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Triple Positive Solutions of Higher-Order Nonlinear Boundary Value Problems

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Abstract: In this paper, by using Leggett-Williams fixed-point theorem and Hölder inequality, we study the existence of three positive solutions for higher-order two-point boundary value problem (BVP):

$$(-1)^n u^{(2n)}(t) = \omega(t) f(t, u(t)), \quad t \in [a,b],$$

$$u^{(2i)}(a) = u^{(2i)}(b) = 0, \quad i \in \{0,1,2,...,n-1\},$$

where $\omega(t)$ is L^p-integrable function for some $1 \le p \le +\infty$, $n \in \mathbb{N}$, $b > a \ge 0$ and $f \in C([a,b]\times[0,+\infty),[0,+\infty))$. The results are illustrated with an example.

Keywords: higher-order differential equation, triple positive solutions, Green's function.

1. Introduction

In this paper, the focus is on the existence of three positive solutions for higher-order two-point boundary value problem such that:

$$(-1)^n u^{(2n)}(t) = \omega(t) f(t, \omega(t)), \quad t \in [a,b], \quad (1)$$

$$u^{(2i)}(a) = u^{(2i)}(b) = 0, \quad i \in \{0,1,2,...,n-1\}, \quad (2)$$

where $\omega(t)$ is L^p-integrable function for some $1 \le p \le +\infty$ and $n \in \mathbb{N}$. In addition f and ω satisfy

(A_1) $\omega(t)$ is L^p-integrable function for some $1 \le p \le +\infty$ and there exists $\lambda > 0$ such that $\omega(t) \ge \lambda$ *a.e. on* $[a,b]$, and

(A_2) $f \in C([a,b]\times[0,+\infty),[0,+\infty))$.

The theory of multi-point boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section is composed of N parts of different densities that can be set up as a multi-point boundary value problem. Many problems in the theory of elastic stability can be handled as multi-point boundary value problems too. Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability, and astronomy, be a mandlong wave theory, induction motors, engineering and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in different areas of applied sciences, see the related studies [1-10] and the references therein. Many authors have studied the higher-order boundary value problems, they only considered that f is non-decreasing or non-increasing in t, or the boundary condition depends only on derivatives of even orders; see the related studies [11-17] and the references therein.

Inspired and motivated by the works mentioned above, in this paper we study the existence of three positive solutions for BVP $(1)-(2)$. The arguments are based upon a fixed point theorem due to Leggett and Williams which deals with fixed points of a cone-preserving operator defined on an ordered Banach space [18] . The current paper is organized as follows. In section 2, we provide some lemmas that will be used to prove our main results. In section 3, the main results of BVP $(1)-(2)$ will be stated and proved, and we give an example to illustrate our results.

2. Preliminaries

We shall consider the Banach space $C[a,b]$ equipped with the norm $\|u\| = \max_{a\le t\le b}|u(t)|$ for any $u \in C[a,b]$ and $C^+[a,b] = \{u \in C[a,b] : u(t) \ge 0, t \in [a,b]\}$.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $K \subset E$ is said to be a cone provided that

(i) $c_1 u + c_2 v \in K$ for all $c_1 > 0, c_2 > 0$, and

(ii) $u \in K, -u \in K$ implies $u = 0$.

Every cone K induces an ordering in E given by $u \le v$ if and only if $v - u \in K$.

Definition 2.2. The map $\beta : K \to [0,+\infty)$ is said to be nonnegative continuous concave functional on a cone K of a real Banach space E provided $\beta : K \to [0,+\infty)$ is continuous and for all $u \in K, v \in K$,

$$\beta(tu + (1-t)v) \ge t\beta(u) + (1-t)\beta(v), \ \forall t \in [a,b].$$

In arriving our results, we need the following four preliminary lemmas. The first is well known.

