Journal of Computer Science & Computational Mathematics, Volume 7, Issue 4, December 2017
DOI: 10.20967/jcscm.2017.04.002

Remarks on Derivable Maps of Semi-prime Rings

Gurninder S. Sandhu*, Deepak Kumar

Department of Mathematics, Punjabi University, Patiala, Punjab-147002, India
*Corresponding author email: sandhugurninder@gmail.com

Abstract: Let R be an associative ring with center $Z(R)$ and a non-zero ideal I. Let G and F are multiplicative (generalized)-derivations of R together with mappings g and f respectively. In this note, we prove that R contains a non-zero central ideal if any one of the following holds for all x, y in I:

1. $G(xy) \pm F(x)F(y) \pm [x,y] \in Z(R)$
2. $G(xy) \pm F(x)F(y) \pm (xoy) \in Z(R)$
3. $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$
4. $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$

Moreover, we investigate the identities $F(x)F(y) \pm yx \in Z(R)$ and $F(xy) \pm yx \in Z(R)$ over a non-zero left ideal of R and improve some known results.

Keywords: semi-prime ring, prime ring, ideal, multiplicative derivation, multiplicative (generalized)-derivation.

1. Introduction

This paper deals with multiplicative derivations of semi-prime and prime rings in the context of central-values. Recall, a ring R is called prime if (0) is the only prime ideal of R and is called semi-prime if it has no non-zero nilpotent ideal. The symbols $[x, y]$ and xoy denotes the commutator $xy - yx$ and anti-commutator $xy + yx$ respectively. The well-known commutator and anti-commutator identities are: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$ and $(x\ o\ yz) = (x\ o\ y)z - y[x, z] = y(x\ o\ z) + [x, y]z$, $(xy\ o\ z) = x(y\ o\ z) - [x, z]y = (x\ o\ z)y + x[y, z]$. An annihilator of a non-empty subset S of R is a set $A_R(S) = \{a \in R \mid as = 0 = sa$ for all $s \in S\}$. A mapping $f: R \rightarrow R$ is called centralizing (or commuting) on R if $[f(x), x] \in Z(R)$ (or $[f(x), x] = 0$) for all $x \in R$. There has been a significant interest in centralizing and commuting mappings in prime and semi-prime rings (for instance, see [1-3]).

A mapping $d: R \rightarrow R$ is said to be a derivation of R if $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ for all x, y in R. The notion of derivations has been generalized in many ways. In 1991, Daif [4] introduced multiplicative derivations by dropping the condition of additivity in derivation. In [5] Goldmann and Šemrl gave a complete study of multiplicative derivations. Recently, Dhara and Ali [6] initiated the study of a mapping $F: R \rightarrow R$ associated with another map $d: R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, which is called a multiplicative (generalized)–derivation of R. In particular, if $d = 0$ then F is called a multiplicative left multiplier of R. Of course, in this definition both F and d are not necessarily additive. If d is additive then it is called a multiplicative generalized derivation which introduced by Daif and Tammam-El-Sayiad in [7]. It is easy to see that multiplicative (generalized)- derivation looks more appropriate as it covers multiplicative derivation and multiplicative left multiplier of R. Throughout this paper a multiplicative (generalized)-derivation is denoted by an order pair (F, f).

During the last seven decades there has been a large amount of results concerning the conditions that force a ring to be commutative (for example see [1,3,8-10] where further references can be found). In [3] Posner proved a classical result*: If R is a prime ring with a nonzero derivation d on R such that d is centralizing on R, then R is commutative*. This theorem has been generalized in many ways. Towards the commutativity of prime rings with derivations Ashraf *et al.* [9] proved: *Let R be a prime ring and I a nonzero ideal of R. Suppose d be a nonzero derivation of R. If $d(xy) \pm xy \in Z(R)$ for all $x, y \in I$, then R is commutative*. In [2], Ashraf *et al.* extend these results for generalized derivations and obtained the following theorem: *Let R be a prime ring and I a nonzero ideal of R. Suppose F is a generalized derivation associated with a derivation d on R. If one of the following:* (*i*) $F(xy) \pm xy \in Z(R)$, (*ii*) $F(xy) \pm yx \in Z(R)$, (*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds for all $x, y \in I$ then R is commutative*. After that, Atteya [11] studied these situations on semi-prime rings and obtained the following results: *Let R be a semi-prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that any one of the following:* (*i*) $F(xy) \pm xy \in Z(R)$, (*ii*) $F(xy) \pm yx \in Z(R)$, (*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds for all $x, y \in I$, then R contains a nonzero central ideal*.

