Statistical Study of the Stochastic Solution Processes of Stochastic Navier-Stokes Equations Using Homotopy-WHEP Technique

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Abstract: This paper indicates the application of the Homotopy-WHEP method to find an approximate for the statistical moments of the stochastic solution processes of 2-D Navier-Stokes equations under the effect of a stochastic excitation. Some cases studies from the results of this method are considered to illustrate some corrections.

Keywords: Stochastic Navier stokes equations, homotopy WHEP technique, Wiener–Hermite Expansion (WHE), homotopy perturbation method (HPM), averages, variance.

1. Introduction

The mathematical theory of the Navier–Stokes equation is of fundamental importance to a deep understanding, prediction and control of turbulence in nature and in technological applications such as combustion dynamics and manufacturing processes. The incompressible Navier–Stokes equation is a well accepted model for atmospheric and ocean dynamics.

The stochastic Navier–Stokes equation has a long history (e.g., see[1,2] for two of the earlier studies) as a model to understand external random forces. In aeronautical applications random forcing of the Navier–Stokes equation models structural vibrations and, in atmospheric dynamics, unknown external forces such as sun heating and industrial pollution can be represented as random forces. In addition to the above reasons there is a mathematical reason for studying stochastic Navier–Stokes equations. It is well known that the invariant measure of the Navier–Stokes equation is not unique.

In this paper using homotopy-WHEP technique (See[3-8]), we indicate the statistical properties of the of the stochastic solution processes of the stochastic Navier-Stokes equations which are formulated in section 2. Also in this article, we introduce the basics of the Wiener–Hermite expansion (WHE) and the homotopy perturbation method (HPM) which are showed in sections 3 and 4 respectively. In Sect. 5, we illustrate the application of WHE to approximate the stochastic system. In Sect. 6, we describe the procedure of (HPM) for solving the deterministic system of section 5. In Section 7, we introduce the application of eigenfuctions expansion to solve the linear system of section 6. In Section 8, some cases studies are presented to indicate the results of the method analysis and the paper are concluded in Sect. 9.

2. The Formulation of the Problem

In this section, we consider the nonlinear stochastic tow-dimension Navier-Stokes equations with external stochastic excitation in the form of Wiener process $W(x;\omega)$ described by the system of the following stochastic differential equations

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\alpha\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]-\frac{\partial p}{\partial x}-\gamma u,\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\alpha\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\frac{\partial p}{\partial y}-\gamma v+\sigma W(x;\omega),\\
&0\le x\le L,\quad 0\le y\le L,\quad t\ge 0,\\
&u(x,y,0)=\phi_1(x,y),\;\; v(x,y,0)=\phi_2(x,y),\\
&u(0,y,t)=u(L,y,t)=0, v(x,0,t)=v(x,L,t)=0,
\end{aligned}\right\}\quad(1)$$

where σ scale of the stochastic term and ω is a random outcome of a triple probability space (Ω,P,B) where Ω is a sample space, B is a σ-algebra associated with Ω and P is a probability measure.

Eliminating p from the second and third equations of the system (1) by differentiation them respectively with respect to y and x, we get

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial y\partial t}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^2 u}{\partial y\partial x}=\alpha\left[\frac{\partial^3 u}{\partial y\partial x^2}+\frac{\partial^3 u}{\partial y^3}\right]\\
&\qquad\qquad -\frac{\partial^2 p}{\partial y\partial x}-\gamma\frac{\partial u}{\partial y},\\
&\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 v}{\partial y\partial x}=\alpha\left[\frac{\partial^3 v}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial x^3}\right]\\
&\qquad\qquad -\frac{\partial^2 p}{\partial y\partial x}-\gamma\frac{\partial v}{\partial x}+\sigma n(x;\omega),
\end{aligned}\right\}\quad(2)$$

where $n(x;\omega)$ is the space white noise process, then reduction (2), we obtain

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+u\nabla^2 v-v\nabla^2 u=\alpha\left[\frac{\partial^3 v}{\partial x\partial y^2}+2\frac{\partial^3 v}{\partial x\partial y^2}-\frac{\partial u^3}{\partial y^3}\right]\\
&\qquad -\gamma\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+\sigma n(x;\omega)
\end{aligned}\right\},\quad(3)$$

from the first equation of system (1), we assume a random function $\psi(x,y,t;\omega)$ which satisfies

$$v=\frac{\partial\psi}{\partial x},\qquad u=-\frac{\partial\psi}{\partial y}\ , \tag{4}$$

then from (3) and (4) the stochastic model (1) tensed to the following

$$\left.\begin{aligned}&\frac{\partial\nabla^2\psi(x,y,t;\omega)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}=\alpha\nabla^4\psi-\gamma\nabla^2\psi+\sigma\, n(x;\omega)\\&\psi(x,y,0)=\phi(x,y),\frac{\partial\psi(0,y,t)}{\partial y}=\frac{\partial\psi(L,y,t)}{\partial y}=0,\\&\frac{\partial\psi(x,0,t)}{\partial x}=\frac{\partial\psi(x,L,t)}{\partial x}=0,\end{aligned}\right\}, \tag{5}$$

where $\frac{\partial(\ldots,\ldots)}{\partial(x,y)}$ is the Jacobian, $\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$

and $\phi_1=-\frac{\partial\phi}{\partial y},\ \ \phi_2=\frac{\partial\phi}{\partial x}$

3. The Wiener–Hermite Expansion (WHE)

The Wiener-Hermite polynomials $H^{(i)}(x_1,x_2,\ldots,x_i)$ (WHPs) are the elements of a complete set of statistically orthogonal random functions (See [9]) and satisfies the following recurrence relation,

$$\begin{aligned}H^{(i)}(x_1,x_2,\ldots,x_i)&=H^{(i-1)}(x_1,x_2,\ldots,x_{i-1})H^{(1)}(x_i)\\&-\sum_{m=1}^{i-1}H^{(i-2)}(x_{i_1},x_{i_2},\ldots,x_{i_{i-2}})\,\delta(x_{i-m}-x_i),\ i\ge 2,\end{aligned} \tag{6}$$

where

$$\left.\begin{aligned}&H^{(0)}=1,\qquad H^{(1)}(x)=n(x),\\&H^{(2)}(x_1,x_2)=H^{(1)}(x_1)H^{(1)}(x_2)-\delta(x_1-x_2),\\&H^{(3)}(x_1,x_2,x_3)=H^{(2)}(x_1,x_2)H^{(1)}(x_3)-H^{(1)}(x_1)\delta(x_2-x_3)\\&\qquad -H^{(1)}(x_2)\delta(x_1-x_3),\\&H^{(4)}(x_1,x_2,x_3,x_4)=H^{(3)}(x_1,x_2,x_3)H^{(1)}(x_4)\\&\qquad -H^{(2)}(x_1,x_2)\delta(x_3-x_4)\\&\qquad -H^{(2)}(x_1,x_3)\delta(x_2-x_4)-H^{(2)}(x_2,x_3)\delta(x_1-x_4),\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}&E\left[H^{(i)}(x_1,x_2,\ldots,x_i)\right]=0\quad\forall\, i\ge 1\\&E\left[H^{(i)}(x_1,x_2,\ldots,x_i)H^{(j)}(x_1,x_2,\ldots,x_j)\right]=0\quad\forall\, i\ne j\end{aligned}\right\}, \tag{8}$$

where $\delta(-)$ is the Dirac delta function, E denotes the ensemble statistical average operator, and $n(x)$ is the stochastic white noise process which has the statistical properties,

$$E[n(x)]=0,\quad E[n(x_1)n(x_2)]=\delta(x_1-x_2)\ . \tag{9}$$

Due to the completeness of the Wiener-Hermite set, any stochastic function $u(x;\omega)$ can be expanded as

$$\left.\begin{aligned}&u(x;\omega)=u^{(0)}(x)+\int_R u^{(1)}(x;x_1)\,H^{(1)}(x_1)dx_1+\\&\int_{R^2}u^{(2)}(x;x_1,x_2)\,H^{(1)}(x_1,x_2)dx_1dx_2+\\&\int_{R^3}u^{(3)}(x;x_1,x_2,x_3)\,H^{(3)}(x_1,x_2,x_3)dx_1dx_2dx_3+\ldots\ldots,\end{aligned}\right\} \tag{10}$$