Lemma 2.1. (see[6]).
Let $y(t,u(t)) \in C([a,b]\times[0,\infty))$, then the BVP

$$\begin{cases} (-1)^n u^{(2n)}(t) = y(t,u(t)), & t\in[a,b], \\ u^{(2i)}(a) = u^{(2i)}(b) = 0, & i\in\{0,1,2,...,n-1\}, \end{cases} \quad (3)$$

has a unique solution

$$u(t) = \int_a^b G_j(t,s)\, y(s,u(s))\, ds, \quad (4)$$

where

$$G_j(t,s) = \int_a^b G(t,\tau) G_{j-1}(\tau,s)\, ds,\ j\in\{2,3,...,n\}, \quad (5)$$

where $G_1(t,s) = G(t,s)$ is such that

$$G_1(t,s) = \begin{cases} \dfrac{(t-a)(b-s)}{b-a}, & a\le t\le s\le b, \\ \dfrac{(s-a)(b-t)}{b-a}, & a\le s\le t\le b. \end{cases}$$
(6)

It is obvious that

$$G_n(t,s) > 0, \quad (t,s)\in(a,b)\times(a,b). \quad (7)$$

Lemma 2.2. For all $(t,s)\in[a,b]\times[a,b]$, we have

$$G_n(t,s) \le \left(\frac{b-a}{6}\right)^{n-1} \frac{(s-a)(b-s)}{b-a},\ n\in\mathbb{N}^*. \quad (8)$$

Proof. Let $(t,s)\in[a,b]\times[a,b]$, it is clear from (6) that

$$G(t,s) \le \frac{(s-a)(b-s)}{b-a} \quad (9)$$

i.e. (8) is true for $n=1$. Assume (8) holds for $n=k\ge 1$. Then, for $(t,s)\in[a,b]\times[a,b]$, it follows from (5), (7) and (9) that

$$G_{k+1}(t,s) = \int_a^b G(t,\tau) G_k(\tau,s)\, d\tau$$

$$\le \int_a^b \frac{(\tau-a)(b-\tau)}{b-a} \left(\frac{b-a}{6}\right)^{k-1} \frac{(s-a)(b-s)}{b-a}\, d\tau$$

$$= \left(\frac{b-a}{6}\right)^{k} \frac{(s-a)(b-s)}{b-a}.$$

Thus (8) is true for $n=k+1$. The proof is complete. □

Lemma 2.3. Let $\delta\in\left(a,\dfrac{a+b}{2}\right)$, then for all $(t,s)\in[\delta,b-\delta]\times[a,b]$, we have

$$G_n(t,s) \ge \theta_n(\delta) \frac{(s-a)(b-s)}{b-a} \quad (10)$$

$$\ge \left(\frac{6}{b-a}\right)^{n-1} \theta_n(\delta) \max_{a\le t\le b} G_n(t,s). \quad (11)$$

Where $0<\theta_n(\delta)<1$ is a constant given by

$$\theta_n(\delta) = (\delta-a)^n \left(\frac{4\delta^3 - 6b\delta^2 + 6ab\delta - 3ab^2 + b^3}{6(b-a)}\right)^{n-1}$$

Proof. For $(t,s)\in[\delta,b-\delta]\times[a,b]$, from (6) we find

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b-s)}{b-a}, & a\le t\le s\le b \\ \dfrac{(s-a)(b-t)}{b-a}, & a\le s\le t\le b \end{cases}$$

$$\ge \begin{cases} \dfrac{(\delta-a)(b-s)}{b-a}, & a\le t\le s\le b \\ \dfrac{(s-a)(b-(b-\delta))}{b-a}, & a\le s\le t\le b \end{cases}$$

$$\ge \frac{(\delta-a)(s-a)(b-s)}{b-a}. \quad (12)$$

Hence (10) is true for $n=1$. Suppose that (11) is true for $n=k\ge 1$. Then, using (5), (7) and (12), for $(t,s)\in[\delta,b-\delta]\times[a,b]$, we get

$$G_{k+1}(t,s) = \int_a^b G(t,\tau) G_k(\tau,s)\, d\tau$$

$$\ge \int_\delta^{b-\delta} G(t,\tau) G_k(\tau,s)\, d\tau$$

$$\ge \int_\delta^{b-\delta} \frac{(\delta-a)(\tau-a)(b-\tau)}{b-a} \theta_k(\delta) \frac{(s-a)(b-s)}{b-a}\, d\tau$$

$$= \theta_{k+1}(\delta) \frac{(s-a)(b-s)}{b-a}.$$

So, (11) is true for $n=k+1$. The proof is complete. □

Define a cone K by

$$K = \{u\in C^+[a,b] : u(t)\ge 0,\ t\in[a,b]\}.$$

It is easy to see that K is closed convex cone in

$C^+[a,b]$.

