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Existence of χ-optimal Solution of Fractional Programming Problem with Interval Parameters

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Abstract: This paper solves a nonlinear fractional programming problem in which the coefficients of the objective function and constraints are interval parameters. Methodology is developed to transform the model into a general optimization problem, which is free from interval uncertainty. Relation between the original problem and the transformed problem is established. Finally, the proposed methodology is explained through a numerical example.

Keywords: fractional programming, interval valued function, interval inequalities, partial order relation.

1. Introduction

The fractional programming problem, i.e., the minimization/maximization of ratio of two functions subject to the given conditions, arises in various optimization problem; linear fractional programming is used in game theory; quadratic fractional programming problem in production planning. A general fractional programming problem is

$$(\mathbf{P}):\quad \min f(x) = \frac{p(x)}{q(x)}$$

subject to $x \in X = \{x | g_j(x) \leq b_j\}, j = 1,2,\dots,m$

where $p, q, g_j: \mathbb{R}^n \to \mathbb{R}, X \subseteq \mathbb{R}^n$ with the assumption $q(x) \neq 0$. $p(x)$ and $q(x)$ are continuous real valued functions of $x \in X$. Depending upon the linearity and nonlinearity of the functions $p(x)$, $q(x)$ and $g_j(x)$, $(\mathbf{P})$ is said to be linear and nonlinear fractional programming problem accordingly. Different approaches exist in literature for obtaining the optimal solution of particular kinds of fractional programming problems. In real life situation, due to presence of uncertainty in data set, the objective function and the constraints cannot be estimated perfectly. To address these uncertainties most of the researchers have exerted stochastic and fuzzy approaches. In stochastic programming problem, the uncertain parameters are supposed to be random numbers with usual probability distribution functions. In the other hand, the parameters are expected as fuzzy numbers with well-known membership functions. However, choosing suitable membership functions for fuzzy programming technique, probability distribution functions for stochastic programming problem is a difficult task for a decision maker. So, one may use interval number to overcome these difficulties for handling the uncertain parameters by different approaches. Lower and upper bound of the interval can be estimated from the historical data. An interval number can be thought as an extension of a real number and also as subset of a real line. If at least one parameter of an optimization problem is an interval, then this is an optimization problem with interval parameters. In interval optimization problem, at least the objective function and constraints is an interval valued function.

Linear fractional programming with interval uncertainty are discussed by [1-3]. Nonlinear fractional programming problem with interval parameters is investigated by [4]. They have developed a technique to solve a nonlinear interval fractional programming problem depending upon the proper selection of weight function. In presence of interval uncertainty in the proposed model, and selection of weight function, derived solution of the problem may not be fully acceptable for the decision maker. Satisfaction level of the solution is not analyzed in [4]. In this paper two aims are targeted: investigate a fractional programming problem with interval parameters which includes both linear and nonlinear interval valued functions in the objective function as well as in the constraints, derive methodology to find solution of this problem. In addition to this there is no burden of choosing weight function in this proposed methodology. Solution obtained by this proposed methodology is acceptable with certain degree of satisfaction.

The paper is structured by the following sections. Section 2 deals with prerequisites on interval analysis, and introduces χ −order relation for the comparison between two intervals as well as two interval vectors. Nonlinear fractional programming problem with interval parameters is proposed in Section 3, and procedure to find the solution of the problem is explained in this section. The proposed solution procedure is explained through a numerical example in Section 4. Section 5 expresses some conclusions of the present work.

2. Prerequisites

We use the following notations to explain the methodology.

- $I(\mathbb{R})$: Set of closed intervals on $\mathbb{R}$. $I(\mathbb{R})$ is the set $\hat{a} = [a^L, a^R]$.
- Λ_k: $\{1,2,\dots,k\}$.
- Mathematical operations $\circledast$ $(* \in \{+,-,.,/\})$ in $I(\mathbb{R})$ can be interpreted as follows.
  For $\hat{a} = [a^L, a^R]$ and $\hat{b} = [b^L, b^R]$ in $I(\mathbb{R})$, $\hat{a} \circledast \hat{b} = \{a * b : a \in \hat{a}, b \in \hat{b}\}$. For $\hat{a} \oslash \hat{b}$, $0 \notin \hat{b}$.
- The spread of the interval $\hat{a}$, is denoted by $\mu(\hat{a}) = a^R - a^L$.
- The set of interval vectors $I(\mathbb{R})^n$ is the set $\{\hat{a}_v : (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)^T, \ \hat{a}_j \in I(\mathbb{R}), j \in \Lambda_n\}$.

2.1. Order relation in $I(\mathbb{R})$

$I(\mathbb{R})$ is a partially ordered set. There are several partial order relations exist in literature (refer [5-7]). Order relations between two intervals $\hat{a}$ and $\hat{b}$ in $I(\mathbb{R})$ can be explained in two ways; first one is an extension of $<$ on real line that is, $\hat{a} \leq \hat{b}$ iff $a^R < b^L$, and the other is an extension of the idea of set inclusion that is, , $\hat{a} \subseteq \hat{b}$ if and only if $a^L \geq b^L$ and $a^R \leq b^R$. These order relations cannot explain ranking between two overlapping intervals.

We introduce $\preccurlyeq_\chi$ –partial order relation in $I(\mathbb{R})$ by which partial order relation for overlapped intervals can be explicated. Using this partial order relation we establish the existence of solution of (**IP**) at subsequent period.