Using the statistical properties of WHPs (See [3]), we introduce some statistical moments of the first order series of the WHE for any stochastic process $u(x;\omega)$ as follow,

$$E[u(x;\omega)]=u^{(0)}(x)+\int_{-\infty}^{\infty}u^{(1)}(x;x_1)\,E[H^{(1)}(x_1)]dx_1=u^{(0)}(x), \tag{11}$$

$$Var[u(x;\omega)]=E\big(u(x;\omega)-E[u(x;\omega)]\big)^2=\int_{-\infty}^{\infty}[u^{(1)}(x;x_1)]^2dx_1, \tag{12}$$

$$\left.\begin{aligned}&E[u(x;\omega)]^3=[u^{(0)}(x)]^3+\\&3[u^{(0)}(x)]^2\int_{-\infty}^{\infty}u^{(1)}(x;x_1)\,E[H^{(1)}(x_1)]dx_1+\\&3u^{(0)}(x)\int_{R^2}\left[\prod_{i=1}^{2}u^{(1)}(x,x_i)\right]E\left[\prod_{i=1}^{2}H^{(1)}(x_i)\right]dx_1dx_2\\&+\int_{R^3}\left[\prod_{i=1}^{3}u^{(1)}(x,x_i)\right]E\left[\prod_{i=1}^{3}H^{(1)}(x_i)\right]\prod_{i=1}^{3}dx_i=\\&(E[u(x;\omega)])^3+3E[u(x;\omega)]Var[u(x;\omega)],\end{aligned}\right\}, \tag{13}$$

$$\left.\begin{aligned}&E[u(x;\omega)]^4=\\&[u^{(0)}(x)]^4+4[u^{(0)}(x)]^3\int_{-\infty}^{\infty}u^{(1)}(x;x_1)\,E[H^{(1)}(x_1)]dx_1\\&+6[u^{(0)}(x)]^2\int_{R^2}\left[\prod_{i=1}^{2}u^{(1)}(x,x_i)\right]E\left[\prod_{i=1}^{2}H^{(1)}(x_i)\right]dx_1dx_2+\\&4[u^{(0)}(x)]^2\int_{R^3}\left[\prod_{i=1}^{3}u^{(1)}(x,x_i)\right]E\left[\prod_{i=1}^{3}H^{(1)}(x_i)\right]\prod_{i=1}^{3}dx_i\\&+\int_{R^4}\left[\prod_{i=1}^{4}u^{(1)}(x,x_i)\right]E\left[\prod_{i=1}^{4}H^{(1)}(x_i)\right]\prod_{i=1}^{4}dx_i\\&\quad=[u^{(0)}(x,t)]^4+6[u^{(0)}(x,t)]^2Var[u(x;\omega)]\ +3\,Var[u(x;\omega)]^2,\end{aligned}\right\} \tag{14}$$

4. The Basic Idea of the Homotopy Perturbation Method (HPM)

The HPM, proposed first by He [10-15], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. The HPM is applied to Volterra's intgro-differential equation [16], to nonlinear oscillators [13], bifurcation of nonlinear problems [17], bifurcation of delay-differential equations [18], nonlinear wave equations [19], boundary value problems [20], and to other fields. This HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. In this technique, a parameter $p\in[0,1]$ is embedded in a homotopy function $v(r,p):\varphi\times[0,1]\to R$ which satisfies

$$H(v,r)=(1-p)[L(v)-L(u_0)]+p[(A(v)-f(r)]=0\,, \tag{15}$$

where u_0 is an initial approximation to the solution of the equation,

$$A(u)-f(r)=0,\ r\in\varphi\,, \tag{16}$$

with boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=0,\ r\in\Gamma \tag{17}$$

in which A is a nonlinear differential operator which can be decompose into a linear operator L and a non-linear operator N, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of φ. The homotopy introduces a continuously deformed solution for the case of $p=0$, $L(v)-L(u_0)=0$ to the case of $p=1$, $A(v)-f(r)=0$, which is the original equation (20). This is the basic idea of the homotopy method which is to deform continuously a simple problem (and easy to solve) into the difficult problem under study. The basic assumption of the HPM method is that the solution of the original equation (15) can be expanded as a power series in p as:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots\dots \quad (18)$$

Now, setting $p=1$, the approximate solution of equation (16) is obtained as:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \dots\dots, \quad (19)$$

the rate of convergence of the method depends greatly on the initial approximation u_0 which is considered as the main disadvantage of HPM.

The homotopy perturbation technique can be applied to a system of nonlinear differential equations. Let us have the following coupled system,

$$\left.\begin{array}{l} L_1(\Phi_1) + N_1(\Phi_1,\Phi_2) = F_1(r) \\ L_1(\Phi_2) + N_2(\Phi_1,\Phi_2) = F_2(r) \end{array}\right\}, \quad (20)$$

where L_1 and L_2 are linear differential operators and N_1 and N_2 are nonlinear operators. The homotopy functions can be constructed as follows:

$$\left.\begin{array}{l} H_1 = L_1(v) - L_1(\varphi_0) + p[L_1(\varphi_0) + N_1(v,u) - F_1(r)] = 0 \\ H_2 = L_2(u) - L_2(\phi_0) + p[L_2(\phi_0) + N_2(v,u) - F_2(r)] = 0 \end{array}\right\}, \quad (21)$$

letting

$$\left.\begin{array}{l} v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots\dots \\ u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots\dots \end{array}\right\}, \quad (22)$$

and then substituting in the original equations (21), enables getting iterative equations in the unknowns v_i and u_i. The solutions are got as

$$\left.\begin{array}{l} \Phi_1 = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \dots\dots \\ \Phi_2 = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \dots\dots \end{array}\right\}. \quad (23)$$

We can generalize the previous technique to n equations in the form

$$L_k(\Phi_k) + N_1(\Phi_1,\Phi_2,\dots\dots,\Phi_n) = F_k(r), \ \ k = 1,2,3,\dots\dots,n \quad (24)$$

and construct the homotopy functions as

$$\begin{array}{l} H_k = L_k(v_k) - L_k(\varphi_0{}^{(k)}) + p[L_k(\varphi_0{}^{(k)}) + \\ \qquad N_1(v_1,v_2,v_3,\dots,v_n) - F_k(r)] = 0, \ \ k = 1,2,\dots,n, \end{array} \quad (25)$$

the basic assumptions are

$$v_k = v_k{}^{(0)} + pv_k{}^{(1)} + p^2 v_k{}^{(2)} + \dots\dots, \ \ k = 0,1,2,\dots,n, \quad (26)$$

and still

$$\Phi_k = \lim_{p\to 1} v_k = v_k{}^{(0)} + v_k{}^{(1)} + v_k{}^{(2)} + v_k{}^{(3)} + \dots., \ \ k = 0,1,2\dots,n \quad (27)$$

5. The Application of WHE to Approximate of the Stochastic Solution Process

From the survey of the Wiener-Hermite polynomials (WHPs) which were discussed in section 3, the first order series of Wiener-Hermite expansion of the stochastic solution process $\psi(x,y,t;\omega)$ of the model (5) takes the following form

$$\psi(x,y,t;\omega) = \psi^{(0)}(x,y,t) + \int_0^L \psi^{(1)}(x,y,t;x_1) H^{(1)}(x_1) dx_1, \quad (28)$$

where $\psi^{(0)}(x,y,t)$ and $\psi^{(1)}(x,y,t;x_1)$ are deterministic functions and the mean and variance of the solutions processes of the original stochastic model (1) obtained by

$$\left.\begin{array}{l} E[u(x,y,t;\omega)] = -\dfrac{\partial\psi^{(0)}(x,y,t)}{\partial y}, \\ E[v(x,y,t;\omega)] = \dfrac{\partial\psi^{(0)}(x,y,t)}{\partial x}, \\ Var[u(x,y,t;\omega)] = \displaystyle\int_0^L \left(\dfrac{\partial\psi^{(1)}(x,y,t,x_1)}{\partial y}\right)^2 dx_1, \\ Var[v(x,y,t;\omega)] = \displaystyle\int_0^L \left(\dfrac{\partial\psi^{(1)}(x,y,t,x_1)}{\partial x}\right)^2 dx_1. \end{array}\right\} \quad (29)$$

Substituting from (28) into (5), we get

$$\left.\begin{array}{l} \left[\left(\dfrac{\partial}{\partial t}+\gamma\right)\nabla^2 - \alpha\nabla^4\right]\left(\psi^{(0)}(x,y,t) + \displaystyle\int_0^L \psi^{(1)} H^{(1)}(x_1)dx_1\right) + \\ \displaystyle\int_0^L \left(\dfrac{\partial(\psi^{(1)},\nabla^2\psi^{(0)})}{\partial(x,y)} + \dfrac{\partial(\psi^{(0)},\nabla^2\psi^{(1)})}{\partial(x,y)}\right) H^{(1)}(x_1)dx_1 + \\ \displaystyle\int_0^L\int_0^L \dfrac{\partial(\psi^{(1)}(x_1),\nabla^2\psi^{(1)}(x_2))}{\partial(x,y)} H^{(1)}(x_1)H^{(1)}(x_2)dx_1dx_2 \\ + \dfrac{\partial(\psi^{(0)},\nabla^2\psi^{(0)})}{\partial(x,y)} - \sigma\, n(x;\omega) = 0 \end{array}\right\}. \quad (30)$$

The deterministic functions $\psi^{(0)}$ and $\psi^{(1)}$ are given from a deterministic system which yield from some statistical averages (See [3]) of the final stochastic equation and takes the form

$$\left.\begin{array}{l} \left[\left(\dfrac{\partial}{\partial t}+\gamma\right)\nabla^2 - \alpha\nabla^4\right]\psi^{(0)}(x,y,t) \\ \qquad + \dfrac{\partial(\psi^{(0)},\nabla^2\psi^{(0)})}{\partial(x,y)} + \displaystyle\int_0^L \dfrac{\partial(\psi^{(1)},\nabla^2\psi^{(1)})}{\partial(x,y)} dx_1 = 0, \\ \left[\left(\dfrac{\partial}{\partial t}+\gamma\right)\nabla^2 - \alpha\nabla^4\right]\psi^{(1)}(x,y,t,x_1) \\ \qquad + \dfrac{\partial(\psi^{(1)},\nabla^2\psi^{(0)})}{\partial(x,y)} + \dfrac{\partial(\psi^{(0)},\nabla^2\psi^{(1)})}{\partial(x,y)} = \sigma\delta(x-x_1), \\ \psi^{(0)}(x,y,0) = \phi(x,y), \ \ \psi^{(1)}(x,y,0,x_1) = 0, \\ \dfrac{\partial\psi^{(0)}(0,y,t)}{\partial y} = \dfrac{\partial\psi^{(0)}(L,y,t)}{\partial y} = 0 \\ \dfrac{\partial\psi^{(0)}(x,0,t)}{\partial x} = \dfrac{\partial\psi^{(0)}(x,L,t)}{\partial y} = \dfrac{\partial\psi^{(1)}(x,0,t,x_1)}{\partial x} = \\ \dfrac{\partial\psi^{(1)}(x,L,t,x_1)}{\partial x} = \dfrac{\partial\psi^{(1)}(0,y,t,x_1)}{\partial y} = \dfrac{\partial\psi^{(1)}(0,L,t,x_1)}{\partial y} = 0 \end{array}\right\} \quad (31)$$