Now, define an integral operator $T: K \to C^+[a,b]$ by

$$(Tu)(t) = \int_a^b G_j(t,s)\omega(s) f(s,u(s))ds, \ j \in \mathbb{N}^*$$
$$= \int_a^b \int_a^b G(t,s)G_{j-1}(s,\tau)\omega(\tau) f(\tau,u(\tau))d\tau ds.$$

We know that $u \in C^+[a,b]$ is a solution of BVP $(1)-(2)$ if and only if u is a fixed point of operator T.

Lemma 2.4. Assume (A_1) and (A_2) hold. Then $T(K) \subset K$ and $T: K \to K$ is completely continuous.

Proof. Since the proof of completely continuous is standard, we need only to prove $T(K) \subset K$. In fact, for $y \in K$ and since $f(t,u(t)) \geq 0$ and $\omega(t) \geq 0$ for any $(t,u(t)) \in [a,b] \times [0,\infty)$, then $Ty \geq 0$. □

Now, let $0 < d < l < r$ be given and let β be nonnegative continuous concave functional on the cone K. Define the convex sets K_l and $K(\beta,l,r)$ by

$$K_l = \{u \in K : \|u\| < l\},$$

and

$$K(\beta,l,r) = \{u \in K : l \leq \beta(u), \|u\| \leq r\}.$$

The key tool in our approach is the following Leggett-Williams fixed point theorem.

Theorem 2.1. (see [18]). Let E be a Banach space and $K \subset E$ be a cone in E. $T: \overline{K_l} \to \overline{K_l}$ be a completely continuous operator and β be nonnegative continuous concave functional on K with $\beta(u) \leq \|u\|$ for all $u \in K_l$. Suppose there exist $0 < d < l < r \leq c$ such that

(i) $u \in \{K(\beta,l,r) : \beta(u) > l\} \neq \emptyset$ and $\beta(Tu) > l$ for all $u \in K(\beta,l,r)$,

(ii) $\|Tu\| < d$ for $\|u\| < d$,

(iii) $\beta(Tu) > l$ for $u \in K(\beta,l,c)$ with $\|Tu\| > r$.

Then T has at least three positive solutions u_1, u_2 and u_3 satisfying
$\|u_1\| < d, l < \beta(u_2), \|u_3\| > d$ and $\beta(u_3) < l$.

We will employ Hölder inequality.

Lemma 2.5. (Hölder). Let $f \in L^p[a,b]$ with $p > 1$, $g \in L^q[a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a,b]$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Let $f \in L^1[a,b]$, $g \in L^\infty[a,b]$. Then $fg \in L^1[a,b]$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

3. Main Results

In this section, we apply Theorem 2.1 and Lemma 2.5 to establish the existence of triple positive solutions for BVP $(1)-(2)$.

We consider the following three cases for $\omega \in L^p[a,b]$, $p > 1$, $p = 1$ and $p = \infty$.

For convenience, we introduce the following notation:

$$f^\infty = \lim_{u\to\infty} \sup \max_{t\in[a,b]} \frac{f(t,u)}{u}, \quad D = \left(\frac{b-a}{6}\right)^{j-1} \|e\|_q \|\omega\|_p.$$

Theorem 3.1. Assume (A_1) and (A_2) hold. Furthermore, suppose that there exist constants $0 < d < l < \frac{l}{\delta^*} \leq c$ such that

(H_1) $f^\infty < \frac{1}{D}$,

(H_2) $f(t,u) > \frac{6l}{\theta_n(\delta)(b-a)\lambda}$,

for $(t,u) \in [\delta, b-\delta] \times [a,b]$, $\delta \in \left[a, \frac{a+b}{2}\right]$, $n \in \mathbb{N}^*$,

(H_3) $f(t,u) < \frac{d}{D}$, $for\ (t,u) \in [a,b] \times [0,d]$.

Then BVP $(1)-(2)$ has at least three positive solutions u_1, u_2 and u_3 satisfying
$\|u_1\| < d, l < \beta(u_2), \|u_3\| > d$ and $\beta(u_3) < l$.

Proof. Let $\beta = \min_{t\in[\delta,b-\delta]} u(t)$. Then $\beta(u)$ is nonnegative continuous concave functional on K with $\beta(u) \leq \|u\|$ for all $u \in K$.