Two intervals may overlap, one interval may lie behind another interval or one interval may include another interval. To explain this idea mathematically, we incorporate a function $\chi: I(\mathbb{R}) \times I(\mathbb{R}) \to \mathbb{R}$ as follows. For $\hat{a}$ and $\hat{b} \in I(\mathbb{R})$,

$$\chi(\hat{a}, \hat{b}) = \begin{cases} 1, & a^R \leq b^L \\ 0, & a^L \geq b^R \\ \dfrac{b^R - a^L}{(b^R - b^L) + (a^R - a^L)}, & a^L < b^R \text{ and } a^R > b^L \end{cases}.$$

$\chi(\hat{a}, \hat{b})$ describes degree of inferiority of $\hat{a}$ with $\hat{b}$. It can be observed that χ is a continuous function and belongs to [0,1]. Additionally, $\chi(\hat{a}, \hat{b})+\chi(\hat{b}, \hat{a})=1$.

On the basis of the idea of degree of inferiority of two intervals, order relation $\preccurlyeq_\chi$ between two intervals can be interpreted.

Definition 2.1. For two intervals $\hat{a}$,$\hat{b} \in I(\mathbb{R})$,

$$\hat{a} \preccurlyeq_\chi \hat{b}, iff\ \mu(\hat{a}) \leq \mu(\hat{b})\ and\ \chi(\hat{a}, \hat{b}) \in \left[\frac{1}{2}, 1\right]$$

$$\hat{a} \prec_\chi \hat{b}, iff\ \mu(\hat{a}) \leq \mu(\hat{b})\ and\ \chi(\hat{a}, \hat{b}) = 1$$

$$\hat{a} = \hat{b}, iff\ \mu(\hat{a}) = \mu(\hat{b})\ and\ \chi(\hat{a}, \hat{b}) = \tfrac{1}{2}.$$

$\chi(\hat{a}, \hat{b}) \in \left[\frac{1}{2}, 1\right]$ means $\chi(\hat{a}, \hat{b}) \geq \chi(\hat{b}, \hat{a})$.

For example, $\mu([1,4]) < \mu([0,4])$ and $\chi([1,4],[0,4]) = \frac{5}{7}$. So $[1,4] \preccurlyeq_\chi [0,4]$ with degree of inferiority $\frac{5}{7}$;

$\chi([1,5],[3,6]) = \frac{5}{7}$, but $[1,5] \preccurlyeq_\chi [3,6]$ is not true, since $\mu([1,5]) \nleq \mu([3,6])$.

$\mu([1,4]) < \mu([5,9])$ and $\mu([1,4],[5,9])$ =1. So $[1,4] \prec_\chi [5,9]$ with degree of inferiority 1;

$\mu([1,4]) = \mu([-3,0])$ but $\chi([1,4],[-3,0]) = 0$, so $[1,4] \preccurlyeq_\chi [-3,0]$ is not true.

It is easy to prove that $\preccurlyeq_\chi$ is a partial order.

2.2. Order relation in $I(\mathbb{R})^n$

For the comparison between two interval vectors, we interpret the partial order relation $\preccurlyeq_\chi{}^n$ in $I(\mathbb{R})^n$.

Definition 2.2. For $\hat{a}_v = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)^T$ and $\hat{b}_v = (\hat{b}_1, \hat{b}_2, \dots, \hat{b}_n)^T \in I(\mathbb{R})^n$,

$$\hat{a}_v \preccurlyeq_\chi{}^n \hat{b}_v\ iff\ \hat{a}_i \preccurlyeq_\chi \hat{b}_i, \quad \forall\, i \in \Lambda_n.$$

Based upon the idea of inferiority between two intervals, degree of inferiority between two interval vectors $\hat{a}_v$ and $\hat{b}_v$ of dimension n can be defined as

$$\chi(\hat{a}_v, \hat{b}_v) = \min_{i \in \Lambda_n}\{\chi(\hat{a}_i, \hat{b}_i)\} \quad (1)$$

Consider the interval vectors $\hat{a}_v = \begin{pmatrix}[0,2]\\ [2,3]\end{pmatrix}$ and $\hat{b}_v = \begin{pmatrix}[1,4]\\ [2,4]\end{pmatrix}$. $\chi([0,2],[1,4]) = \frac{4}{5}$ and $\chi([2,3],[2,4]) = \frac{2}{3}$.

$\chi(\hat{a}_v, \hat{b}_v) = \min\{\chi([0,2],[1,4]),\ \chi([2,3],[2,4])\} = \frac{2}{3}$.

Hence we say $\hat{a}_v \preccurlyeq_\chi{}^2\ \hat{b}_v$ with degree of inferiority 66%.

2.3. Interval valued function

Interval valued function is defined by many researchers in several manners (refer [5-8]). In this present work we have considered the interval valued function as defined in recent work [7].

2.4. Interval inequalities

Given two intervals $\hat{a} = [a^L, a^R]$ and $\hat{b} = [b^L, b^R]$, decision x satisfying the interval equation $\hat{a}x = \hat{b}$ is given by $\{x \in \mathbb{R} | ax = b, a \in \hat{a}, b \in \hat{b}\}$, provided $0 \notin \hat{a}$. For example, decision x satisfying $[1,4]x = [4,6]$ is $\{x \in \mathbb{R} | ax = b, 1 \leq a \leq 4, 4 \leq b \leq 6\} = [1,6]$. Similar interpretation for interval inequality can be described.

$$\{x \in \mathbb{R} | \hat{a}x \preccurlyeq \hat{b}\} = \{x \in R | ax \leq b, a \in \hat{a}, b \in \hat{b}\}. \quad (2)$$

In our proposed model in Section 3, the constraints are interval inequalities, and the objective function is a fraction of two interval valued functions.