6. The Application of the HPM to Approximate the Deterministic System

From the basics which were showed in section 4, we propose the application of HPM to solve the non-linear deterministic system which reduced from the pervious section. Let us put the system (31) in the following form

$$\left.\begin{aligned} &L(\psi^{(0)})+N_1(\psi^{(0)},\psi^{(1)})=0,\\ &L(\psi^{(1)})+N_2(\psi^{(0)},\psi^{(1)})=\sigma\delta(x-x_1)\end{aligned}\right\}, \qquad (32)$$

where L is linear differential operators and N_1 and N_2 are non-linear operators which are defined from the following,

$$\left.\begin{aligned} &L=\left(\frac{\partial}{\partial t}+\gamma\right)\nabla^2-\alpha\nabla^4,\\ &N_1(\psi^{(0)},\psi^{(1)})=\frac{\partial(\psi^{(0)},\nabla^2\psi^{(0)})}{\partial(x,y)}+\int_0^L\frac{\partial(\psi^{(1)},\nabla^2\psi^{(1)})}{\partial(x,y)}dx_1,\\ &N_2(\psi^{(0)},\psi^{(1)})=\frac{\partial(\psi^{(1)},\nabla^2\psi^{(0)})}{\partial(x,y)}+\frac{\partial(\psi^{(0)},\nabla^2\psi^{(1)})}{\partial(x,y)}\end{aligned}\right\} \quad (33)$$

then homotopy functions can be constructed as follows

$$\left.\begin{aligned} &H_1=L(v)-L(\varphi_0)+p\left[L_1(\varphi_0)+N_1(v,u)\right]=0\\ &H_2=L(u)-L(\phi_0)+p\left[L_1(\phi_0)+N_1(v,u)-\sigma\delta(x-x_1)\right]=0\end{aligned}\right\} \quad (34)$$

where φ_0 and ϕ_0 arbitrary functions satisfy the initial and boundary conditions in the original equations (5) and take the following choice form

$$\left.\begin{aligned} &\varphi_0=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Ae^{\beta_{n,m}t}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\\ &A=\frac{4}{L^2}\int_0^L\int_0^L\phi(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)dxdy,\\ &\phi_0=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^t e^{\mu_{n,m}(t-\tau)}d\tau\,\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)\sin\left(\frac{n\pi}{L}x_1\right)\end{aligned}\right\} \quad (35)$$

then, from the application of the perturbation theory, we consider v and u in the third order approximation of ε as follow,

$$\left.\begin{aligned} &v=v_0+pv_1+p^2v_2+p^3v_3\\ &u=u_0+pu_1+p^2u_2+p^3u_3\end{aligned}\right\}, \qquad (36)$$

such that

$$\left.\begin{aligned} &\psi^{(0)}(x,y,t)=\lim_{p\to1}v=v_0+v_1+v_2+v_3+.......\\ &\psi^{(1)}(x,y,t;x_1)=\lim_{p\to1}u=u_0+u_1+u_2+u_3+.......\end{aligned}\right\}, \qquad (37)$$

substituting in the homotopy functions (34) and equating the equal powers of p in both sides of the equations, we get the following iterative equations:

$$\left.\begin{aligned} &L_1(v_0)-L_1(\varphi_0)=0, \qquad v_0(x,y,0)=\phi(x,y),\\ &\frac{\partial v_0(0,y,t)}{\partial y}=\frac{\partial v_0(L,y,t)}{\partial y}=\frac{\partial v_0(x,0,t)}{\partial x}=\frac{\partial v_0(x,L,t)}{\partial x}=0\end{aligned}\right\}, \qquad (38)$$

$$\left.\begin{aligned} &L_1(u_0)-L_1(\phi_0)=0, \qquad u_0(x,y,0,x_1)=\frac{\partial u_0(x,0,t,x_1)}{\partial x}=\\ &\frac{\partial u_0(x,L,t,x_1)}{\partial x}=\frac{\partial u_0(0,y,t,x_1)}{\partial y}=\frac{\partial u_0(0,L,t,x_1)}{\partial y}=0\end{aligned}\right\}, \qquad (39)$$

$$\left.\begin{aligned} &L(v_1)+L(\varphi_0)+\frac{\partial(v_0,\nabla^2v_0)}{\partial(x,y)}+\int_0^L\frac{\partial(u_0,\nabla^2u_0)}{\partial(x,y)}dx_1=0, v_1(x,y,0)=0,\\ &\frac{\partial v_1(0,y,t)}{\partial y}=\frac{\partial v_1(L,y,t)}{\partial y}=\frac{\partial v_1(x,0,t)}{\partial x}=\frac{\partial v_1(x,L,t)}{\partial x}=0\end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned} &L(u_1)+L(\phi_0)+\frac{\partial(u_0,\nabla^2v_0)}{\partial(x,y)}+\frac{\partial(v_0,\nabla^2u_0)}{\partial(x,y)}-\sigma\delta(x-x_1)=0,\\ &u_1(x,y,0,x_1)=\frac{\partial u_1(x,0,t,x_1)}{\partial x}=\frac{\partial u_1(x,L,t,x_1)}{\partial x}\\ &\qquad=\frac{\partial u_1(0,y,t,x_1)}{\partial y}=\frac{\partial u_1(0,L,t,x_1)}{\partial y}=0,\end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned} &L(v_2)+\frac{\partial(v_0,\nabla^2v_1)}{\partial(x,y)}+\frac{\partial(v_1,\nabla^2v_0)}{\partial(x,y)}\\ &\qquad+\int_0^L\left[\frac{\partial(u_0,\nabla^2u_1)}{\partial(x,y)}+\frac{\partial(u_1,\nabla^2u_0)}{\partial(x,y)}\right]dx_1=0\\ &v_2(x,y,0)=\frac{\partial v_2(0,y,t)}{\partial y}=\frac{\partial v_2(L,y,t)}{\partial y}=\frac{\partial v_2(x,0,t)}{\partial x}=\frac{\partial v_2(x,L,t)}{\partial x}=0\end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned} &L(u_2)+\frac{\partial(u_1,\nabla^2v_0)}{\partial(x,y)}+\frac{\partial(u_0,\nabla^2v_1)}{\partial(x,y)}+\frac{\partial(v_0,\nabla^2u_1)}{\partial(x,y)}+\frac{\partial(v_1,\nabla^2u_0)}{\partial(x,y)}=0,\\ &u_2(x,y,0,x_1)=\frac{\partial u_2(x,0,t,x_1)}{\partial x}=\frac{\partial u_2(x,L,t,x_1)}{\partial x}\\ &\qquad=\frac{\partial u_2(0,y,t,x_1)}{\partial y}=\frac{\partial u_2(0,L,t,x_1)}{\partial y}=0,\end{aligned}\right\} \quad (43)$$

$$\left.\begin{aligned} &L(v_3)+\frac{\partial(v_0,\nabla^2v_2)}{\partial(x,y)}+\frac{\partial(v_2,\nabla^2v_0)}{\partial(x,y)}+\frac{\partial(v_1,\nabla^2v_1)}{\partial(x,y)}+\\ &\int_0^L\left[\frac{\partial(u_0,\nabla^2u_2)}{\partial(x,y)}+\frac{\partial(u_0,\nabla^2u_2)}{\partial(x,y)}+\frac{\partial(u_1,\nabla^2u_1)}{\partial(x,y)}\right]dx_1=0,\\ &v_3(x,y,0)=\frac{\partial v_3(0,y,t)}{\partial y}=\frac{\partial v_3(L,y,t)}{\partial y}=\frac{\partial v_3(x,0,t)}{\partial x}=\frac{\partial v_3(x,L,t)}{\partial x}=0,\end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned} &L(u_3)+\frac{\partial(u_2,\nabla^2v_0)}{\partial(x,y)}+\frac{\partial(u_0,\nabla^2v_2)}{\partial(x,y)}+\frac{\partial(u_1,\nabla^2v_1)}{\partial(x,y)}+\frac{\partial(v_0,\nabla^2u_2)}{\partial(x,y)}\\ &+\frac{\partial(v_2,\nabla^2u_0)}{\partial(x,y)}+\frac{\partial(v_1,\nabla^2u_1)}{\partial(x,y)}=0, \quad u_3(x,y,0,x_1)=0,\\ &\frac{\partial u_3(x,0,t,x_1)}{\partial x}=\frac{\partial u_3(x,L,t,x_1)}{\partial x}=\frac{\partial u_3(0,y,t,x_1)}{\partial y}=\frac{\partial u_3(0,L,t,x_1)}{\partial y}=0\end{aligned}\right\} \quad (45)$$