We denote $r = \frac{l}{\delta^*}$. From (H_1), there exist

$0<\sigma<\frac{1}{D}$ and $l>0$ such that $f(t,u)\le\sigma u$ and $u\ge l$. Let $\eta=\max_{0\le u\le l,\, a\le t\le b} f(t,u)$. Then

$$f(t,u)\le\sigma u+\eta,\ t\in[a,b],\ 0\le u\le+\infty, \quad (13)$$

Set $c>\max\left\{\frac{D\eta}{1-D\sigma},\frac{l}{\left(\frac{6}{b-a}\right)^{j-1}\theta_j(\delta)}=\frac{l}{\delta^*}\right\}$ and $e(s)=\frac{(s-a)(b-s)}{b-a}$. Then, for $u\in\overline{K_c}$, it follows from (8) and (13) that

$$\begin{aligned}(Tu)(t)&=\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds\\
&\le\int_a^b\left(\frac{b-a}{6}\right)^{j-1}\frac{(s-a)(b-s)}{b-a}(\sigma u+\eta)\omega(s)ds\\
&\le\int_a^b\left(\frac{b-a}{6}\right)^{j-1}\frac{(s-a)(b-s)}{b-a}(\sigma\|u\|+\eta)\omega(s)ds\\
&\le\left(\frac{b-a}{6}\right)^{j-1}(\sigma\|u\|+\eta)\int_a^b\frac{(s-a)(b-s)}{b-a}\omega(s)ds\\
&\le\left(\frac{b-a}{6}\right)^{j-1}(\sigma c+\eta)\int_a^b\frac{(s-a)(b-s)}{b-a}\omega(s)ds\\
&\le\left(\frac{b-a}{6}\right)^{j-1}(\sigma c+\eta)\|\omega\|_p\|e\|_q\\
&<c.\end{aligned}$$

Which shows that $Tu\in K_c$.

Hence, we have shown that if (H_1) holds, then T maps $\overline{K_c}$ into K_c.

We verify that $\{u/u\in K(\beta,l,r):\beta(u)>l\}\ne\varnothing$ and $\beta(Tu)>l$ for all $u\in K(\beta,l,r)$.

Take $\varphi_0(t)=\frac{(\delta^*+1)l}{\delta^*},\ for\ t\in[a,b]$. Then

$$\varphi_0\in\left\{u/u\in K\left(\beta,l,\frac{l}{\delta^*}\right),\beta(u)>l\right\}.$$

This shows that $\{u/u\in K(\beta,l,r):\beta(u)>l\}\ne\varnothing$.

Therefore, from (H_2) and (11), we have

$$\begin{aligned}\beta(Tu)&=\min_{t\in[\delta,b-\delta]}(Tu)(t)\\
&=\min_{t\in[\delta,b-\delta]}\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds\\
&\ge\theta_j(\delta)\int_a^b\omega(s)\frac{(s-a)(b-s)}{b-a}f(s,u(s))ds\\
&\ge\frac{6l}{(b-a)}\int_a^b\frac{(s-a)(b-s)}{b-a}ds\ge l.\end{aligned}$$

If $u\in\overline{K_d}$, then it follows from (H_3) that

$$\begin{aligned}(Tu)(t)&=\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds\\
&\le\left(\frac{b-a}{6}\right)^{j-1}\left(\frac{d}{D}\right)\int_a^b\frac{(s-a)(b-s)}{b-a}\omega(s)ds\\
&\le\left(\frac{b-a}{6}\right)^{j-1}\left(\frac{d}{D}\right)\|\omega\|_p\|e\|_q=d.\end{aligned}$$

Finally, we assert that if $u\in K(\beta,l,c)$ and $\|Tu\|>r$, then $\beta(Tu)>l$.

Suppose $u\in K(\beta,l,c)$ and $\|Tu\|>r$, then it follows from (11) that

$$\begin{aligned}\beta(Tu)&=\min_{t\in[\delta,b-\delta]}(Tu)(t)\\
&=\min_{t\in[\delta,b-\delta]}\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds\\
&\ge\theta_j(\delta)\int_a^b\omega(s)\frac{(s-a)(b-s)}{b-a}f(s,u(s))ds\\
&\ge\left(\frac{6}{b-a}\right)^{j-1}\theta_j(\delta)\int_a^b\max_{t\in[a,b]}G_j(t,s)\omega(s)f(s,u(s))ds\\
&\ge\delta^*\|Tu\|>l.\end{aligned}$$

To sum up, the hypotheses of Theorem 2.1 hold. Therefore, BVP $(1)-(2)$ has at least three positive solutions u_1,u_2 and u_3 such that

$\|u_1\|<d,\ l<\beta(u_2),\ \|u_3\|>d$ and $\beta(u_3)<l$. □

The following corollarys deals with the case $p=+\infty$.