3. Nonlinear Fractional Programming Problem with Interval Parameters

Consider the fractional programming problem with bounded parameters (**IP**) as

(**IP**): min $\hat{p}(x) \oslash \hat{q}(x)$

subject to $\hat{g}_k(x) \preccurlyeq \hat{b}_k, k \in \Lambda_m$, (3)

$x \geq 0$. (4)

Here $\hat{p}, \hat{q}, \hat{g}_k: \mathbb{R}^n \to I(\mathbb{R})$ are given by $\hat{p}(x) \triangleq [p^L(x), p^R(x)]$, $\hat{q}(x) \triangleq [q^L(x), q^R(x)]$ and $\hat{g}_k(x) = [g_k^L(x), g_k^R(x)], \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$. Note that in (**IP**), $q^L(x) > 0$. In (**IP**) all the functions $\hat{p}(x), \hat{q}(x), \hat{g}_k(x)$'s are interval valued nonlinear functions.

3.1. Transformation of the Model

The objective function of (**IP**) is a fraction of two interval valued functions, which increases difficulty solving (**IP**). For this reason, it is required to convert the objective function into a function, which cannot be stated as the fraction of two functions.

The objective function of (**IP**) is

$$\hat{p}(x) \oslash \hat{q}(x), q^L(x) > 0$$
$$\triangleq [p^L(x), p^R(x)] \otimes \frac{1}{[q^L(x), q^R(x)]}$$

$\triangleq [p^L(x), p^R(x)] \otimes \hat{t}$, where $\hat{t} = [t^L, t^R] = \left[\frac{1}{q^R(x)}, \frac{1}{q^L(x)}\right]$

$\triangleq [p^L(x), p^R(x)] \times t$, where $\frac{1}{q^R(x)} \leq t \leq \frac{1}{q^L(x)}, q^L(x) > 0$

$\triangleq t\left[p^L\left(\frac{y}{t}\right), p^R\left(\frac{y}{t}\right)\right]$ with $tq^R\left(\frac{y}{t}\right) \geq 1, tq^L\left(\frac{y}{t}\right) \leq 1, y = xt, y \in \mathbb{R}^n$.

Hence the transformed objective function is

$t\left[p^L\left(\frac{y}{t}\right), p^R\left(\frac{y}{t}\right)\right]$ with $tq^R\left(\frac{y}{t}\right) \geq 1, tq^L\left(\frac{y}{t}\right) \leq 1$, $y = xt, y \in \mathbb{R}^n$.

Using the transformation $xt = y$, the interval inequalities (3) can be transformed into

$$\left[g_k^L\left(\frac{y}{t}\right), g_k^R\left(\frac{y}{t}\right)\right] \preccurlyeq [b_k^L, b_k^R] \quad (5)$$

So the transformed feasible region is given by

$$S=\left\{(y,t):\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]\preccurlyeq[b_k^L,b_k^R]\ \forall\, k,\ tq^R\left(\frac{y}{t}\right)\geq 1,\ tq^L\left(\frac{y}{t}\right)\leq 1,\ y\in\mathbb{R}^n,t\in\mathbb{R}\right\}. \quad (6)$$

Hence the problem becomes

$$\begin{aligned}(\mathbf{IP})(y,t):\ \min\quad & t\left[p^L\left(\frac{y}{t}\right),p^R\left(\frac{y}{t}\right)\right]\\ \text{subject to}\quad & \left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]\preccurlyeq[b_k^L,b_k^R]\ \forall\, k\\ & tq^R\left(\frac{y}{t}\right)\geq 1,\ tq^L\left(\frac{y}{t}\right)\leq 1,\\ & y=xt,\ t>0. \qquad (7)\end{aligned}$$

We can see that uncertainties are associated with $(\mathbf{IP})(y,t)$ in the following forms.

i. A point $(y,t)\in\mathbb{R}^n\times\mathbb{R}$ is a feasible solution of $(\mathbf{IP})(y,t)$ if (y,t) satisfies the interval inequalities $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]\preccurlyeq[b_k^L,b_k^R]\ \forall\, k$. So for every (y,t), the interval $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]$ may lie before or after $[b_k^L,b_k^R]$, or overlap with $[b_k^L,b_k^R]$. Hence feasibility of (y,t) depends upon the degree of inferiority of these two intervals, which is associated with an uncertain factor.

ii. A point $(y,t)\in\mathbb{R}^n\times\mathbb{R}$ with certain degree of inferiority of the intervals $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]$ with $[b_k^L,b_k^R]$ can be a solution of $(\mathbf{IP})(y,t)$ if it minimizes the objective function $t\left[p^L\left(\frac{y}{t}\right),p^R\left(\frac{y}{t}\right)\right]$. Since the objective function $t\left[p^L\left(\frac{y}{t}\right),p^R\left(\frac{y}{t}\right)\right]$ is an interval valued function, so for different feasible solution (y,t), the interval $t\left[p^L\left(\frac{y}{t}\right),p^R\left(\frac{y}{t}\right)\right]$ has to be compared with respect to a partial ordering. In this paper, we use χ −partial ordering, which is discussed in Subsection 2.1 and 2.2. In this way uncertainty is associated with objective function as well.

The uncertainties of the problem can be addressed individually in the subsequent subsections.