Apparently, a generalized derivation is a multiplicative (generalized)-derivation but the converse is not necessarily true. Thus, it would be a fact of interest to prove the results of generalized derivation for multiplicative (generalized)-derivations. In this direction many results has been obtained during last five years (see [6, 12-14]). Recently, Tiwari *et al.* [12] studied the following identities involving generalized derivations on some appropriate subsets of prime rings: (i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$, (ii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$, (iii) $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$. In this paper we investigate these identities for multiplicative (generalized)-derivations on semi-prime rings.

2. Main Results

The following facts are of great importance to prove our objectives:

Fact 2.1. [[*10*] ***Lemma 2.***] If R is a prime ring with a non-zero central ideal, then R is commutative.

Fact 2.2. [[*15*] ***Corollary 2.***] If R is a semi-prime ring and I

is an ideal of R, then $I \cap A_R(I) = (0)$.

Fact 2.3. [[*15*] ***Corollary pg no. 7***] Let R be a semi-prime ring and let $I \neq 0$ be a right ideal of R. If I is commutative as a ring, then $I \subseteq Z(R)$.

Fact 2.4. Let R be a semi-prime ring, I a non-zero left ideal of R. If $[I[x,z]_2, z] = 0\ \forall\, x, z \in I$ then $I[I, I] = (0)$.

Proof: By our hypothesis

$$[y[[z, x], z], z] = 0\ \ \forall\ x, y, z \in I \tag{I}$$

On substituting xy for y in (I) , we get $x[y[[z, x], z], z] + [x, z]y[[z, x], z] = 0$ where $x, y, z \in I$. Equation (I) implies that $[x, z]y[[x, z], z] = 0$ where $x, y, z \in I$. Substituting $y = ry$ in last relation, we obtain $[x, z]ry[[x, z], z] = 0$ where $x, y, z \in I$ and $r \in R$. Therefore, $y[[x, z], z]Ry[[x, z], z] = (0)$ where $x, y, z \in I$ and $r \in R$. Semi-primeness of R yields that

$$y[[x, z], z] = 0\ \forall\ x, y, z \in I \tag{II}$$

Linearizing w.r.t.z to obtain

$$y[[x, z], t] + y[[x, t], z] = 0\ \ \forall\ x, y, z, t \in I \tag{III}$$

Replacing z by zt in (III) we have $y[[x, z], t]t + y[z[x, t], t] + yz[[x, t], t] + y[[x, t], z]t = 0$ where $x, y, z, t \in I$. Using (II) and (III) and we get

$$y[z[x, t], t] = 0\ \forall\ x, y, z, t \in I \tag{IV}$$

Putting $z = xz$ in (IV) to obtain $yx[z[x, t], t] + y[x, t]\, z[x, t] = 0$ where $x, y, z, t \in I$. Relation (IV) yields that $y[x, t]\, z[x,t] = 0$ where $x, y, z, t \in I$. On substituting $z=ry$ in last relation to obtain $y[x, t]Ry[x, t] = (0)$ where $x, y, t \in I$. Since R is semi-prime ring we get $I[I, I] = (0)$. ☐

2.1 On Two Sided Ideals

Theorem 2.1.1. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R s.t. $G(xy) + F(x)F(y) - [x, y] \in Z(R)\ \forall\ x, y \in I$, then R contains a non-zero central ideal.