7. The Application of Eigenfunctions Expansion Method

The iterative linear partial differential equations (38-45) which reduced from the pervious section have the following form

$$\left.\begin{aligned} &\left[\left(\frac{\partial}{\partial t}+\gamma\right)\nabla^2-\alpha\nabla^4\right]\Psi(x,y,t)+f(x,y,t)=0,\\ &0\le x\le L, \quad 0\le y\le L, \quad t\ge0,\\ &\Psi(x,y,0)=\Phi(x,y), \quad \Psi_y(0,y,t)=\Psi_y(L,y,t)=\\ &\Psi_x(x,0,t)=\Psi_x(x,L,t)=0,\end{aligned}\right\}, \qquad (46)$$

and from the application of eigenfunctions expansion method (See [21]) , the final form has the general solution

$$\left.\begin{array}{l}\Psi(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[I_{n.m}(t)+T_{n.m}\,e^{-\alpha\lambda_{m,n}t}\right]\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)\\ \lambda_{n,m}=\gamma+\left(\frac{\pi}{L}\right)^2(m^2+n^2),\quad I_{n.m}(t)=\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}F_{n.m}(\tau)d\tau,\\ T_{n.m}=\frac{4}{L^2}\int_0^L\int_0^L\Phi(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)dxdy,\\ F_{n.m}(t)=\frac{4}{\pi^2(m^2+n^2)}\int_0^L\int_0^L f(x,y,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)dxdy,\end{array}\right\}\quad(47)$$

then the system (38-45) has the following solutions respectively,

$$\left.\begin{array}{l}v_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}r_1(n,m,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\\ r_1(n,m,t)=Ae^{\beta_{n,m}t},\ \beta_{n,m}=-\alpha_1(n^4+m^4)\end{array}\right\}\quad(48)$$

$$\left.\begin{array}{l}u_0(x,y,t;x_1)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}r_2(n,m,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\\ r_2(n,m,t)=\int_0^t e^{\mu_{n,m}(t-\tau)}d\tau,\ \mu_{n,m}=-\alpha_2(n^3+m^3)\end{array}\right\},\quad(49)$$

$$\left.\begin{array}{l}v_1(x,y,t)=\left(\sum_{n=1}^{4}q_n(t)\sin\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{2\pi}{L}x\right)\\ \quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}r_3(n,m,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)\end{array}\right\},\quad(50)$$

$$\left.\begin{array}{l}q_1(t)=\frac{3}{20}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{1,2}(t-\tau)}r_4(\tau)d\tau,\\ q_2(t)=\frac{1}{32}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{2,2}(t-\tau)}r_5(\tau)d\tau,\\ q_3(t)=\frac{-1}{52}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{3,2}(t-\tau)}r_4(\tau)d\tau,\\ q_4(t)=\frac{-1}{10}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{4,2}(t-\tau)}r_5(\tau)d\tau,\end{array}\right\},\quad(51)$$

$$\left.\begin{array}{l}r_4(t)=3\left(\frac{\pi}{L}\right)^4\left[r_1(1,1,t)r_1(1,2,t)\right],\\ r_5(t)=\frac{16\pi^4}{L^3}\left[r_2(1,1,t)r_2(1,2,t)\right]\\ r_3(n,m,t)=\int_0^t e^{\lambda_{n,m}(t-\tau)}r_6(n,m,\tau)d\tau,\\ r_6(n,m,t)=-\frac{\pi^2}{4}(n^2+m^2)\left[\lambda_{m,n}r_1(n,m,t)+\frac{dr_1(n,m,t)}{dt}\right],\end{array}\right\}\quad(52)$$

$$u_1(n,m,t;x_1)=\left(\sum_{n=1}^{5}V_n(t)\sin\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{L}x_1\right)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\begin{array}{l}r_7(n,m,t)\sin\left(\frac{m\pi}{L}y\right)+\\ r_8(n,m,t)\sin\left(\frac{(2m-1)\pi}{L}y\right)\end{array}\right]\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x_1\right)\quad(53)$$

$$\left.\begin{array}{l}V_1(t)=\frac{L^2}{8}\left[2r_{13}(2,1,t)-r_{10}(2,1,t)\right]\\ V_2(t)=\frac{L^2}{8}\left[r_9(2,2,t)-2r_{11}(2,2,t)\right],\\ V_3(t)=\frac{L^2}{8}\left[r_{10}(2,3,t)-r_{12}(2,3,t)\right],\\ V_4(t)=\frac{L^2}{8}r_{11}(2,4,t),\ V_5(t)=\frac{L^2}{8}r_{12}(2,5,t)\end{array}\right\},\quad(54)$$

$$\left.\begin{array}{l}r_9(n,m,t)=\frac{4}{\pi^2(n^2+m^2)}\int_0^t e^{-\lambda_{n,m}(t-\tau)}\rho_1(\tau)d\tau,\\ r_{10}(n,m,t)=\frac{4}{\pi^2(n^2+m^2)}\int_0^t e^{-\lambda_{n,m}(t-\tau)}\rho_2(\tau)d\tau,\\ r_{11}(n,m,t)=\frac{4}{\pi^2(n^2+m^2)}\int_0^t e^{-\lambda_{n,m}(t-\tau)}\rho_3(\tau)d\tau,\\ r_{12}(n,m,t)=\frac{4}{\pi^2(n^2+m^2)}\int_0^t e^{-\lambda_{n,m}(t-\tau)}\rho_4(\tau)d\tau,\\ r_{13}(n,m,t)=\frac{4}{\pi^2(n^2+m^2)}\int_0^t e^{-\lambda_{n,m}(t-\tau)}\rho_5(\tau)d\tau,\end{array}\right\},\quad(55)$$

$$\left.\begin{array}{l}\rho_1(t)=-2\left(\frac{\pi}{L}\right)^4\left[r_1(1,1,t)r_2(1,2,t)\right],\\ \rho_2(t)=-\frac{1}{2}\left(\frac{\pi}{L}\right)^4 r_1(1,2,t)\left[7r_2(1,1,t)+25r_2(1,2,t)\right],\\ \rho_3(t)=\frac{13}{2}\rho_1(t),\quad \rho_4(t)=-\frac{25}{2}\left(\frac{\pi}{L}\right)^4\left[r_1(1,2,t)r_2(1,2,t)\right],\\ \rho_5(t)=\frac{1}{2}\left(\frac{\pi}{L}\right)^4 r_1(1,2,t)\left[3r_2(1,1,t)-10r_2(1,2,t)\right],\end{array}\right\}\quad(56)$$

$$\left.\begin{array}{l}r_7(n,m,t)=\frac{2}{\pi^2(n^2+m^2)}\int_0^t e^{\lambda_{n,m}(t-\tau)}r_{14}(n,m,\tau)d\tau,\\ r_{14}(n,m,t)=-\frac{\pi^2}{4}(n^2+m^2)\left[\lambda_{m,n}r_2(n,m,t)+\frac{dr_2(n,m,t)}{dt}\right],\\ r_8(n,m,t)=-\frac{32\sigma}{(2m-1)L\pi^3([2m-1)^2+n^2)}\int_0^t e^{-\alpha\lambda_{n,2m-1}(t-\tau)}d\tau,\end{array}\right\}\quad(57)$$

$$\left.\begin{array}{l}v_2(x,y,t)=\left(\sum_{n=1}^{4}s_n(t)\sin\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{2\pi}{L}x\right)\\ \quad+\left(\sum_{n=1}^{2}S_n(t)\sin\left(\frac{(2n-1)\pi}{L}x\right)\right)\sin\left(\frac{2\pi}{L}y\right),\end{array}\right\},\quad(58)$$

$$\left.\begin{array}{l}s_1(t)=\frac{3}{20}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{2,1}(t-\tau)}r_{16}(\tau)d\tau,\\ s_2(t)=\frac{1}{32}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{2,2}(t-\tau)}r_{15}(\tau)d\tau,\\ s_3(t)=\frac{-1}{52}\left(\frac{L}{\pi}\right)^2\int_0^t e^{-\lambda_{2,3}(t-\tau)}r_{16}(\tau)d\tau,\end{array}\right\}\quad(59)$$

$$\left.\begin{aligned}
s_4(t) &= \frac{-1}{160}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,4}(t-\tau)} r_{16}(\tau)d\tau, \\
s_1(t) &= \frac{3}{20}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,1}(t-\tau)} r_{16}(\tau)d\tau, \\
s_2(t) &= \frac{1}{32}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,2}(t-\tau)} r_{15}(\tau)d\tau, \\
s_3(t) &= \frac{-1}{52}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,3}(t-\tau)} r_{16}(\tau)d\tau, \\
s_4(t) &= \frac{-1}{160}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,4}(t-\tau)} r_{16}(\tau)d\tau, \\
S_1(t) &= \frac{3}{20}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{1,2}(t-\tau)} r_{17}(\tau)d\tau, \\
S_2(t) &= \frac{-1}{52}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{3,2}(t-\tau)} r_{17}(\tau)d\tau,
\end{aligned}\right\}, \tag{60}$$

$$\left.\begin{aligned}
r_{15}(t) &= -2\left(\frac{\pi}{L}\right)^4 \left[r_2(1,1,t) r_8(1,2,t)\right], \\
r_{16}(t) &= 3\left(\frac{\pi}{L}\right)^4 \left[r_1(1,1,t) r_3(1,1,t) + \frac{L}{2} r_1(1,1,t) r_7(1,2,t)\right], \\
r_{17}(t) &= -3\left(\frac{\pi}{L}\right)^4 \left[r_1(1,1,t) q_1 + \frac{L}{2} r_2(1,1,t) V_1\right],
\end{aligned}\right\}, \tag{61}$$