3.2. Uncertainty in Feasible Region

Feasible region of $(\mathbf{IP})(y,t)$ is the set S. It can be observed that the uncertainty, present in S, is associated with m interval inequalities $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]\preccurlyeq[b_k^L,b_k^R]\ \forall\, k\in\wedge_m$. For the interval inequalities $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]\preccurlyeq[b_k^L,b_k^R]\ \forall\, k\in\wedge_m$, the interval $\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right]$ may lie before or overlap or exceed $[b_k^L,b_k^R]$ for every k. Accordingly the feasibility of (y,t) can be measured as follows.

One may observe that,

i. $(y,t)\in S$ is a completely acceptable feasible solution if $(y,t)\in S_1=\left\{(y,t):g_k^R\left(\frac{y}{t}\right)\leq b_k^L\ \forall\, k\right\}$;

ii. (y,t) is not at all an acceptable feasible point if (y,t) goes beyond the region $\left\{(y,t):g_k^L\left(\frac{y}{t}\right)\leq b_k^R\ \forall\, k\right\}$, i.e., $(y,t)\in S'=\left\{(y,t):g_k^L\left(\frac{y}{t}\right)\geq b_k^R\ \forall\, k\right\}$;

iii. (y,t) is a slightly acceptable feasible point if it consists in $S\setminus S_1$.

This can also be explained by the degree of inferiority between two interval vectors $\begin{pmatrix}\left[g_1^L\left(\frac{y}{t}\right),g_1^R\left(\frac{y}{t}\right)\right]\\ \left[g_2^L\left(\frac{y}{t}\right),g_2^R\left(\frac{y}{t}\right)\right]\\ \vdots\\ \left[g_m^L\left(\frac{y}{t}\right),g_m^R\left(\frac{y}{t}\right)\right]\end{pmatrix}$ and $\begin{pmatrix}[b_1^L,b_1^R]\\ [b_2^L,b_2^R]\\ \vdots\\ [b_m^L,b_m^R]\end{pmatrix}$.

Employing the idea of degree of inferiority discussed in Subsection 2.1, the acceptable feasible degree of (y,t) satisfying the interval inequalities (5) is given by

$$\min_{1\leq k\leq m}\left\{\chi_k^F\left(\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right],[b_k^L,b_k^R]\right)\right\}, \quad (8)$$

where $\chi_k^F: I(\mathbb{R})\times I(\mathbb{R})\to\mathbb{R}$ is defined by

$$\chi_k^F\left(\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right],[b_k^L,b_k^R]\right)=\begin{cases}1, & g_k^R\left(\frac{y}{t}\right)\leq b_k^L\\ 0, & g_k^L\left(\frac{y}{t}\right)\geq b_k^R\\ \dfrac{b_k^R-g_k^L\left(\frac{y}{t}\right)}{(b_k^R-b_k^L)+\left(g_k^R\left(\frac{y}{t}\right)-g_k^L\left(\frac{y}{t}\right)\right)}, & elsewhere.\end{cases} \quad (9)$$

Define a set

$$\boldsymbol{S'}=\left\{((y,t),\tau):\tau=\min_k\chi_k^F\left(\left[g_k^L\left(\frac{y}{t}\right),g_k^R\left(\frac{y}{t}\right)\right],[b_k^L,b_k^R]\right),tq^R\left(\frac{y}{t}\right)\geq 1,tq^L\left(\frac{y}{t}\right)\leq 1,t>0\right\}.$$

For $((y,t),\tau)\in\boldsymbol{S'}$ with $\tau\in\left[\frac{1}{2},1\right]$, we say (y,t) as an acceptable feasible point with acceptable degree of feasibility τ and $\boldsymbol{S'}$ is the acceptable feasible region.

Feasible point of $(\mathbf{IP})(y,t)$ related to acceptable degree of feasibility can be clarified as the following definition.

Definition 3.1 $(y,t)\in\mathbb{R}^n\times\mathbb{R}$ is called an acceptable feasible point of $(\mathbf{IP})(y,t)$ with acceptable feasible degree τ if $((y,t),\tau)\in\boldsymbol{S'}$ with $\tau\in\left[\frac{1}{2},1\right]$. We denote an acceptable feasible point as $((y,t):\tau)$.

3.3. Uncertainty in objective function

The objective function $t\left[p^L\left(\frac{y}{t}\right),p^R\left(\frac{y}{t}\right)\right]$ of $(\mathbf{IP})(y,t)$ is in interval form. In this paper the minimization in the problem $(\mathbf{IP})(y,t)$ can be interpreted with respect to $\preccurlyeq_\chi$ partial order relation (defined in Subsection 2.1).

Following $\preccurlyeq_\chi$ partial order relation, we define the solution of $(\mathbf{IP})(y,t)$ as follows. We call the solution as χ-optimal solution of $(\mathbf{IP})(y,t)$.