Proof: Let us assume that

$$G(xy) + F(x)F(y) - [x, y] \in Z(R)\ \forall\ x, y \in I \tag{1}$$

Substituting yz for y in (1) to obtain $(G(xy) + F(x)F(y) - [x, y])z + xyg(z) + F(x)yf(z) - y[x, z] \in Z(R)\ \forall\ x, y, z \in I$. On commuting with z, we have

$$[xyg(z), z] + [F(x)yf(z), z] - [y[x, z], z] = 0\ \forall\ x, y, z \in I \tag{2}$$

Replace x by xz in (2) to get

$$[xzyg(z), z] + [F(x)zyf(z), z] + [xf(z)yf(z), z] - [y[x, z]z, z] = 0\ \ \forall\ x, y, z \in I \tag{3}$$

Replace y by zy in (2) and we get

$$[xzyg(z), z] + [F(x)zyf(z), z] - [zy[x, z], z] = 0 \tag{4}$$

Subtract (4) form (3) to obtain

$$[xf(z)yf(z), z] - [[y[x, z], z], z] = 0\ \ \forall\ x, y, z \in I \tag{5}$$

On replacing x by xz in (5) we have

$$[xzf(z)yf(z), z] - [[y[x, z], z], z]z = 0\ \ \forall\ x, y, z \in I \tag{6}$$

Multiply (5) from right side with z and subtract it from (6) to get

$$[x[f(z)yf(z), z], z] = 0\ \ \forall\ x, y, z \in I \tag{7}$$

Substituting $f(z)yf(z)x$ for x in (7) and expand it, by using (7) it reduces to

$$[f(z)yf(z), z]x[f(z)yf(z), z] = 0\ \ \forall\ x, y, z \in I \tag{8}$$

Since I is an ideal of R, we have $x[f(z)yf(z),z]Rx[f(z)yf(z),z] = (0)$ where $x, y, z \in I$. Semi-primeness of R forces that $x[f(z)yf(z), z] = 0$ for all $x, y, z \in I$.

$$xf(z)yf(z)z = xzf(z)yf(z)\ \ \forall\ x, y, z \in I \tag{9}$$

Replacing y by $yf(z)t$ in (9) to get

$$xf(z)yf(z)tf(z)z = xzf(z)yf(z)tf(z)\ \ \forall\ x, y, z, t \in I \tag{10}$$

Using (9) in (10) we have $xf(z)yzf(z)tf(z) = xf(z)yf(z)ztf(z)$ where $x, y, z, t \in I$. That is, $xf(z)y[z, f(z)]tf(z) = 0$ where $x, y, z, t \in I$. It implies that $x[f(z), z]y[f(z), z]t[f(z), z] = 0$ where $x, y, z, t \in I$. In particular, we have $(I[f(z), z])^3 = (0)$ where $z \in I$. Since R is a semi-prime ring, we must have $I[f(z), z] = (0)$ where $z \in I$. Thus, $I[f(z), z] \in A_R(I) \cap I$, hence by Fact 2.2 $[f(z), z] = 0$ where $z \in I$. In this way, Fact 2.2 implicitly states that every non-zero ideal of a semi-prime ring is a semi-prime ring itself.

On replacing y by yz in (2) we have

$$[xyzg(z), z] + [F(x)yzf(z), z]-[yz[x, z], z] = 0\ \forall\ x, y, z \in I \tag{11}$$

Right multiply (2) by z, we obtain

$$[xyg(z), z]z + [F(x)yf(z), z]z-[y[x, z], z]z = 0\ \forall\ x, y, z \in I \tag{12}$$

Subtract (11) from (12) and using the fact that $[f(z), z] = 0$ to get

$$[xy[g(z), z], z] - [y[[x, z], z], z] = 0\ \ \forall\ x, y, z \in I \tag{13}$$

Substitute zy for y in (13), we have

$$[xzy[g(z), z], z] - z[y[[x, z], z], z] = 0\ \ \forall\ x, y, z \in I \tag{14}$$

