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Application of the Galerkin Method with Chebyshev Polynomials for Solving the Integral Equation

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Abstract: In this paper, we apply Galerkin method with Chebyshev polynomials to solve the integral equation of the second kind with degenerate kernel. We compare the exact solution with an approximate solution obtained by Galerkin method on numerical examples. Graphical comparison between the exact solution and the approximate solution is carried out. The results show that the Galerkin method with Chebyshev polynomials is efficient and can be applied to solve other problems.

Keywords: approximation, Galerkin method, integral equations, Chebyshev polynomials.

1. Introduction

Projection method has been applying for a long time, and its general abstract treatment goes back to the fundamental theory of Kantorovich [1]. Kantorovich gave a general schema for defining and analyzing the projection method to solve the linear operator equations. To solve the numerical solution of Fredholm integral equation (FIE) Elliott collocation method based on the Chebyshev polynomials and Chebyshev expansions is applied [1].

For numerical resolution of the integral equations

$$g(x) = f(x) + \lambda \int_{-1}^{1} k(x,t) g(t) dt \qquad (1)$$

We develop Galerkin method with Chebyshev polynomials. Sarita Poonia [2], Nik Long [1], have applied the Galerkin method with Laguerre polynomials for obtaining the approximate solution of infinite or finite integral equation.

To solve approximately the integral equation (1), we usually choose a finite dimensional family of function which contains a function $g_n(x)$ close on exact solution $g(x)$.

The favorite approximate solution $g_n(x)$ is designated by forcing it for satisfying the equation (1). There exist various means through which $g_n(x)$ is assumed for satisfying equation (1) approximately, and this results in producing different kind of methods ([1, 2, 3, 4]).

Many researchers have developed the approximate method to solve the integral equation (1) when the limit of integration is finite [1, 5] and literature cited therein.

In this paper, we are doing the study the Galerkin method using Chebyshev polynomials to solve (1). As Chebyshev polynomials are orthogonal by weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $]-1;1[$, it properly fits the density function $g(x)$. The method details is presented in section 2. For degenerate kernel $k(x,t)$, the exact solution is sketched in section 3. Lastly, for different kernel $k(x,t)$ and $f(x)$ some numerical examples are given in section 4.

2. Galerkin Method with Chebyshev Polynomials

Let integral equation

$$g(x) = f(x) + \lambda \int_{-1}^{1} k(x,t) g(t) dt$$

where $f(x)$ is a continuous function and the $k(x,t)$ is the Kernel have singularity in the region

$$D = \{(x,t), -1 < x, t < 1\}$$

and $g(x)$ is a function to be determined.

Consider Chebyshev base function as $\{P_0(x), P_1(x), ...P_n(x)\}$ where

$$P_0(x) = 1, ..P_1(x) = x, .., P_2(x) = 2x^2 - 1...$$

$$P_n(x) = \frac{n}{2} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}$$

$$= \cos(nar\cos x), ..\forall n \in \mathbb{N}.$$

The system $\{P_0(x), P_1(x), ...P_n(x)\}$ form an orthogonal base in $L^2((-1,1), w(x)dx)$, with the following properties

$$< P_n(x), P_m(x) >= \int_{-1}^{1} \frac{P_n(x) P_m(x)}{\sqrt{1-x^2}} dx = 0, .n \neq m.$$

$$\|P_n(x)\|_\infty = 1, ...n = 0, 1, 2, ...$$

The system is called system polynomial Chebyshev [5]. To solve the integral equation (1), we use the projection method by the approximation $g_n(x)$ of the solution of the equation (1) which is a combination linear finished of orthogonal system $\{P_n(x),....n\in\mathbb{N}\}$ and also solution of the integral equation.

$$g_n(x)=f(x)+\lambda\int_{-1}^{1}k(x,t)g_n(t)dt. \qquad (2)$$

By taking the linear combination of Chebyshev polynomials,

$$g_n(x)=\sum_{j=0}^{n}c_jP_j(x). \qquad (3)$$

By replacing (3) with (2), we obtain

$$\sum_{j=0}^{n}c_jP_j(x)=f(x)+\lambda\int_{-1}^{1}k(x,t)(\sum_{j=0}^{n}c_jP_j(t))dt. \qquad (4)$$