$$\left.\begin{aligned}
&u_2(x,y,t;x_1) = \\
&\quad \left[w_1(t)\sin\left(\frac{\pi}{L}x\right) + w_2(t)\sin\left(\frac{3\pi}{L}x\right)\right]\sin\left(\frac{2\pi}{L}y\right)\sin\left(\frac{\pi}{L}x_1\right),
\end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned}
w_1(t) &= \frac{3}{20}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{1,2}(t-\tau)} r_{18}(\tau)d\tau, \\
w_2(t) &= -\frac{1}{52}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{3,2}(t-\tau)} r_{18}(\tau)d\tau, \\
r_{18}(\tau) &= -3\left(\frac{\pi}{L}\right)^4 \left[r_2(1,1,t) q_1 + \frac{L}{2} r_1(1,1,t) V_1\right],
\end{aligned}\right\}, \tag{63}$$

$$\left.\begin{aligned}
v_3(x,y,t) &= \left[d_1(t)\sin\left(\frac{\pi}{L}x\right) + d_2(t)\sin\left(\frac{3\pi}{L}x\right)\right]\sin\left(\frac{2\pi}{L}y\right) \\
&+ \left[b_1(t)\sin\left(\frac{\pi}{L}y\right) + b_2(t)\sin\left(\frac{3\pi}{L}y\right)\right]\sin\left(\frac{2\pi}{L}x\right),
\end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned}
d_1(t) &= \frac{3}{20}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{1,2}(t-\tau)} r_{19}(\tau)d\tau, \\
d_2(t) &= -\frac{1}{52}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{3,2}(t-\tau)} r_{19}(\tau)d\tau, \\
b_1(t) &= \frac{3}{20}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,1}(t-\tau)} r_{20}(\tau)d\tau, \\
b_2(t) &= -\frac{1}{52}\left(\frac{L}{\pi}\right)^2 \int_0^t e^{-\lambda_{2,3}(t-\tau)} r_{20}(\tau)d\tau,
\end{aligned}\right\}, \tag{65}$$

$$\left.\begin{aligned}
r_{19}(\tau) &= 3\left(\frac{\pi}{L}\right)^4 \left[\begin{aligned} &r_3(1,1,t) q_1 + r_1(1,1,t) s_1 + \\ &\frac{L}{2}[r_8(1,1,t) - r_7(1,1,t)] V_1 \end{aligned}\right], \\
r_{20}(t) &= 3\left(\frac{\pi}{L}\right)^4 \left[r_1(1,1,t) S_1 + \frac{L}{2} r_2(1,1,t) w_1\right]
\end{aligned}\right\} \tag{66}$$

8. Cases Study

In this section, we introduce some cases study which indicate the corrections related to the approximations of the statistical moments of the stochastic solution process.

Case study 1. This case illustrates the change of the first statistical mom- ent of the stochastic solution processes $u(x,y,t;\omega)$ and $v(x,y,t;\omega)$ at fixed value for one independent variable from the following figures,

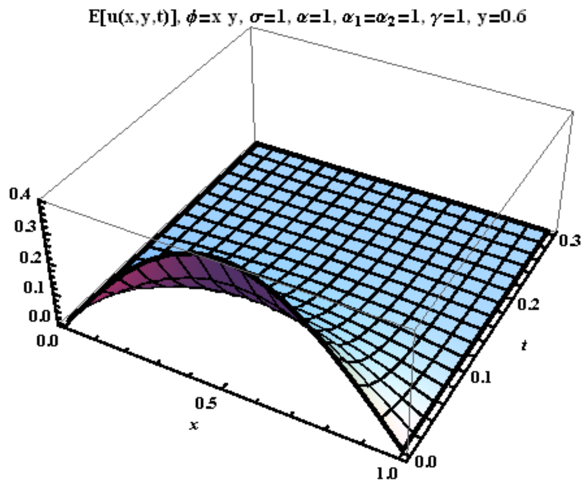

Figure 1. The change of the 3^{rd} correction $E[u(x,y,t;\omega)]$ at fixed value of y, $L=1$

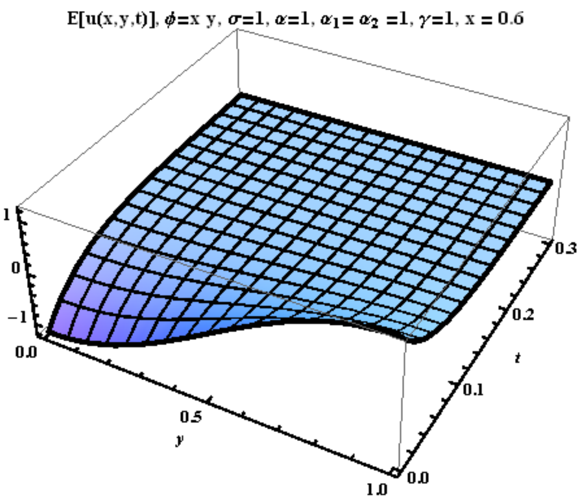

Figure 2. The change of the 3^{rd} correction $E[u(x,y,t;\omega)]$ at fixed value of x, $L=1$

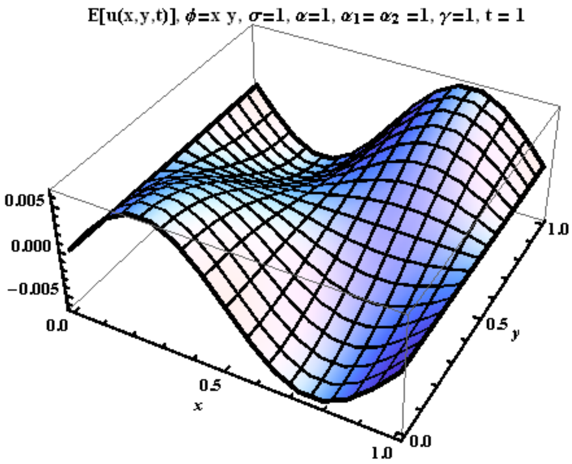

Figure 3. The change of the 3^{rd} correction $E[u(x,y,t;\omega)]$ at fixed value of t, $L=1$

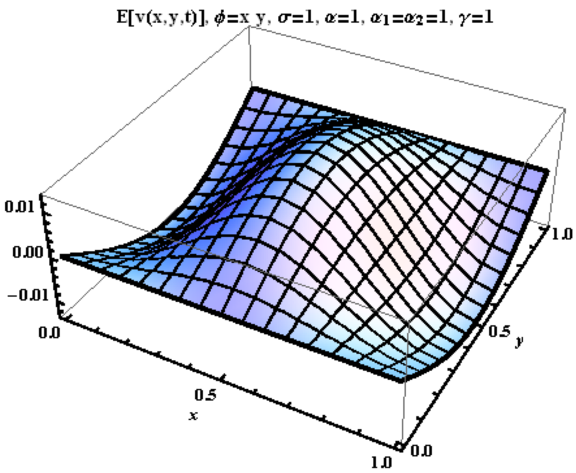

Figure 4. The change of the 3^{rd} correction $E[v(x,y,t;\omega)]$ at fixed value of t, $L=1$

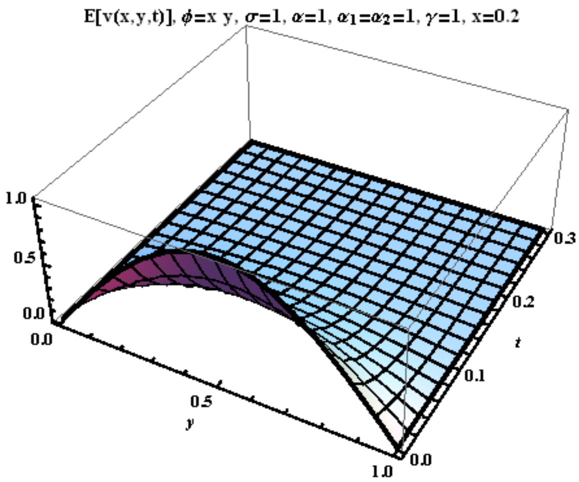

Figure 5. The change of the 3^{rd} correction $E[v(x,y,t;\omega)]$ at fixed value of x, $L=1$

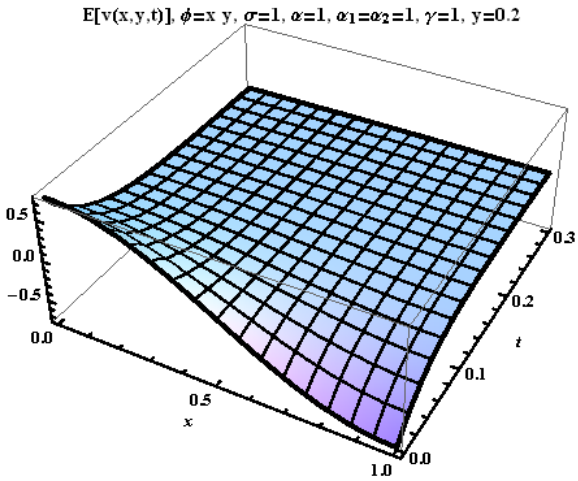

Figure 6. The change of the 3^{rd} correction $E[v(x,y,t;\omega)]$ at fixed value of y, $L=1$

Case Study 2. The change of the variance of the stochastic process $u(x,y,t;\omega)$ and $v(x,y,t;\omega)$ at fixed value for one independent variable is depicted by from the following figures