Multiply (13) from left side by z and subtract it from (14) to obtain

$$[[x, z]y[g(z), z], z] = 0\ \ \forall\ x, y, z \in I \tag{15}$$

Replace x by xt in (15) and expand to get $[[x, z]ty[g(z), z], z] + [x[t, z]y[g(z), z], z] = 0$ where $x, y, z, t \in I$. Relation (15) reduces it to $[x[t, z]y[g(z), z], z] = 0$ where $x, y, z, t \in I$. On expanding it, we obtain $x[[t, z]y[g(z), z], z] + [x, z][t, z]y[g(z), z] = 0$ where $x, y, z, t \in I$. Again using (15) to get

$$[x, z][t, z]y[g(z), z] = 0 \ \forall \ x, y, z, t \in I \quad (16)$$

On substituting $g(z)x$ for x in (16) we have $g(z)[x, z][t, z]y[g(z), z] + [g(z), z]x[t, z]y[g(z), z] = 0$ where $x, y, z, t \in I$. Using (16) we obtain $[g(z), z]x[t, z]y[g(z), z] = 0$ where $x, y, z, t \in I$. Semi-primeness of I yields that

$$[t, z]y[g(z), z] = 0 \ \forall \ y, z, t \in I \quad (17)$$

Replace t by $tg(z)$ in (17) and expand we have $t[g(z), z]y[g(z), z] + [t, z]g(z)y[g(z), z] = 0$ where $y, z, t \in I$. Using (17) to obtain $(I[g(z), z])^2 = (0)$ where $z \in I$. Since R is a semi-prime ring, we have $I[g(z), z] = (0)$ where $z \in I$. Therefore, $[g(z), z] = 0$ for all $z \in I$.

Expression (13) implies that $[y[[x, z], z], z] = 0$ where $x, y, z \in I$. Now by Fact 2.4, it follows that I[I, I] = (0). Since I be an ideal of semi-prime ring R, we have [I, I] = (0). By Fact 2.3, $I \subseteq Z(R)$.

Similarly, we can prove the same conclusions for $G(xy) + F(x)F(y) + [x, y] \in Z(R)$ where $x, y \in I$ with some necessary variations. ☐

Theorem 2.1.2. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) - F(x)F(y) \pm [x, y] \in Z(R) \ \forall \ x, y \in I$, then R contains a non-zero central ideal.

Proof: It is easy to see that if (G, g) is a multiplicative (generalized)-derivation of R then $(-G, -g)$ is so. On replacing (G, g) by $(-G, -g)$ in Theorem 2.1.1, we will obtain the same conclusions for $G(xy) - F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in I$. ☐

Theorem 2.1.3. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) \pm F(x)F(y) \pm yx \in Z(R) \ \forall \ x, y \in I$, then R contains a non-zero central ideal.

Proof: The mapping $(G \mp I_d, g)$ is a multiplicative (generalized) derivation of R provided (G, g) is so. Replace (G, g) by $(G \mp I_d, g)$ in Theorem 2.1.1 and Theorem 2.1.2 where I_d is the identity map of R, to obtain the same conclusions for $G(xy) + F(x)F(y) \pm yx \in Z(R)$ and $G(xy) + F(x)F(y) \pm yx \in Z(R)$ respectively $\forall \ x, y \in I$. ☐

Remark: Note that in our Theorem 2.1.3, if we fix $F = 0$ and R a prime ring, in the light of Fact 2.1, it proves Theorem 2.3 and Theorem 2.4 of [9] for multiplicative (generalized)-derivations. Moreover, if we take $G = 0$ and R a prime ring then it proves Theorem 2.6 of [9] for multiplicative (generalized)-derivations.

Theorem 2.1.4. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) + F(x)F(y) \pm (xoy) \in Z(R) \ \forall \ x, y \in I$, then R contains a non-zero central ideal.