Let

$$h_j(x)=\int_{-1}^{1}k(x,t)P_j(t)dt$$

Then equation (4) can be written as

$$\sum_{j=0}^{n}c_j(P_j(x)-\lambda h_j(x))=f(x) \qquad (5)$$

Multiplying (5) by $P_i(x)$ it follow

$$\sum_{j=0}^{n}c_j<P_j(x)-\lambda h_j(x),P_i(x)>=<f(x),P_i(x)>. \qquad (6)$$

By orthogonality condition, the equation (6) can be written as

$$c_i-\lambda\sum_{j=0}^{n}c_j<h_j(x),P_i(x)>=<f(x),P_i(x)>,i=0,1..n \qquad (7)$$

The determinant of this system is the following

$$D(\lambda)=\begin{vmatrix}1-\lambda<h_0(x),P_0(x)> & <h_1(x),P_0(x)> & \cdots & -\lambda<h_n(x),P_0(x)>\\ -\lambda<h_0(x),P_1(x)> & 1-<h_1(x),P_1(x)> & \cdots & -\lambda<h_n(x),P_1(x)>\\ \vdots & \vdots & \vdots & \vdots\\ -\lambda<h_0(x),P_n(x)> & -\lambda<h_1(x),P_n(x)> & \cdots & 1-<h_n(x),P_n(x)>\end{vmatrix}$$

The equation (7) system has an exclusive solution if λ is not eigenvalues $(D(\lambda)\neq 0)$. This allows given coefficients $(c_j)_{(0\leq j\leq n)}$.

3. The exact solution for the degenerate kernel

Consider the degenerate kernel

$$k(x,t)=p_1(x)p_2(t)$$

then the equation (1) becomes

$$g(x)=f(x)+\lambda p_1(x)\int_{-1}^{1}p_2(t)g(t)dt. \qquad (8)$$

Denoting the integral side of (8) by

$$c=\int_{-1}^{1}p_2(t)g(t)dt \qquad (9)$$

we get

$$g(x)=f(x)+\lambda cp_1(x) \qquad (10)$$

Substitution of (10) with (9) gives

$$c=\frac{\int_{-1}^{1}p_2(t)f(t)dt}{1-\lambda\int_{-1}^{1}p_1(t)p_2(t)dt}. \qquad (11)$$

We get from (10) and (11)

$$g(x)=f(x)+\frac{p_1(x)\int_{-1}^{1}p_2(t)f(t)dt}{1-\lambda\int_{-1}^{1}p_1(t)p_2(t)dt}. \qquad (12)$$

where

$$\int_{-1}^{1}p_1(t)p_2(t)dt\neq\frac{1}{\lambda}.$$

4. Numerical examples

Example 1. Let $\lambda=1$ and

$$k(x,t)=t\sqrt{1-x^2},..f(x)=x^5-\frac{2}{7}\sqrt{1-x^2}.$$

We have $p_1(x)=\sqrt{1-x^2}$ and $p_2(t)=t$, due to (12) the exact solution is $g(x)=x^5$.

Now, we calculate the approximate solution, for $n=5$.

First, we calculate the $h_j(x)$ and the coefficients c_j.

