \subseteq A$ such that $p \notin \overline{B}$ and $\overline{B}$ is not compact.

Lemma 3.1. [6] Let X be a topological space. X is T_R if given any set $A \subseteq X$ and two distinct points p, $q \in \overline{A}$, there is $B \subseteq A$ such that $p \in \overline{B}$ and $q \notin \overline{B}$.

Theorem 3.2. T_R-space is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and all proper subspaces are T_R. Let $A \subseteq X$ and p, q are two distinct points in $\overline{A}$. Since $|X| \geq 3$, so there is $x \in X - \{p, q\}$. Moreover, $X - \{x\}$ is a T_R proper subspace, then there is $B \subseteq A$ such that $p \in \overline{B}^{X-\{x\}}$ and $q \notin \overline{B}^{X-\{x\}}$, thus $p \in \overline{B}$ and $q \notin \overline{B}$.

It is easy to see that S_1 is PHP. The next Theorem show that S_2 is PHP. □

Theorem 3.3. S_2 is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ such that all proper subspaces are S_2. Assume that $p \neq q$ in X with $p \notin \overline{\{q\}}$, since $|X| \geq 3$, then there is $x \in X - \{p, q\}$.

Case 1: $x \notin \overline{\{q\}}$, hence there are disjoint open sets U_p and U_q in $X - \{x\}$ containing p and q, respectively. Also, there are disjoint open sets V_x and V_q in $X - \{p\}$ containing x and q, respectively. Thus, there are open sets U_1, U_2, V_1 and V_2 in X such that $U_p = U_1 \cap X - \{x\}$, $U_q = U_2 \cap X - \{x\}$, $V_x = V_1 \cap X - \{x\}$ and $V_q = V_2 \cap X - \{x\}$. Now, $U = U_1$ and $V = U_2 \cap V_2$ are disjoint open sets containing p and q, respectively.

Case 2: $x \in \overline{\{q\}}$, then $p \notin \overline{\{x\}}$, since $X - \{x\}$ and $X - \{q\}$ are proper S_2-subspaces, then there are disjoint open sets U_p and U_q in $X - \{x\}$ containing p and q, respectively, and disjoint open sets V_p and V_x in $X - \{q\}$ containing p and x, respectively. Hence, there are open sets U_1, U_2, V_1 and V_2 in X such that $U_p = U_1 \cap X - \{x\}$, $U_q = U_2 \cap X - \{x\}$, $V_p = V_1 \cap X - \{x\}$ and $V_x = V_2 \cap X - \{x\}$. Now, $U = U_1 \cap V_1$ and $V = U_2$ are disjoint open sets containing p, q, respectively. □

Theorem 3.4. P_R-space is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and all proper subspaces are P_R. Let $A \subseteq X$ and p, q be two distinct points in A such that $p \in \overline{A} - \overline{\{q\}}$. Since $|X| \geq 3$, so there is $x \in X - \{p, q\}$. Since $X - \{x\}$ is a proper P_R-subspace, so there is $B_x \subseteq A - \{x\}$ such that $p \in \overline{B_x}^{X-\{x\}}$ and $q \notin \overline{B_x}^{X-\{x\}}$, thus there is a closed set $B \subseteq X$ such that $\overline{B_x} = B \cap X - \{x\}$ and $B \subseteq \overline{B_x} \cup \{x\}$, moreover $q \notin \overline{B_x} \cup \{x\}$, implies $q \notin B$. □

Theorem 3.4. Q_R-space is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and all proper subspaces are Q_R. Let $A \subseteq X$ and $p \neq q$ be distinct points in $\overline{A}$ such that $p \in \overline{A}$ and $q \notin \overline{\{p\}}$. Since $|X| \geq 3$, so there is $x \in X - \{p, q\}$. Since, $X - \{x\}$ is a proper Q_R-space, so there is $B_x \subseteq A - \{x\}$ such that $p \in \overline{B_x}^{X-\{x\}}$ and $q \in \overline{B_x}^{X-\{x\}}$, thus there is a closed set $B \subseteq X$ such that $\overline{B_x} = B \cap X - \{x\}$ and $B \subseteq \overline{B_x} \cup \{x\}$, moreover $q \notin \overline{B_x} \cup \{x\}$, implies $q \notin B$. □

To prove the next result we shall give our next observation.

Lemma 3.6. A space X is compact if there exists a point p in X such that each member in the class $\{U^c : U$ is an open set containing $p\}$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of X. Then, there is $U_p \in \mathcal{U}$ containing p, thus U_p^c is compact and $\mathcal{V} = \mathcal{U} - \{U_p\}$ covering U_p^c, so there is finite subcover $\{V_1, V_2, \dots, V_n\}$ of $\mathcal{V}$, so the class $\{V_1, V_2, \dots, V_n\} \cup \{U_p\}$ covering X, therefore X is compact. □

A similar observation is valid for Lindelöf and countably compact.