Definition 3.2 An acceptable feasible point $((y^*,\tau^*):\tau^*)$ with acceptable degree of feasibility τ^* of $(\mathbf{IP})(y,t)$ is said to be a χ -optimal solution of $(\mathbf{IP})(y,t)$if there does not exist any acceptable feasible point with $\tau\leq\tau^*$ of $(\mathbf{IP})(y,t)$ such that

$$\left[tp^L\left(\frac{y}{t}\right),t\,p^R\left(\frac{y}{t}\right)\right]\prec_\chi\left[t^*p^L\left(\frac{y^*}{t^*}\right),t^*\,p^R\left(\frac{y^*}{t^*}\right)\right].$$

To address the uncertainty in objective function of $(\mathbf{IP})(y,t)$, we will assign goal to the objective function. Goal can be supplied by decision makers. Let $[l,u]$ be a preassigned goal of the objective function, which cannot be completely acceptable for a decision maker. The goal is more acceptable for the decision maker if $t\,p^R\left(\frac{y}{t}\right) \leq l$, and it is not acceptable if $tp^L\left(\frac{y}{t}\right) \geq u$, and for the other case the goal is partially acceptable. The acceptability of the goal can be presented by the acceptable degree of achievement of the goal of the objective function. That is, every acceptable feasible point (y,t) satisfies $\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right] \preccurlyeq_\chi [l,u]$. This can be addressed using the idea of degree of inferiority between two intervals. The degree of inferiority between the intervals $\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right]$ and $[l,u]$ is defined by

$$\chi^O\left(\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right],[l,u]\right)$$

$$=\begin{cases} 1, & t\,p^R\left(\frac{y}{t}\right) \leq l \\ 0, & tp^L\left(\frac{y}{t}\right) \geq u \\ \dfrac{u-tp^L\left(\frac{y}{t}\right)}{(u-l)+\left(t\,p^R\left(\frac{y}{t}\right)-tp^L\left(\frac{y}{t}\right)\right)}, & elsewhere \end{cases} \tag{10}$$

3.4. Uncertainty in Feasible Region and Objective Function Taken Together

The objective function is specified by its acceptable degree of achievement of the goal, $\chi^O\left(\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right],[l,u]\right)$, and the constraints are specified by their acceptable degree of feasibility $\tau = \min\limits_k\left\{\chi_k^F\left(\left[g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{y}{t}\right), g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)\right],[b_k^L, b_k^R]\right)\right\}$. So in this uncertain scenario, a decision $(y,t) \in S$ for $(\mathbf{IP})(y,t)$ is the choice of activities that satisfies together the objective function and constraints with certain degree of satisfaction. Hence, degree of satisfaction of (y,t) is

$$\min\left\{\chi^O\left(\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right],[l,u]\right) : ((y,t),\tau) \in \boldsymbol{S'}\right\}$$

$$= \min_{((y,t),\tau)\in \boldsymbol{S'}} \frac{u-tp^L\left(\frac{y}{t}\right)}{(u-l)+\left(t\,p^R\left(\frac{y}{t}\right)-tp^L\left(\frac{y}{t}\right)\right)}. \tag{11}$$

This problem can be represented as

$$\max \theta$$

$$\text{subject to} \quad \theta \leq \chi^O\left(\left[tp^L\left(\frac{y}{t}\right), t\,p^R\left(\frac{y}{t}\right)\right],[l,u]\right)$$

$$\theta \leq \chi_k^F\left(\left[g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{y}{t}\right), g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)\right],[b_k^L, b_k^R]\right)$$

$$tq^R\left(\frac{y}{t}\right) \geq 1, tq^L\left(\frac{y}{t}\right) \leq 1, t > 0$$

$$\frac{1}{2} \leq \theta \leq 1,$$

which is equivalent to the problem,

$$(\overline{\mathbf{IP}})(y,t):\ \max \quad \theta$$

subject to

$$\theta \leq \frac{u-tp^L\left(\frac{y}{t}\right)}{(u-l)+\left(t\,p^R\left(\frac{y}{t}\right)-tp^L\left(\frac{y}{t}\right)\right)} \tag{12}$$

$$\theta \leq \frac{b_k^R - g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{y}{t}\right)}{(b_k^R - b_k^L)+\left(g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)-g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{y}{t}\right)\right)} \tag{13}$$

$$tq^R\left(\frac{y}{t}\right) \geq 1, tq^L\left(\frac{y}{t}\right) \leq 1, t > 0 \tag{14}$$

$$\frac{1}{2} \leq \theta \leq 1.$$

It is a general nonlinear programming, which can be solved using any nonlinear programming technique. Let (y^*,t^*,θ^*) be a solution of the problem$(\overline{\mathbf{IP}})$. The following theorem establishes the relation between the solution of $(\mathbf{IP})(y,t)$ and $(\overline{\mathbf{IP}})(y,t)$.

Theorem 3.1. If (y^*,t^*,θ^*) be an optimal solution of the problem $(\overline{\mathbf{IP}})(y,t)$, then (y^*,t^*) is an χ-optimal solution of the problem $(\mathbf{IP})(y,t)$ with degree of satisfaction θ^***.**

Proof: (y^*,t^*,θ^*) is an optimal solution of $(\overline{\mathbf{IP}})(y,t)$. So, (y^*,t^*) and θ^* satisfy (12-14). This means, (y^*,t^*) is an acceptable feasible point of $(\mathbf{IP})(y,t)$ with acceptable degree of feasibility

$$\min\left\{\frac{b_k^R-g_k^L\left(\frac{y^*}{t^*}\right)}{(b_k^R-b_k^L)+\left(g_k^R\left(\frac{y^*}{t^*}\right)-g_k^L\left(\frac{y^*}{t^*}\right)\right)}\right\} \geq \theta^*.$$

Suppose (y^*,t^*) is not an χ-optimal solution of $(\mathbf{IP})(y,t)$ and there exists $(\bar{y},\bar{t}) \neq (y^*,t^*)$, which is an χ -optimal solution of $(\mathbf{IP})(y,t)$ with degree of satisfaction θ^*. Since $(\bar{y},\bar{t})$ is an acceptable feasible point of $(\mathbf{IP})(y,t)$, then there exist

$$\tau = \min\left\{\frac{b_k^R - g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{\bar{y}}{\bar{t}}\right)}{(b_k^R - b_k^L)+\left(g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{\bar{y}}{\bar{t}}\right)-g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{\bar{y}}{\bar{t}}\right)\right)}\right\} \in \left[\frac{1}{2},1\right].$$