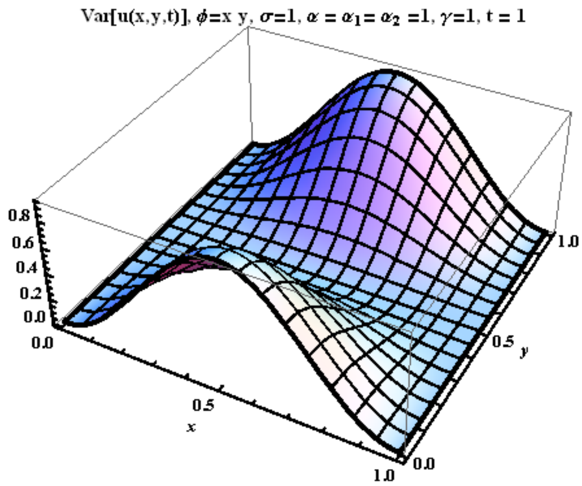

Figure7. The change of the 2^{nd} correction $Var[u(x,y,t;\omega)]$ at fixed value of t, $L=1$

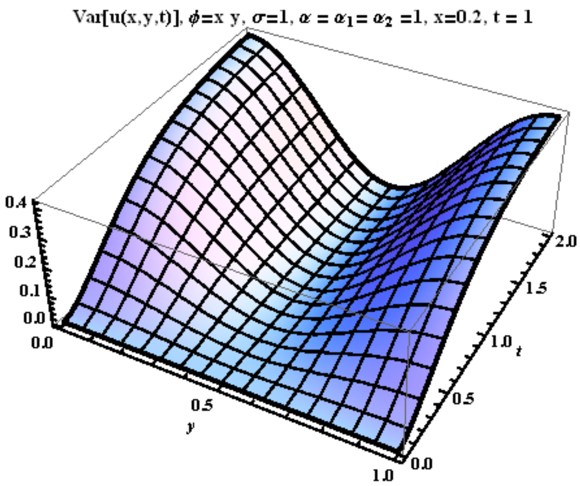

Figure 8. The change of the 2^{nd} correction $Var[u(x,y,t;\omega)]$ at fixed value of x, $L=1$

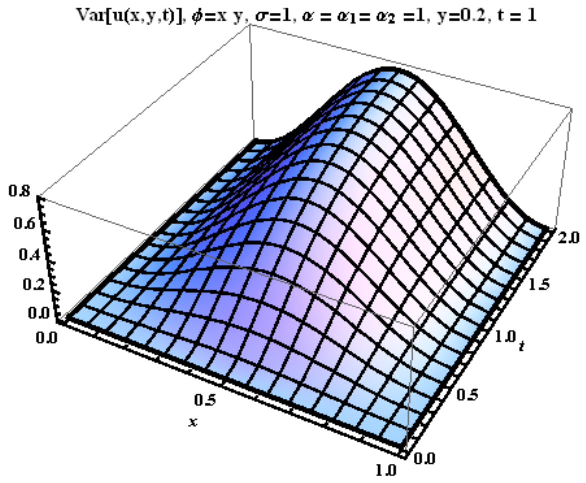

Figure9. The change of the 2^{nd} correction $Var[u(x,y,t;\omega)]$ at fixed value of y , $L=1$

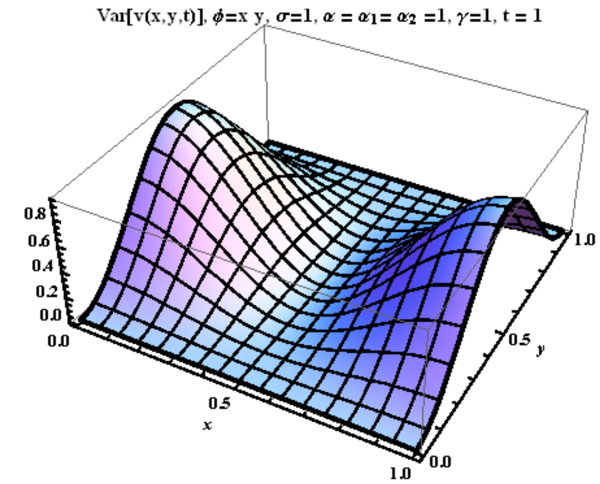

Figure 10. The change of the 2^{nd} correction $Var[v(x,y,t;\omega)]$ at fixed value of t , $L=1$

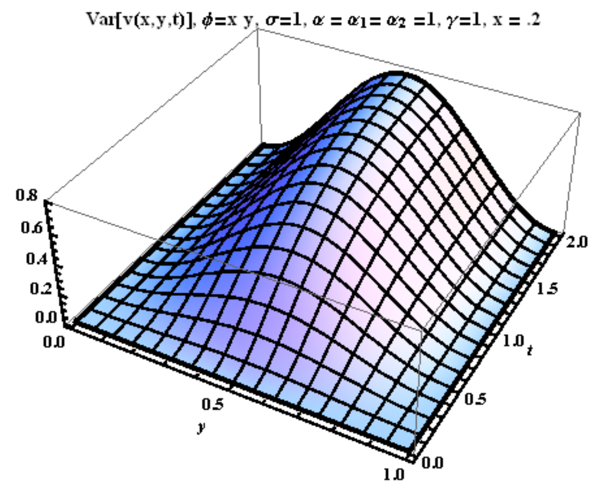

Figure 11. The change of the 2^{nd} correction $Var[v(x,y,t;\omega)]$ at fixed value of x , $L=1$

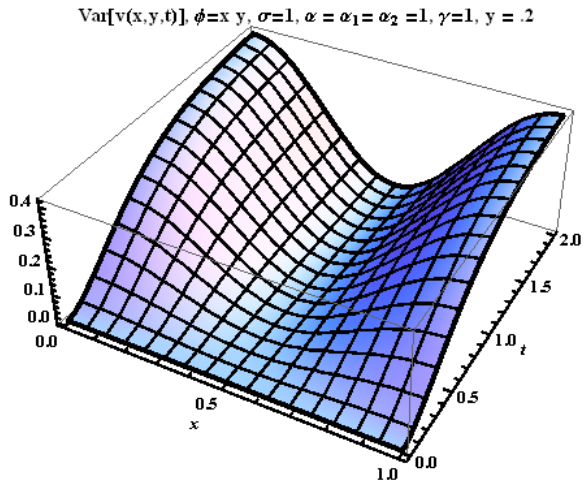

Figure 12: The change of the 2^{nd} correction $Var[v(x,y,t;\omega)]$ at fixed value of y , $L=1$

Case study 3. In this case, the variation of the third statistical moment of the solution process at fixed value for one independent variable is presented from the following figures,

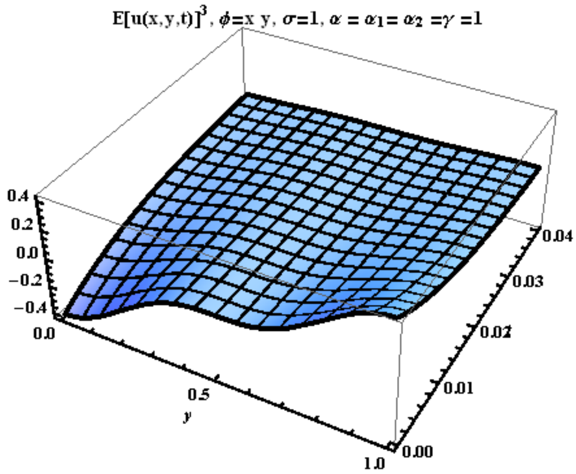

Figure 13. The 2^{nd} correction $E[u(x,y,t;\omega)]^3$ at fixed value of x , $L=1, x=0.2$

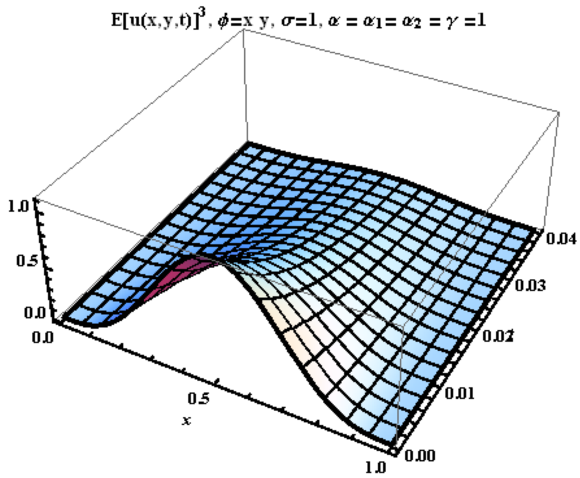

Figure 14. The 2^{nd} correction $E[u(x,y,t;\omega)]^3$ at fixed value of y , $L=1, y=0.8$

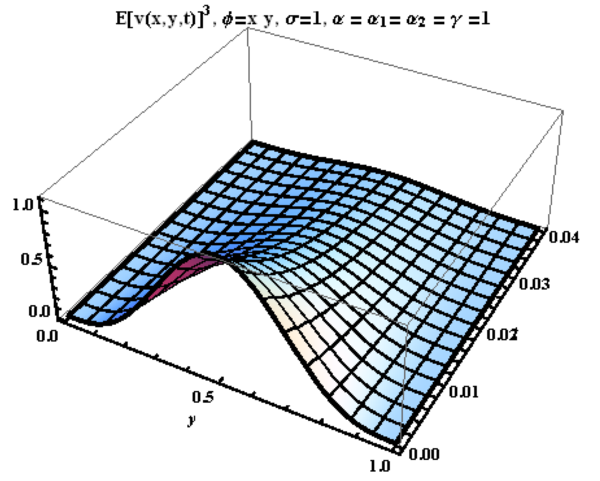

Figure 15. The 2^{nd} correction $E[v(x,y,t;\omega)]^3$ at fixed value of x, $L=1, x=0.2$