Corollary 3.1. Assume $(A_1),(A_2),(H_1),(H_2)$ and (H_3) hold. Then BVP $(1)-(2)$ has at least three positive solutions u_1,u_2 and u_3 such that $\|u_1\|<d,\ l<\beta(u_2),\ \|u_3\|>d$ and $\beta(u_3)<l$.

Proof. Let $\|\omega\|_\infty\|e\|_1$ replace $\|\omega\|_p\|e\|_q$ and repeat the argument above. □

Finally, we consider the case $p=1$. Let

(H_4) $\quad f^{\infty}<\frac{1}{D'}$,

(H_5) $\quad f(t,u)\le \frac{d}{D'},\ for\ (t,u)\in[a,b]\times[0,d]$,

where

$$D'=\left(\frac{b-a}{6}\right)^{j-1}\|\omega\|_1.$$

Corollary 3.2. Assume $(A_1),(A_2),(H_4)$ and (H_5) hold. Then BVP $(1)-(2)$ has at least three positive solutions u_1,u_2 and u_3 such that

$\|u_1\|<d,\ l<\beta(u_2),\ \|u_3\|>d$ and $\beta(u_3)<l$.

Proof. Set

$$c'>\max\left\{\frac{D'\eta}{1-D'\sigma'},\frac{l}{\left(\frac{6}{b-a}\right)^{j-1}\theta_j(\delta)}=\frac{l}{\delta^*}\right\} \text{ and}$$

$$e(s)=\frac{(s-a)(b-s)}{b-a},\ \text{ where } 0<\sigma'<\frac{1}{D'}.$$

Then, for $u\in K_{c'}$, it follows from (8) and (13) that

$$(Tu)(t)=\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds$$
$$\le\int_a^b\left(\frac{b-a}{6}\right)^{j-1}\frac{(s-a)(b-s)}{b-a}(\sigma' u+\eta)\omega(s)ds$$
$$\le\int_a^b\left(\frac{b-a}{6}\right)^{j-1}\frac{(s-a)(b-s)}{b-a}(\sigma'\|u\|+\eta)\omega(s)ds$$
$$\le\left(\frac{b-a}{6}\right)^{j-1}(\sigma' c'+\eta)\int_a^b\frac{(s-a)(b-s)}{b-a}\omega(s)ds$$
$$\le\left(\frac{b-a}{6}\right)^{j-1}(\sigma' c'+\eta)\|\omega\|_1<c'.$$

Which shows that $Tu\in K_{c'}$. Hence, we have shown that if (H_4) holds, then T maps $\overline{K}_{c'}$ into $K_{c'}$.

If $u\in\overline{K_d}$, then it follows from (H_5) that

$$(Tu)(t)=\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds$$
$$\le\left(\frac{b-a}{6}\right)^{j-1}\left(\frac{d}{D'}\right)\int_a^b\frac{(s-a)(b-s)}{b-a}\omega(s)ds$$
$$\le\left(\frac{b-a}{6}\right)^{j-1}\left(\frac{d}{D'}\right)\|\omega\|_1=d.$$

Finally, we assert that if $u\in K(\beta,l,c)$ and $\|Tu\|>r$, then $\beta(Tu)>l$.

Suppose $u\in K(\beta,l,c)$ and $\|Tu\|>r$, then it follows from (11) that

$$\beta(Tu)=\min_{t\in[\delta,b-\delta]}(Tu)(t)$$
$$=\min_{t\in[\delta,b-\delta]}\int_a^b G_j(t,s)\omega(s)f(s,u(s))ds$$
$$\ge\theta_j(\delta)\int_a^b\omega(s)\frac{(s-a)(b-s)}{b-a}f(s,u(s))ds$$
$$\ge\left(\frac{6}{b-a}\right)^{j-1}\theta_j(\delta)\int_a^b\max_{t\in[a,b]}G_j(t,s)\omega(s)f(s,u(s))ds$$
$$\ge\delta^*\|Tu\|>l.$$

To sum up, the hypotheses of Theorem 2.1 hold. Therefore, BVP $(1)-(2)$ has at least three positive solutions u_1,u_2 and u_3 such that

$\|u_1\|<d,\ l<\beta(u_2),\ \|u_3\|>d$ and $\beta(u_3)<l$. □

Remark 3.1.