Proof: We consider $G(xy) + F(x)F(y) + (xoy) \in Z(R) \ \forall \ x, y \in I$. Putting $y = yz$ in this identity to get $G(xy) + F(x)F(y) + (x \, o \, y)z + xyg(z) + F(x)yf(z) - y[x, z] \in Z(R)$. On commuting with z and using our hypothesis, we have $[xyg(z), z] + [F(x)yf(z), z] - [y[x, z], z] = 0$ where $x, y, z \in I$. This expression coincides with (2), the proof follows from Theorem 2.1.1. ☐

By substituting $(-G, -g)$ for (G, g) in Theorem 2.1.4 we get the following:

Theorem 2.1.5. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) - F(x)F(y) \pm (xoy) \in Z(R) \ \forall \ x, y \in I$, then R contains a non-zero central ideal.

Corollary 2.1.1 Let I be a non-zero ideal of a prime ring R. Suppose that (F, f) and (G, g) are multiplicative (generalized) –derivations of R satisfying any one of the following:

1. $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$
2. $G(xy) \pm F(x)F(y) \pm (x \, o \, y) \in Z(R)$
3. $G(xy) \pm F(x)F(y) \pm yx \in Z(R) \ \forall \ x, y \in I$

then, R is commutative.

Theorem 2.1.6. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) + F(x)F(y) \pm xy \in Z(R) \ \forall \ x, y \in I$, then f and g are commuting maps on I.

Proof: Let us assume that

$$G(xy) + F(x)F(y) \pm xy \in Z(R) \ \forall \ x, y \in I \quad (18)$$

Putting $y = yz$ in (18) to get $(G(xy) + F(x)F(y) \pm xy)z + xyg(z) + F(x)yf(z) \in Z(R)$ where $x, y, z \in I$. On commuting with z, our hypothesis yields that

$$[xyg(z), z] + [F(x)yf(z), z] = 0 \ \forall \ x, y, z \in I \quad (19)$$

Replacing x by xz in (19) and expanding to obtain

$$[xzyg(z), z]+[F(x)zyf(z), z]+[xf(z)yf(z), z]=0 \ \forall \ x, y, z \in I \quad (20)$$

On substituting $y = zy$ in (19) we have

$$[xzyg(z), z] + [F(x)zyf(z), z] = 0 \ \forall \ x, y, z \in I \quad (21)$$

Combining (20) and (21) we obtain

$$[xf(z)yf(z), z] = 0 \ \forall \ x, y, z \in I \quad (22)$$

On replacing x by $f(z)yf(z)x$ in (22) and expanding to get $f(z)yf(z)[xf(z)yf(z), z] + [f(z)yf(z), z]xf(z)yf(z) = 0$ where $x, y, z \in I$. Relation (22) reduces it to $[f(z)yf(z), z]xf(z)yf(z) = 0$

where $x, y, z \in I$. It implies that $[f(z)yf(z), z]x[f(z)yf(z), z] = 0$ where $x, y, z \in I$, which is same as (8). Hence, theorem 2.1.1 ensures that $[f(z), z] = 0$ for all $z \in I$.

Next, replacing y by yz in (19) to obtain

$$[xyzg(z), z] + [F(x)yzf(z), z] = 0 \quad \forall\ x, y, z \in I \qquad (23)$$

On multiplying (19) from right side with z, we have

$$[xyg(z)z, z] + [F(x)yf(z)z, z] = 0 \quad \forall\ x, y, z \in I \qquad (24)$$

Subtract (23) form (24) to obtain $[xy[g(z), z], z] + [F(x) y[f(z), z], z] = 0$ where $x, y, z \in I$. The fact $[f(z), z] = 0$ forces that $[xy[g(z), z], z] = 0$ for all $x, y, z \in I$. From this expression, we can easily obtain (15) and by repeating the same argument we get $[g(z), z] = 0$ where $z \in I$, as desired. ☐

By substituting ($-G$, $-g$) for (G, g) in Theorem 2.1.6 we get the following:

Theorem 2.1.7. Let I be a non-zero ideal of a semi-prime ring R. If (F, f) and (G, g) are multiplicative (generalized) derivations of R s.t. $G(xy) - F(x)F(y) \pm xy \in Z(R)\ \forall\ x, y \in I$, then f and g are commuting maps on I.