Again $(\bar{y},\bar{t})$ is an χ -optimal solution of $(\mathbf{IP})(y,t)$, then there exist

$$\bar{\theta} = \min\left\{\tau, \frac{u-tp^L\left(\frac{\bar{y}}{\bar{t}}\right)}{(u-l)+\left(t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right)-tp^L\left(\frac{\bar{y}}{\bar{t}}\right)\right)}\right\}.$$

So

$$\bar{\theta} \leq \frac{u-tp^L\left(\frac{\bar{y}}{\bar{t}}\right)}{(u-l)+\left(t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right)-tp^L\left(\frac{\bar{y}}{\bar{t}}\right)\right)}$$

$$\bar{\theta} \leq \frac{b_k^R - g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{\bar{y}}{\bar{t}}\right)}{(b_k^R - b_k^L)+\left(g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{\bar{y}}{\bar{t}}\right)-g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{\bar{y}}{\bar{t}}\right)\right)}.$$

Also $(\bar{y},\bar{t})$ satisfy (14). Hence $(\bar{y},\bar{t})$ is a feasible point of $(\overline{\mathbf{IP}})(y)$.

Since (y^*,t^*) is not an χ -optimal solution, according to **Definition 3.2**,

$$\left[tp^L\left(\frac{\bar{y}}{\bar{t}}\right), t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right)\right] \prec_\chi \left[tp^L\left(\frac{y^*}{t^*}\right), t\,p^R\left(\frac{y^*}{t^*}\right)\right].$$

Then from **Definition 2.1**,

$$\mu\left(\left[tp^L\left(\frac{\bar{y}}{\bar{t}}\right), t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right)\right]\right) \leq \mu\left(\left[tp^L\left(\frac{y^*}{t^*}\right), t\,p^R\left(\frac{y^*}{t^*}\right)\right]\right)$$

and $\quad t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right) \leq tp^L\left(\frac{y^*}{t^*}\right)$.

This implies,

$$p^L\left(\frac{\bar{y}}{\bar{t}}\right) \leq t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right) \leq tp^L\left(\frac{y^*}{t^*}\right) \leq tp^R\left(\frac{y^*}{t^*}\right).$$

So $\quad u - t\,p^L\left(\frac{\bar{y}}{\bar{t}}\right) \geq u - tp^L\left(\frac{y^*}{t^*}\right) \quad (15)$

Since $[l,u]$ is a non-degenerate interval, we have

$$0 < (u-l)+\left(t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right)-t\,p^L\left(\frac{\bar{y}}{\bar{t}}\right)\right)$$

$$\leq (u-l) + \left(tp^R\left(\frac{y^*}{t^*}\right)-tp^L\left(\frac{y^*}{t^*}\right)\right). \tag{1 6}$$

From (15) and (16),

$$\frac{u - t\,p^L\left(\frac{\bar{y}}{\bar{t}}\right)}{(u-l) + \left(t\,p^R\left(\frac{\bar{y}}{\bar{t}}\right) - t\,p^L\left(\frac{\bar{y}}{\bar{t}}\right)\right)}$$

$$\geq \frac{u - tp^L\left(\frac{y^*}{t^*}\right)}{(u-l) + \left(tp^R\left(\frac{y^*}{t^*}\right) - tp^L\left(\frac{y^*}{t^*}\right)\right)}. \quad (17)$$

This implies $\bar{\theta} \geq \theta^*$, where $\bar{\theta} = \theta(\bar{y}, \bar{t})$. $\bar{\theta} > \theta^*$ contradicts that θ^* is an optimal solution of $(\overline{\mathbf{IP}})(y,t)$. If, $\bar{\theta} = \theta^*$, then it is clear that $\bar{\theta}$ is an alternative optimal solution of $(\overline{\mathbf{IP}})(y,t)$, which is also an χ-optimal solution. Hence (y^*, t^*) is an χ-optimal solution of $(\mathbf{IP})(y,t)$ with acceptable degree of feasibility θ^*. □

Now the proposed methodology is illustrated through a numerical example in the next section.

4. Numerical Example

Example 4.1 Consider the following optimization problem.

$$(\mathbf{IP}):\ \min \quad \hat{p}(x) \oslash \hat{q}(x)$$
$$\text{subject to } [1,2]x_1 \oplus [3,3]x_2 \preccurlyeq [1,10] \quad (18)$$
$$[-2,8]x_1 \oplus [4,6]x_2 \preccurlyeq [4,6] \quad (19)$$
$$[1,2]x_1^2 \oplus [0,1]x_2^2 \preccurlyeq [3,4] \quad (20)$$
$$2x_1^2 + 4x_2^2 > 0, x_1, x_2 \geq 0,$$

where
$\hat{p}(x) = [-10,-6]x_1 \oplus [2,3]x_2 \oplus [4,10]x_1^2 \oplus [-1,1]x_1x_2 \oplus [10,20]x_2^2$ and $\hat{q}(x) = [2,4]x_1^2 \oplus [0,1]x_1x_2 \oplus [4,6]x_1^2$.

Solution: Step 1: Transformation of the model

Suppose $\hat{t} = [t^L, t^R] = \frac{1}{[2,4]x_1^2 \oplus [0,1]x_1x_2 \oplus [4,6]x_1^2}$.