Theorem 3.7. S_∞-space is PHP.
Proof. Let X be a topological space such that all proper subspaces are S_∞. Assume that $A \subseteq X$ such that $\overline{A}$ is not compact and $p \in \overline{A}$. In each of the two Cases: $\overline{A} = X$ and $\overline{A} \neq X$, there is B in A such that $p \notin B$ and $\overline{B}$ is not compact according to Lemma (3.6). □

4. Connected Spaces

In this section we shall show the properties of: locally connected, pathwise-connected, strongly connected, locally strongly connected and totally disconnected are PHP.

Definition 4.1. [7] A space X is *locally connected* if X has base consisting of a connected set.

Lemma 4.1. Let X be a topological space and A be a subspace of X. If B is a connected set in A, then B is connected in X.

Arenas showed that local connectedness T_1 is properly open hereditary property. Next, we will show that local connected is properly hereditary property.

Theorem 4.2. Locally connected is PHP.
Proof. Let (X, τ) be a topological space with $|X| \geq 2$ such that every proper subspace is locally connected. Assume that $a \in X$ and U is an open set in X containing a, then there is $b \neq a$, and since $X - \{b\}$ is proper subspace, thus there is an open connected set B in $X - \{b\}$ containing a such that $a \in B \subseteq U - \{b\}$. It implies that B is an open connected set in X according to Lemma (4.1). □

Theorem 4.3. Pathwise connected is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ such that all proper subspaces are pathwise connected. Assume that x and y are two distinct points in X, since $|X| \geq 3$, there is $a \in X - \{x, y\}$, as $X - \{a\}$ is a proper pathwise connected subspace, hence there is a continuous function $g: I \to X - \{a\}$ such that $g(0) = x$ and $g(1) = y$, thus the function $f: I \to X$ defined by $f(t) = g(t)$ for all $t \in I$ is continuous. Therefore, X is pathwise connected. □

Definition 4.2. [8] A space X is *strongly connected* if any continuous function $f: X \to (\mathbb{Z}, \tau_{cof})$ is constant. A subset A of X is strongly connected if A is a strongly connected subspace.

Theorem 4.4. Strongly connected is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ such that all proper subspaces are strongly connected. Suppose that $f: X \to (\mathbb{Z}, \tau_{cof})$ be a continuous function which is not constant, so there exist a and b in X such that $f(a) \neq f(b)$, hence the restriction function on the proper subspace $\{a, b\}$ is a non-constant continuous function, hence the proper subspace $\{a, b\}$ is not strongly connected, contradiction. Therefore, X is strongly connected. □

Definition 4.3 [8] Space X is *locally strongly connected* if it has a basis consisting of strongly connected open sets.

Lemma (4.1) showed a feature for a connected set. It is easy to see that the strongly connected set has the same feature.

Theorem 4.5. Locally strongly connected is PHP.
Proof. Let (X, τ) be a topological space with $|X| \geq 2$ such that every proper subspace is local strongly connected. Assume that $a \in X$ and U be open in X containing a, then there is $q \neq a$, since $X - \{q\}$ is a proper subspace, thus there is an open strongly connected set B containing a and $a \in B \subseteq U - \{q\}$, implies B is strongly connected open in X. □

Definition 4.4. [3] X is *totally disconnected* if the only nonempty connected subsets of X are the one point sets. In other words, X is totally disconnected if all subspaces A with $|A| > 1$ of X are disconnected.

Theorem 4.6. Totally disconnected is PHP.
Proof. Let X be a topological space with $|X| \geq 3$ and each proper subspace is totally disconnected. Thus, each proper subspace A with $|A| > 1$ is disconnected, then there exists $a \in X$ such that $X - \{a\}$ is disconnected, so there are two open sets U_1 and U_2 in X such that $U_1 \cap U_2 \subseteq \{a\}$

Case 1: $a \notin (U_1 \cap U_2)$, then $A = U_1 \cup \{a\}$ is a proper disconnected subspace, so there are two disjoint open sets V_1 and V_2 in A such that $V_1 \cup V_2 = A$, hence $a \in V_1$ or $a \in V_2$. Without loss of generality, assume that $a \in V_2$, thus V_1 is an open set in U_1, then V_1 is an open set in X, since V_2 is an open set in A, so there is an open set G in X such that $V_2 = G \cap A$ and $G \cup U_2 \subseteq V_2 \cup A^c \cup U_2 = V_2 \cup U_2 \subseteq G \cup U_2$, thus $G \cup V_2$ is an open set in X, therefore $V_2 \cup U_2$ and V_1 are open sets in X such that $V_1 \cap (V_2 \cup U_2) = \emptyset$ and $V_1 \cup (V_2 \cup U_2) = X$.

Case 2: $a \in U_1 \cap U_2$, thus $|U_i| > 1$ for every $i = 1,2$, then U_1 is disconnected, equivalently there are two open sets V_1 and V_2 in U_1 such that $V_1 \cup V_2 = U_1$ and $V_1 \cap V_2 = \emptyset$. Since, U_1 is an open set in X, then V_1 and V_2 are open sets in X. Without loss of generality, Assume that $a \in V_2$, implies V_1 and $V_2 \cup U_2$ are open sets in X such that $V_1 \cap (V_2 \cup U_2) = \emptyset$ and $V_1 \cap (V_2 \cup U_2) = X$. So, X is disconnected, therefore X is totally disconnected. □

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