In the view of Posner's [3] theorem, our hypothesis proves the following result:

Corollary 2.1.2. Let R be a prime ring and (F, f), (G, g) are multiplicative generalized derivations of R satisfying $G(xy) \pm F(x)F(y) \pm xy \in Z(R)\ \forall\ x, y \in I$ then one of the following is true:

(i) $f = 0$ and $g = 0$ on R.
(ii) R is commutative.

2.2 On One Sided Ideals

In [6] Dhara and Ali proved "*Let R be a semi-prime ring, L be a nonzero left ideal of R and F : R → R be a multiplicat - ive (generalized)-derivation associated with the map g of R. If F(xy) ± yx ∈ Z(R) for all x, y ∈ L, then x[x,L] is central for all x ∈ L and L[g(x), x] = (0) for all x ∈ L. Moreover, if R is 3-torsion free, then L[L,L] = (0)*" and "*Let R be a semi - prime ring, L be a nonzero left ideal of R and F : R → R be a multiplicative (generalized)-derivation associated with the map g : R → R. If F(x)F(y) ± yx ∈ Z(R) for all x, y ∈ L, then L[g(x), x] = (0) for all x ∈ L.*" In this section, we sharp these outcomes.

Theorem 2.2.1. Let I be a non-zero left ideal of a semi-prime ring R. If (F, f) be a multiplicative (generalized) derivations of R s.t. $F(x)F(y) \pm yx \in Z(R)\ \forall\ x, y \in I$, then $I[I,I]=(0)$.

Proof: Let us assume that $F(x)F(y) + yx \in Z(R)$ where $x, y \in I$. Replacing y by yz to obtain $(F(y)F(x) + yx)z + F(x)yf(z) + y[z, x] \in Z(R)$ where $x, y, z \in I$. On commuting with z and using our hypothesis, we have

$$[F(x)yf(z), z] + [y[z, x], z] = 0\ \forall\ x, y, z \in I \qquad (25)$$

By [[6] Theorem 2.15] we obtain $I[f(z), z] = (0)$ where $z \in I$. Putting $y = yz$ in (25) to get

$$[F(x)yzf(z), z] + [yz[z, x], z] = 0\ \forall\ x, y, z \in I \qquad (26)$$

Right multiply (25) by z, we have

$$[F(x)yf(z)z, z] + [y[z, x]z, z] = 0\ \forall\ x, y, z \in I \qquad (27)$$

Subtracting (26) from (27) and using the fact that $I[f(z), z] = (0)$ to obtain $[y[[z, x], z], z] = 0$ for all $x, y, z \in I$. Hence by Fact 2.4, the result follows. ☐

Similarly, with certain changes we can prove the same conclusions for $F(x)F(y) - yx \in Z(R)\ \forall\ x, y \in I$.

Theorem 2.2.2. Let I be a non-zero left ideal of a semi-prime ring R. If (F, f) be a multiplicative (generalized) derivations of R s.t. $F(xy) \pm yx \in Z(R)\ \forall\ x, y \in I$, then $I[I,I]=(0)$.

Proof: Let us assume that $F(xy) + yx \in Z(R)$ where $x, y \in I$. Replace y by yz and we obtain $(F(xy) + yx)z + xyf(z) + y[z, x] \in Z(R)$ where $x, y, z \in I$. On commuting with z, our hypothesis forces that

$$[xyf(z), z] + [y[z, x], z] = 0 \quad \forall\ x, y, z \in I \qquad (28)$$

By [[6], Theorem 2.11], $I[f(z), z] = (0)$ where $z \in I$. Replace y by yz in (28) to get

$$[xyzf(z), z] + [yz[z, x], z] = 0 \quad \forall\ x, y, z \in I \qquad (29)$$

Right multiply (28) by z, we have

$$[xyf(z)z, z] + [y[z, x]z, z] = 0 \quad \forall\ x, y, z \in I \qquad (30)$$

Subtracting (29) form (30) and using the fact that $I[f(z), z] = 0$, we get $[y[[z, x], z], z] = 0$ where $x, y, z \in I$. Now by Fact 2.4, the conclusion is proved. ☐

The following example is showing that the restriction imposed on the ring in our Theorems 2.1.1- 2.1.7 are not redundant.