By inducting if calculus, we obtain the following results

j	$h_j(x)=\int_{-1}^{1}k(x,t)P_j(t)dt$
0	**0**
1	$\frac{2}{3}\sqrt{1-x^2}$
2	**0**
3	$-\frac{2}{3}\sqrt{1-x^2}$
4	**0**
5	$-\frac{2}{21}\sqrt{1-x^2}$

Calculus of $c_{j(0\leq j\leq n)}$:

i	$c_i-\lambda\sum_{j=0}^{5}c_j<h_j(x),P_i(x)>=<f(x),P_i(x)>$
0	$c_0-\left(\frac{4}{3}c_1-\frac{4}{3}c_3-\frac{4}{21}c_5\right)=-\frac{4}{7}$
1	$c_1=\frac{5\pi}{16}$
2	$c_2-\left(-\frac{4}{9}c_1+\frac{4}{9}c_3+\frac{4}{63}c_5\right)=\frac{4}{21}$
3	$c_3=\frac{5\pi}{32}$
4	$c_4-\left(-\frac{4}{45}c_1+\frac{4}{45}c_3+\frac{4}{315}c_5\right)=\frac{4}{105}$

5	$c_5 = \frac{\pi}{32}$

We obtain the following system

$$\begin{cases} c_0 - \left(\frac{4}{3}c_1 - \frac{4}{3}c_3 - \frac{4}{21}c_5\right) = -\frac{4}{7} \\ c_1 = \frac{5\pi}{16} \\ c_2 - \left(-\frac{4}{9}c_1 + \frac{4}{9}c_3 + \frac{4}{63}c_5\right) = \frac{4}{21} \\ c_3 = \frac{5\pi}{32} \\ c_4 - \left(-\frac{4}{45}c_1 + \frac{4}{45}c_3 + \frac{4}{315}c_5\right) = \frac{4}{105} \\ c_5 = \frac{\pi}{32} \end{cases} \qquad (13)$$

The solution of the system (13) is

$$\begin{cases} c_0 = \frac{17\pi-28}{42} \\ c_1 = \frac{5\pi}{16} \\ c_2 = \frac{48-17\pi}{756} \\ c_3 = \frac{5\pi}{32} \\ c_4 = \frac{48-17\pi}{1260} \\ c_5 = \frac{\pi}{32} \end{cases}$$

So, the approximate solution is :

$$g_5(x) = c_0 + c_1P_1(x) + c_2P_2(x) + c_3P_3(x)$$
$$+c_4P_4(x) + c_5P_5(x)$$
$$= \frac{17\pi}{42} - \frac{2}{3} + \frac{5\pi}{16}x + \frac{48\pi-17}{756}(2x^2-1) + \frac{5\pi}{32}(4x^3-x)$$
$$+\frac{48\pi-17}{1260}(8x^4-8x^2+1) + \frac{\pi}{32}(16x^5-20x^3+5x)$$

Numerical results are shown in Figure 2.

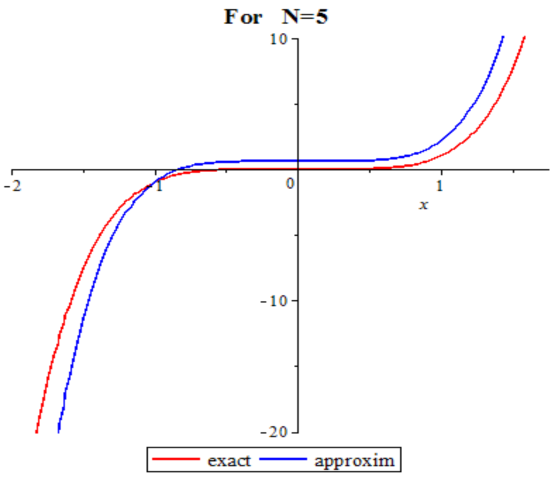

Figure 1. Comparison between the exact solution and the approximate solution.

Example 2. Let $\lambda = 1$ and

$$k(x,t) = (t^3 + t^2)\sqrt{1-x^2},\; f(x) = x^4 - \frac{2}{7}\sqrt{1-x^2}.$$

We have $p_1(x) = \sqrt{1-x^2}$ and $p_2(t) = t^3 + t^2$, due to (12) the exact solution is $g(x) = x^4$.