We remark that the condition (H_3) in Theorem 3.1 can be replaced by the following condition

$(H_3)'$ $\quad f_0^d\le\frac{1}{D}$, where

$$f_0^d=\max\left\{\max_{t\in[a,b]}\frac{f(t,u)}{d}:u\in[0,d]\right\}.$$

$(H_3)''$ $\quad f^0\le\frac{1}{D}$.

Corollary 3.3. If the condition (H_3) in Theorem 3.1 is replaced by $(H_3)'$ or $(H_3)''$, respectively, then the conclusion of Theorem 3.1 also holds.

We construct an example to illustrate the applicability of the results presented.

Example 1. Let $\delta=\frac{1}{4}, n=3, a=0, b=1$ and $p=1$.

It follows from $p=1$ that $q=\infty$. Consider the following boundary value problem

$$\begin{cases} -u^{(6)}(t)=\omega(t)\,f\left(t,u(t)\right), & t\in I=[0,1], \\ u^{(2i)}(0)=u^{(2i)}(1)=0, & i=0,1,2, \end{cases}$$

where $\omega(t)=t+2$ and

$$g=\begin{cases} d, & (t,u)\in I\times[0,d] \\ u+(10)^3\left(\dfrac{l}{\delta^*}\right)\left(\dfrac{u-d}{l-d}\right), & (t,u)\in I\times[d,l] \\ (10)^3\left(\dfrac{l}{\delta^*}\right), & (t,u)\in I\times\left[l,\dfrac{l}{\delta^*}\right] \\ (10)^3\left(\dfrac{l}{\delta^*}\right)+\left(u-\dfrac{l}{\delta^*}\right)t, & (t,u)\in I\times\left[\dfrac{l}{\delta^*},\infty\right) \end{cases}$$

where $g=f\left(t,u(t)\right)$.

It is easy to see by calculating that

$$\omega(t)\geq\lambda=2,\ for\ a.e.\,t\in[0,1].$$

By simple calculation, we obtain

$$\theta_3(\delta)=\frac{1}{64}\left(\frac{11}{96}\right)^2,\quad \delta^*=\frac{1}{64}\left(\frac{11}{16}\right)^2.$$

It follows from $\omega(t)=t+2$ and $e(t)=t(1-t)$ that

$$\|\omega\|_1=\int_0^1(t+2)\,dt=\frac{5}{2},$$

and

$$\|e\|_q=\|e\|_\infty=\lim_{q\to\infty}\left(\int_0^1(t)^q(1-t)^q dt\right)^{\frac{1}{q}}=1.$$

It is easy to verify that

$$D'=\left(\frac{1}{6}\right)^2\|\omega\|_1=\frac{5}{72},\quad f^\infty=1.$$

Choosing $0<d<l<\dfrac{l}{\delta^*}\leq c$, we have

$$f^\infty=1<\frac{72}{5}=\frac{1}{D'},$$

$$f(t,u)=\frac{10^3 l}{\delta^*}=\frac{16384000}{121}l$$
$$>\frac{1769472}{121}l=\frac{6l}{\theta_3(\delta)(b-a)\lambda},$$

for all $(t,u)\in\left[\dfrac{1}{4},\dfrac{3}{4}\right]\times\left[l,\dfrac{l}{\delta^*}\right]$,

and

$$f(t,u)=d<\frac{d}{D'}=\frac{72d}{5},\quad \forall(t,u)\in[0,1]\times[0,d],$$

which shows $(H_1)-(H_4)$ and (H_5) hold.

Thus all assumptions and conditions of Theorem 2.1 are satisfied. Hence, Corollary 3.2 implies that BVP $(1)-(2)$ has at least three positive solutions u_1,u_2 and u_3 such that

$\|u_1\|<d,\ l<\beta(u_2),\ \|u_3\|>d$ and $\beta(u_3)<l$.

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