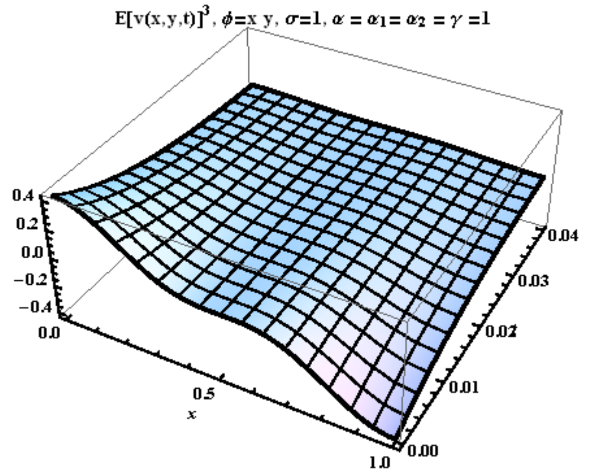

Figure 16. The 2^{nd} correction $E[v(x,y,t;\omega)]^3$ at fixed value of y, $L=1, y=0.8$

Case study 4. From the results of the homotopy –WHEP technique, we present in the following figures a comparison between different corrections for the first statistical moment at fixed values for x and y

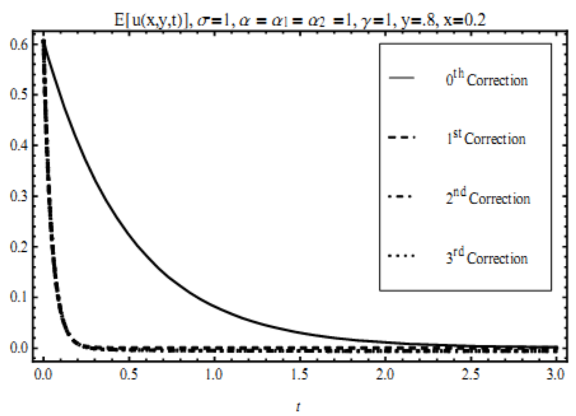

Figure 17. Different corrections of $E[u(x,y,t;\omega)]$, $L=1$

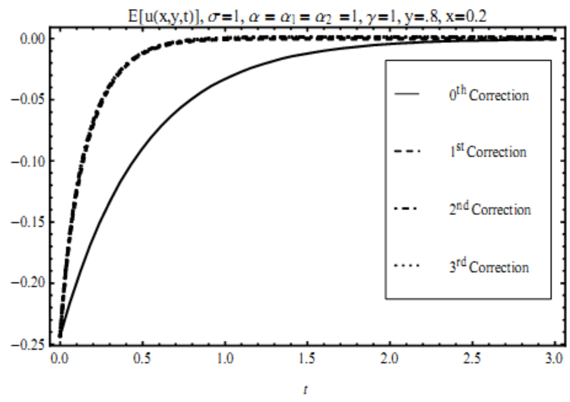

Figure18. Different corrections of $E[u(x,y,t;\omega)]$, $L=2$

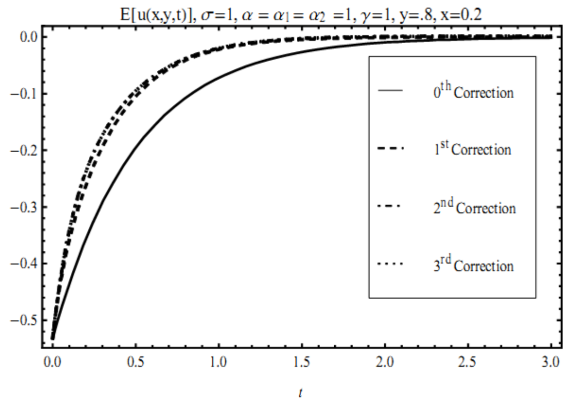

Figure 19. Different corrections of $E[u(x,y,t;\omega)]$, $L=3$

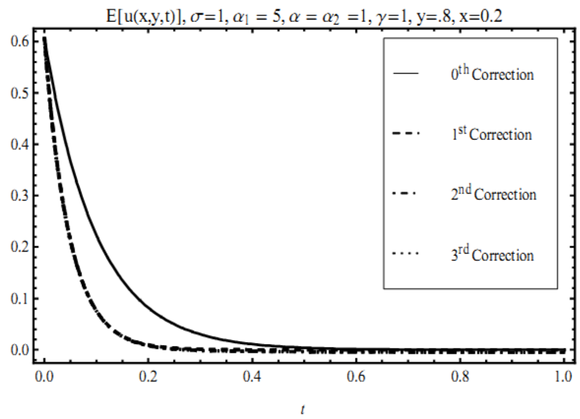

Figure 20. Different corrections of $E[u(x,y,t;\omega)]$, $L=1, \alpha_1=5$

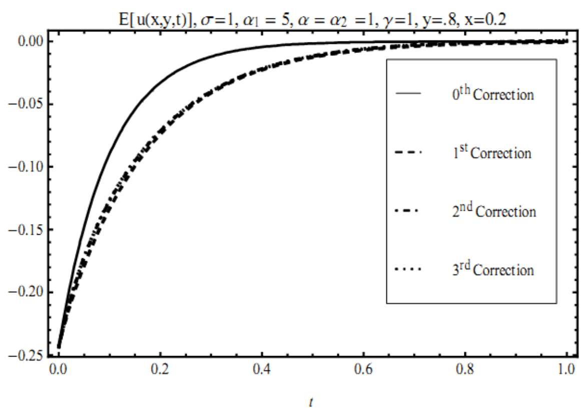

Figure 21. Different corrections of $E[u(x,y,t;\omega)]$, $L=2, \alpha_1=5$

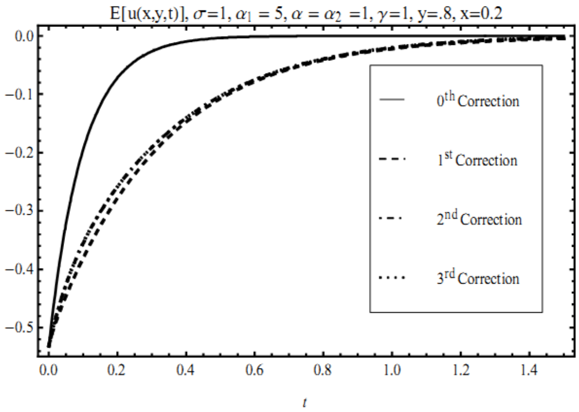

Figure 22. Different corrections of $E[u(x,y,t;\omega)]$, $L=3, \alpha_1 = 5$

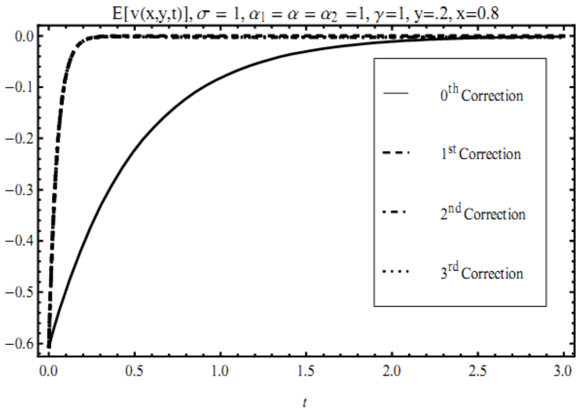

Figure 23. Different corrections of $E[v(x,y,t;\omega)]$, $L=1$

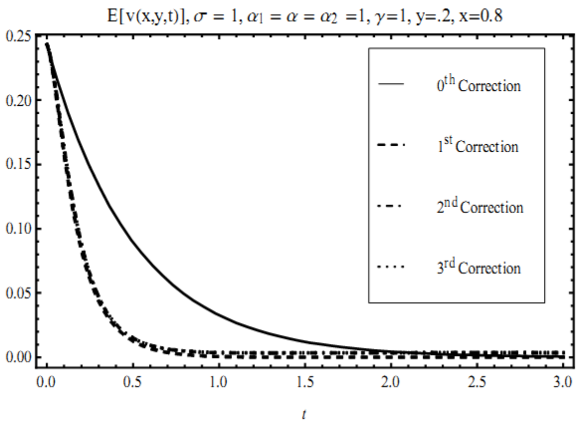

Figure 24. Different corrections of $E[v(x,y,t;\omega)]$, $L=2$

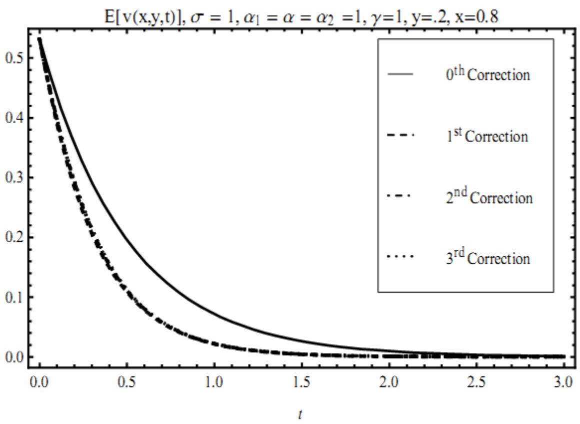

Figure 25. Different corrections of $E[v(x,y,t;\omega)]$, $L=3$

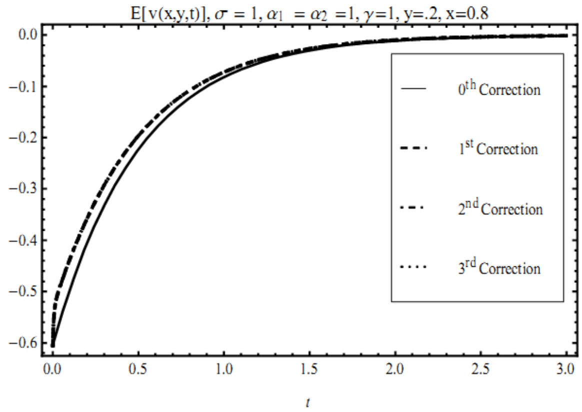

Figure 26. Different corrections of $E[v(x,y,t;\omega)]$, $L=1$, $\alpha=8$

Case study 5. In the following figures, a comparison between different corrections for the variance and some higher statistical moments of the solution process is considered for fixed values of x and y,