Now, we calculate the approximate solution, for $n = 4$. Similarly,

j	$h_j(x) = \int_{-1}^{1} k(x,t)P_j(t)dt$
0	$\frac{2}{3}\sqrt{1-x^2}$
1	$\frac{2}{5}\sqrt{1-x^2}$
2	$\frac{2}{15}\sqrt{1-x^2}$
3	$-\frac{2}{35}\sqrt{1-x^2}$
4	$-\frac{26}{105}\sqrt{1-x^2}$

Calculus of $c_{j(0\le j\le n)}$:

i	$c_i - \lambda\sum_{j=0}^{5} c_j \langle h_j(x), P_i(x)\rangle = \langle f(x), P_i(x)\rangle$
0	$-\frac{1}{3}c_0 - \frac{4}{5}c_1 - \frac{4}{15}c_2 + \frac{4}{35}c_3 + \frac{52}{105}c_4 = -\frac{4}{7} + \frac{3\pi}{8}$
1	$c_1 = 0$
2	$\frac{4}{9}c_0 + \frac{4}{15}c_1 + \frac{49}{45}c_2 - \frac{4}{105}c_3 - \frac{52}{315}c_4 = \frac{4}{21} + \frac{\pi}{4}$
3	$c_3 = 0$
4	$-\frac{4}{45}c_0 + \frac{4}{75}c_1 + \frac{49}{75}c_2 - \frac{4}{525}c_3 - \frac{1523}{1575}c_4 = \frac{4}{105} + \frac{\pi}{16}$

We obtain the following system

$$\begin{cases} -\frac{1}{3}c_0 - \frac{4}{15}c_2 + \frac{52}{105}c_4 = -\frac{4}{7} + \frac{3\pi}{8} \\ c_1 = 0 \\ \frac{4}{9}c_0 + \frac{49}{45}c_2 - \frac{52}{315}c_4 = \frac{4}{21} + \frac{\pi}{4} \\ c_3 = 0 \\ -\frac{4}{45}c_0 + \frac{49}{75}c_2 - \frac{1523}{1575}c_4 = \frac{4}{105} + \frac{\pi}{16} \end{cases} \qquad (14)$$

The solution of the system (14) by using the elimination of GAUSS is

$$\begin{cases} c_0 = \frac{40500}{13009} - \frac{11373\pi}{52036} \\ c_1 = 0 \\ c_2 = -\frac{13500}{13009} + \frac{4200\pi}{13009} \\ c_3 = 0 \\ c_4 = \frac{4980}{13009} + \frac{5565\pi}{208144} \end{cases}$$

the approximate solution is

$$g_4(x) = c_0 + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + c_4P_4(x)$$
$$= \frac{40500}{13009} - \frac{11373\pi}{52036} + \left(-\frac{13500}{13009} + \frac{4200\pi}{13009}\right)(2x^2-1)$$
$$+\left(\frac{4980}{13009} + \frac{5565\pi}{208144}\right)(8x^4-8x^2+1).$$

Numerical results are shown in Figure 2.

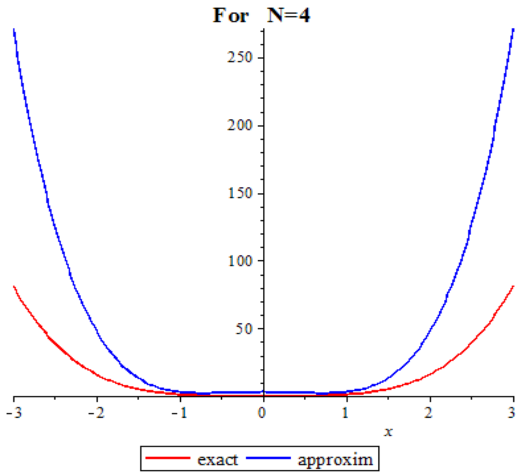

Figure 2. Comparison between the exact solution and the approximate solution

5. Conclusion

Because of the orthogonality property on the interval $]-1,1[$ with square root weight function, the Chebyshev polynomials as an approximate solution have been used . To find the unknown coefficients, Galerkin method is used. For obtaining the approximate solution Maple software is used. The graph in Figure 1 and Figure 2 show a good convergence on interval $]-1,1[$.

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