Let $t = \hat{t}$. So $\frac{1}{4x_1^2 + x_1x_2 + 6x_1^2} \leq t \frac{1}{2x_1^2 + 4x_1^2}$.

Transformed objective function is $t\left[p^L\left(\frac{y}{t}\right), p^R\left(\frac{y}{t}\right)\right]$, where

$$p^L\left(\frac{y}{t}\right) = -10\left(\frac{y_1}{t}\right) + 2\left(\frac{y_2}{t}\right) + 4\left(\frac{y_1}{t}\right)^2 - \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 10\left(\frac{y_2}{t}\right)^2$$

$$p^R\left(\frac{y}{t}\right) = -6\left(\frac{y_1}{t}\right) + 3\left(\frac{y_2}{t}\right) + 10\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 20\left(\frac{y_2}{t}\right)^2,$$

with $t\left(4\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 6\left(\frac{y_2}{t}\right)^2\right) \geq 1$,
$t\left(2\left(\frac{y_1}{t}\right)^2 + 4\left(\frac{y_2}{t}\right)^2\right) \leq 1$ and $y_i = x_i t,\ t > 0$.

Interval inequalities (18), (20) and (21) can be transformed into

$$[1,2]\left(\frac{y_1}{t}\right) \oplus [3,3]\left(\frac{y_2}{t}\right) \preccurlyeq [1,10]$$
$$[-2,8]\left(\frac{y_1}{t}\right) \oplus [4,6]\left(\frac{y_2}{t}\right) \preccurlyeq [4,6]$$
$$[1,2]\left(\frac{y_1}{t}\right)^2 \oplus [0,1]\left(\frac{y_2}{t}\right)^2 \preccurlyeq [3,4]$$
$$\frac{y_1}{t}, \frac{y_2}{t} \geq 0, t > 0.$$

The transformed problem is given by

$(\mathbf{IP})(y,t)$: $\min \quad t\left[p^L\left(\frac{y}{t}\right), p^R\left(\frac{y}{t}\right)\right]$

subject to
$$[1,2]\left(\frac{y_1}{t}\right) \oplus [3,3]\left(\frac{y_2}{t}\right) \preccurlyeq [1,10]$$
$$[-2,8]\left(\frac{y_1}{t}\right) \oplus [4,6]\left(\frac{y_2}{t}\right) \preccurlyeq [4,6]$$
$$[1,2]\left(\frac{y_1}{t}\right)^2 \oplus [0,1]\left(\frac{y_2}{t}\right)^2 \preccurlyeq [3,4]$$
$$t\left(4\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 6\left(\frac{y_2}{t}\right)^2\right) \geq 1$$
$$t\left(2\left(\frac{y_1}{t}\right)^2 + 4\left(\frac{y_2}{t}\right)^2\right) \leq 1$$
$$\frac{y_1}{t}, \frac{y_2}{t} \geq 0, t > 0.$$

Here

$$p^L\left(\frac{y}{t}\right) = -10\left(\frac{y_1}{t}\right) + 2\left(\frac{y_2}{t}\right) + 4\left(\frac{y_1}{t}\right)^2 - \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 10\left(\frac{y_2}{t}\right)^2$$

$$p^R\left(\frac{y}{t}\right) = -6\left(\frac{y_1}{t}\right) + 3\left(\frac{y_2}{t}\right) + 10\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 20\left(\frac{y_2}{t}\right)^2$$

$$g_1^L\left(\frac{y}{t}\right) = \left(\frac{y_1}{t}\right) + 3\left(\frac{y_2}{t}\right)$$
$$g_1^R\left(\frac{y}{t}\right) = 2\left(\frac{y_1}{t}\right) + 3\left(\frac{y_2}{t}\right)$$
$$g_2^L\left(\frac{y}{t}\right) = -2\left(\frac{y_1}{t}\right) + 4\left(\frac{y_2}{t}\right)$$
$$g_2^R\left(\frac{y}{t}\right) = 8\left(\frac{y_1}{t}\right) + 6\left(\frac{y_2}{t}\right)$$
$$g_3^L\left(\frac{y}{t}\right) = \left(\frac{y_1}{t}\right)^2$$
$$g_3^R\left(\frac{y}{t}\right) = 2\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_2}{t}\right)^2$$
$$b_1^L = 1, b_1^R = 10, b_2^L = 4, b_2^R = 6, b_3^L = 3, b_3^R = 4.$$

Step 2: Addressing uncertainties in feasible region

The acceptable feasible region of $(\mathbf{IP})(y,t)$ is given by

$\boldsymbol{S'} = \Big\{((y,t):\tau): \tau = \min_k \left\{\chi_k^F\left(\left[g_{\boldsymbol{k}}^{\boldsymbol{L}}\left(\frac{y}{t}\right), g_{\boldsymbol{k}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)\right], [b_k^L, b_k^R]\right)\right\}, t\left(4\left(\frac{y_1}{t}\right)^2 + \left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right) + 6\left(\frac{y_2}{t}\right)^2\right) \geq 1, t\left(2\left(\frac{y_1}{t}\right)^2 + 4\left(\frac{y_2}{t}\right)^2\right) \leq 1, \frac{y_1}{t}, \frac{y_2}{t} \geq 0, t > 0\Big\}$, where

$$\chi_1^F\left(\left[g_{\boldsymbol{1}}^{\boldsymbol{L}}\left(\frac{y}{t}\right), g_{\boldsymbol{1}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)\right], [b_1^L, b_1^R]\right) = \begin{cases} 1, & 2\frac{y_1}{t} + 3\frac{y_2}{t} \leq 1 \\ 0, & \frac{y_1}{t} + 3\frac{y_2}{t} \geq 10 \\ \dfrac{10 - \left(\frac{y_1}{t} + 3\frac{y_2}{t}\right)}{9 + \frac{y_1}{t}}, & elsewhere \end{cases}$$