Example. Let R = $\left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in S\right\}$, where S denotes the ring of integers. Note that R is a subring of $M_3[S]$. Clearly, R is not semi-prime and $I = \left\{\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : b,c \in S\right\}$ be a non-zero ideal of R. Let us define mappings $G, g: R \to R$

and $F, f: R \to R$ respectively by $G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & bc \\ 0 & 0 & 0 \end{pmatrix}$, $g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & ba \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to check that (G, g) and (F, f) are multiplicative (generalized)-derivations which are satisfying the identities assumed in Theorem 2.1.1 to Theorem 2.1.7, but I is not a central ideal. Hence, we can't drop the condition of semi-primeness in our outcomes.

References

[1] H. E. Bell, M. N. Daif, "On derivations and commutativity in prime rings", *Acta Mathematica Hungarica*, vol. 66, no. 4, pp. 337-343, 1995.

[2] H. E. Bell, W. S. Martindale III, "Centralizing mapping of semiprime rings", *Canadian Mathematical Society*, vol. 30, pp. 92-101, 1987.

[3] E. Ponser, "Derivations in prime rings", *American Mathematical Society*, vol. 8, no. 6, pp. 1093-1100, 1957.

[4] M. N. Daif, "When is a multiplicative derivation additive?", *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 615-618, 1991.

[5] H. Goldmann, P. Semrl, "Multiplicative derivations on C(x)", *Monatshefte für Mathematik*, vol. 121, no. 3, pp. 189-197, 1996.

[6] B. Dhara, S. Ali, "On Multiplicative (generalized)-derivations in prime and semiprime rings", *Aequations Mathematicae*, vol. 86, no. 1-2, pp. 65-79, 2013.

[7] M. N. Daif, M. S. Tamman El-Sayiad, "Multiplicative Generalized derivations which are additive", *East-West Journal of Mathematics*, vol. 9, no. 1, pp. 33-37, 2007.

[8] M. Ashraf, N. Rehman, "On derivation and commutativity in prime rings", *East-West Journal of Mathematics*, vol. 3, no. 1, pp. 87-91, 2001.

[9] M. Ashraf, A. Ali, S. Ali, "Some commutativity theorems for rings with generalized derivations", *Southeast Asian Bulletin of Mathematics*, vol. 31, no. 3, pp. 415-421, 2007.

[10] M. N. Daif, H. E. Bell, "Remarks on derivations on semiprime rings", *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 1, pp. 205-206, 1992.

[11] M. J. Atteya, "On generalized derivations of semiprime rings", *International Journal of Algebra*, vol. 4, no. 9-12, pp. 591-598, 2010.

[12] S. Tiwari, R. Sharma, B. Dhara, "Identities related to generalized derivations on ideal in prime rings", *Contributions to Algebra and Geometry*, vol. 57, no. 4, pp. 809-821, 2016.

[13] D. Kumar, G. S. Sandhu, "On commutativity of semiprime rings with multiplicative (generalized)-derivations", *Journal of Mathematics Research* , vol. 9, no. 2, pp. 9-17, 2017.

[14] D. Kumar, G. S. Sandhu, "On multiplicative (generalized)-derivations in semiprime rings", *International Journal of Pure and Applied Mathematics*, vol. 106, no. 1, pp. 249-257, 2016.

[15] I. N. Herstein, *Rings with involutions*, The University of Chicago Press, Chicago, USA, 1976.