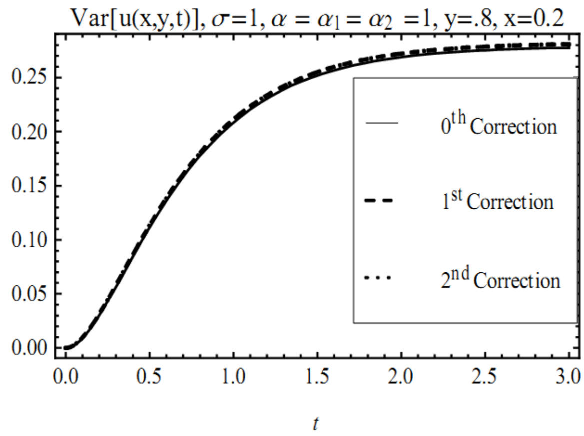

Figure 27. Different corrections of $Var[u(x,y,t;\omega)]$, $L=1$

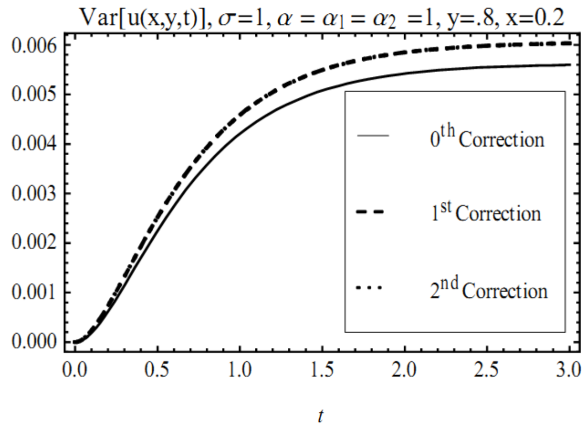

Figure 28. Different corrections of $Var[u(x,y,t;\omega)]$, $L=2$

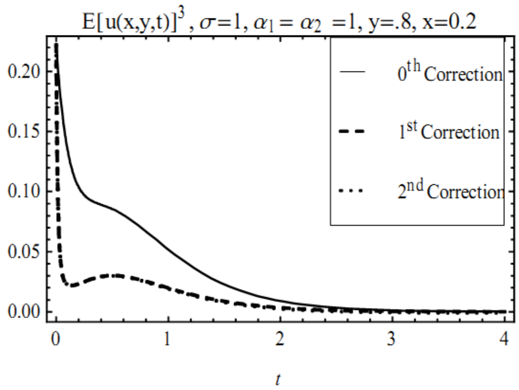

Figure 29. Different corrections of $E[u(x,y,t;\omega)]^3$, $L=1, \alpha=1$

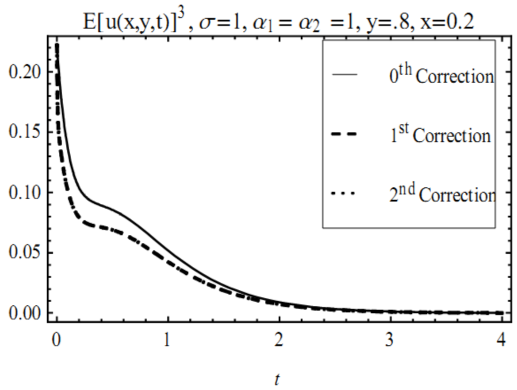

Figure 30. Different corrections of $E[u(x,y,t;\omega)]^3$, $L=1, \alpha=10$

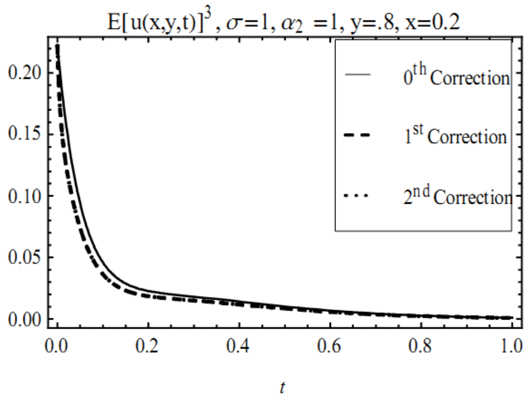

Figure 31. Different corrections of $E[u(x,y,t;\omega)]^3$, $L=1, \alpha=10, \alpha_1=3$

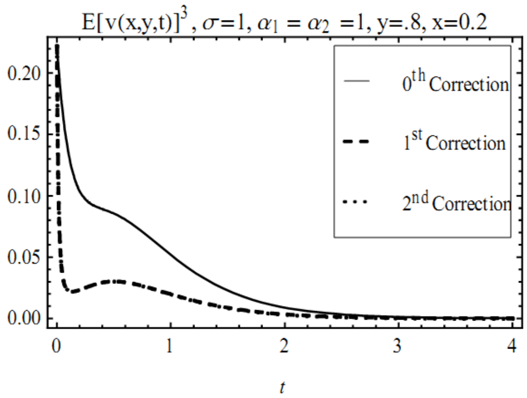

Figure 32. Different corrections of $E[v(x,y,t;\omega)]^3$, $L=1, \alpha=2$

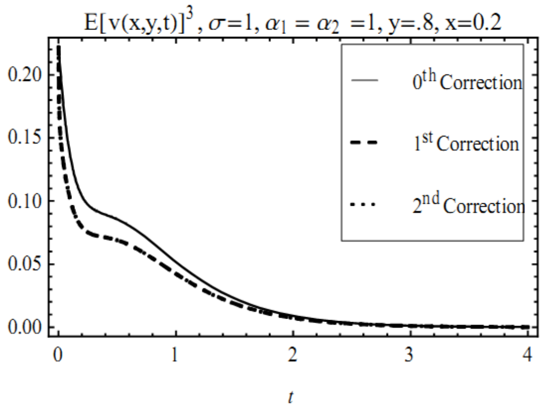

Figure 33. Different corrections of $E[v(x,y,t;\omega)]^3$, $L=1, \alpha=10$

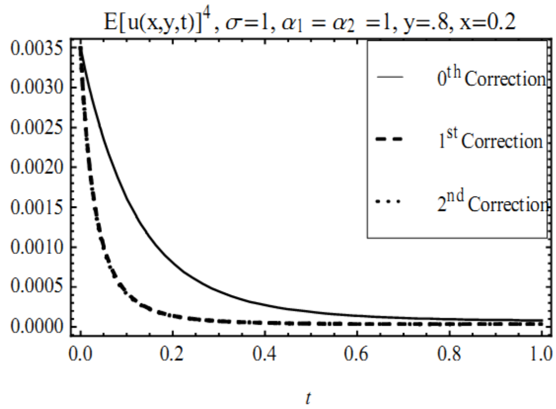

Figure 34. Different corrections of $E[u(x,y,t;\omega)]^4$, $L=2, \alpha=2$

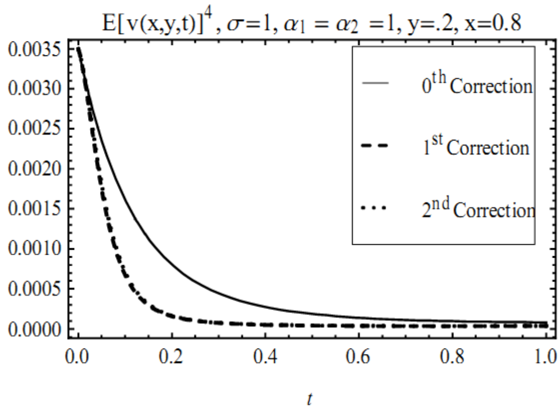

Figure 35. Different corrections of $E[v(x,y,t;\omega)]^4$, $L=2, \alpha=2$

9. Conclusion

In this paper, the homotopy-WHEP was employed to give a statistical analytic solution of the 2-D stochastic Navier-Stokes equations. The application of this method was performed in two phases, the first phase indicated the approximation of the stochastic model using the first order series of the Wiener Hermite expansion of the stochastic solution process and the second phase presented the application of the homotopy perturbation method (HPM) to approximate the deterministic system which reduced from the first phase using the statistical properties of WHPs. From the results of the method analysis, some cases studies indicated some corrections of the approximation process for the statistical moments of the solution process.

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