$$\chi_2^F\left(\left[g_{\boldsymbol{2}}^{\boldsymbol{L}}\left(\frac{y}{t}\right), g_{\boldsymbol{2}}^{\boldsymbol{R}}\left(\frac{y}{t}\right)\right], [b_2^L, b_2^R]\right) = \begin{cases} 1, & 8\frac{y_1}{t} + 6\frac{y_2}{t} \leq 4 \\ 0, & -2\frac{y_1}{t} + 4\frac{y_2}{t} \geq 6 \\ \dfrac{6 - \left(2\frac{y_1}{t} + 4\frac{y_2}{t}\right)}{2 + 10\frac{y_1}{t} + 2\frac{y_2}{t}}, & elsewhere \end{cases}$$

$$\chi_3^F\left(\left[g_3^L\left(\frac{y}{t}\right), g_3^R\left(\frac{y}{t}\right)\right], [b_3^L, b_3^R]\right) = \begin{cases} 1, & 2\left(\frac{y_1}{t}\right)^2 + 6\left(\frac{y_2}{t}\right)^2 \leq 3 \\ 0 & \left(\frac{y_1}{t}\right)^2 \geq 4 \\ \dfrac{4-\left(\frac{y_1}{t}\right)^2}{1+\left(\frac{y_1}{t}\right)^2+\left(\frac{y_2}{t}\right)^2}, & elsewhere. \end{cases}$$

Step 3: Addressing uncertainties in objective function

Let goal of the objective function of $(\mathbf{IP})(y,t)$ be given by $[4,6]$. Then the acceptable degree of achievement of the goal of the objective function is

$$\chi^O\left(t\left[p^L\left(\frac{y}{t}\right), p^R\left(\frac{y}{t}\right)\right], [4,6]\right) = \begin{cases} 1, & tp^R\left(\frac{y}{t}\right) \leq 4 \\ 0, & tp^L\left(\frac{y}{t}\right) \geq 6 \\ \dfrac{6-tp^L\left(\frac{y}{t}\right)}{2+t\left(p^R\left(\frac{y}{t}\right)-p^L\left(\frac{y}{t}\right)\right)}, & elsewhere. \end{cases}$$

Step 4: The deterministic model

Combining Step 2 and Step 3, the deterministic model becomes

$$(\overline{\mathbf{IP}})(y,t): \max \theta$$

$$subject\ to\ \ \theta \leq \frac{6-tp^L\left(\frac{y}{t}\right)}{2+t\left(p^R\left(\frac{y}{t}\right)-p^L\left(\frac{y}{t}\right)\right)}$$

$$\theta \leq \frac{10-\left(\frac{y_1}{t}+3\frac{y_2}{t}\right)}{9+\frac{y_1}{t}}$$

$$\theta \leq \frac{6-\left(2\frac{y_1}{t}+4\frac{y_2}{t}\right)}{2+10\frac{y_1}{t}+2\frac{y_2}{t}}$$

$$\theta \leq \frac{4-\left(\frac{y_1}{t}\right)^2}{1+\left(\frac{y_1}{t}\right)^2+\left(\frac{y_2}{t}\right)^2}$$

$$t\left(4\left(\frac{y_1}{t}\right)^2+\left(\frac{y_1}{t}\right)\left(\frac{y_2}{t}\right)+6\left(\frac{y_2}{t}\right)^2\right) \geq 1$$

$$t\left(2\left(\frac{y_1}{t}\right)^2+4\left(\frac{y_2}{t}\right)^2\right) \leq 1$$

$$\frac{y_1}{t}, \frac{y_2}{t} \geq 0, t > 0, \frac{1}{2} \leq \theta \leq 1.$$

Considering $[4,6]$ as the goal of the objective function, solution of this problem is found as $y_1^* = 0.596$, $y_2^* = 0.00$, $t^* = 1.192$ and $\theta^* = 1$ in Lingo 11.0. So, χ −optimal solution of the problem $(\mathbf{IP})$ is $(x_1^*, x_2^*) = (0.50, 0.00)$ with degree of acceptability 1. Objective value is $[-6.8202, 0.9983]$ which satisfies the goal of the objective functions. This methodology provides one solution of the problem, which is feasible and optimal up to certain acceptable degree. For different choices of goals, different solution of the problem can be found.

5. Conclusions

This paper explains a method to find the solution of a nonlinear fractional programming problem with varying parameters. The proposed problem is converted to a general optimization problem and it is theoretically justified that the solution of the converted problem is an χ-optimal solution of the interval optimization problem. Solution concepts and theoretical development in this paper are derived with respect to χ- partial order relation. The methodology of the present work is applicable to both linear and nonlinear interval fractional programming problems. Interval nonlinear fractional programming problem has also been discussed by [4]. In [4] a solution technique to solve a fractional programming problem which depends upon the selection of appropriate weight function is developed. In presence of uncertainty in the optimization model, and selection of appropriate weight function, solution of this model cannot be totally acceptable for the decision maker. Satisfaction level of the decision is not described in [4]. In this paper, the proposed methodology provides one solution of the problem, which is feasible and efficient up to certain acceptable degree. The theoretical developments of this paper can be applied in finance, management and engineering optimization models when the lower and upper bound of the parameters are provided.

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