Synchronization between Fractional-Order Lorenz-Stenflo Systems Based on Open-Plus-Closed-Loop Control

Mohammad Hadi Atabakzadeh[1*], Gholam Hussian Erjaee[1], Mahmoud Boutefnouchet[2]

[1]Shiraz University, Department of Mathematics, Shiraz, Iran
[2]Qatar University, Mathematics, Physics and Statistic Department, Doha, Qatar

[*]Corresponding author email: mh_atabak@shirazu.ac.ir

Abstract: In this article, we apply open-plus-closed-loop control method to drive the synchronization between two fractional-order Lorenz-Stenflo systems. Based on the stability theorems, sufficient condition for synchronization is proposed. Numerical simulations are presented to demonstrate the application of the theoretical results.

Keywords: fractional differential equation, open-plus-closed-loop control, synchronization, Caputo derivative, Lorenz-Stenflo system.

1. Introduction

Chaotic dynamics have been investigated and studied in many real-world applications such as secure communication, data encryption, power systems, medicine, biology and chemical reactors. Similar to their integer-order counterparts, fractional-order differential systems can display chaotic dynamics. Motivated by potential applications in chaos synchronization, control chaotic dynamics has attracted significant interest. Some work has been done in the field of the chaos and control in fractional-order systems, including Chua system [1], fractional-order Chen system [2], fractional-order Lorenz system [3], fractional-order Rossler system [4] and Newton-Leipnic system [5].

In addition to the above works, several methods relating to chaos control is provided by the researchers; such as linear state feedback [6], the more general synthesis Lyapunov-type method [7], adaptive control [8, 9], the bang-bang controller [10], the inverse system method [11], Linear Matrix Inequalities (LMI) based control [12].

Recently, Jackson and Grosu have developed the concept of the Open-Plus-Closed-Loop (OPCL) method of control [13, 14]. OPCL method uses Taylor expansion for obtaining a general coupling term for synchronization of two systems. The aim of the OPCL method is to entrain complex dynamics to arbitrary given goal dynamics, by adding a suitable control term to the system. Recently, this method has been applied and developed for some related problems of control [13, 14]. The method has been also used for chaotic control [15] and chaos synchronization [16]. Moreover, Tian et al. [17] have proposed a nonlinear OPCL control in which the controlled system is expanded to a higher order in terms of the goal dynamics.

In this article, OPCL method has been used to control and analyze Lorenz-Stenflo chaotic system presented by fractional differential equations. For this purpose, the rest of article has organized as follows. Section 2 has been devoted to the OPCL scheme to control a chaotic system. In Section 3, the fractional-order Lorenz-Stenflo system has been studied. The fixed point of this system and necessary conditions for chaos has been stated in this section. In Section 4, the accuracy of the method has been presented by using numerical simulation. Finally, the conclusions have been drawn in Section 5.

2. Open-Plus-Closed-Loop Scheme

OPCL scheme has first developed by Jackson and Grosu [13, 14] in 1995. This method gives precise deriving for any continuous system in order to reach any desired dynamics. Consider a fractional-order system

$$D_*^{\alpha} y(t) = F(y(t)), \quad y \in R^n, \tag{1}$$

where, $D_*^{\alpha} y(t)$, Caputo's fractional-order derivative of y(t), is defined as [18]:

$$D_*^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\tau} \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{2}$$

for $y \in C^n[0,t]$. Here, α is the order of the derivative and $n = \lceil \alpha \rceil$, when ceiling function $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

The OPCL method offers a scheme for system (1) to attain a dynamics $x(t) \in R^n$. To explain this scheme, suppose operator $D(y,x)$ decomposes as

$$D(y,x) = D_1(x) + D_2(y,x),$$

where

$$D_1(x) = D_*^{\alpha} x - F(x),$$

$$D_2(y,x) = (H - \frac{\partial F}{\partial x})(y - x),$$

and H is a matrix with negative real part eigenvalues (Hurwitz matrix), whose elements can be chosen as simple as possible. If $(\frac{\partial F}{\partial x})_{ij}$ is a constant, we choose $H_{ij} = (\frac{\partial F}{\partial x})_{ij}$ such that $(H - \frac{\partial F}{\partial x})_{ij} = 0$. Otherwise, we need to introduce one or more parameters in the matrix H so that this matrix is a Hurwitz matrix. Now, according to this scheme, system defined by

$$D_*^\alpha y(t) = F(y) + D(y,x)$$

has an asymptotic behavior if $\|y(0) - x(0)\|$ is sufficiently small.

To apply OPCL scheme in synchronization of two identical chaotic systems, we let

$$D_*^\alpha x(t) = F(x), \tag{3}$$

as derive (master) system and define response (slave) system as

$$D_*^\alpha y(t) = F(y) + (H - \frac{\partial F}{\partial x})(y - x). \tag{4}$$

Defining $e(t) = y(t) - x(t)$, the error dynamic between (3) and (4) is then obtained by

$$D_*^\alpha e(t) = F(y(t)) - F(x(t)) + (H - \frac{\partial F}{\partial x})\, e(t). \tag{5}$$

Using Taylor series expansion

$$F(y(t)) = F(x(t)) + \frac{\partial F}{\partial x}(y(t) - x(t)) + \cdots, \tag{6}$$

and keeping the first order term in (6) and putting (6) into (5), we obtain

$$D_*^\alpha e(t) = H\, e(t). \tag{7}$$

In order to analyze the stability of two fractional-order systems which are synchronized by OPCL method, we may consider the following theorem.

Theorem 1. The commensurate fractional-order system $D_*^\alpha x(t) = A\,x(t)$ with $0 < \alpha < 1, x \in R^n$ and $A \in R^{n\times n}$ is

- asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues λ of A,
- stable if and only if it is asymptotically stable, or those critical eigenvalues which satisfy $|\arg(\lambda)| = \frac{\alpha\pi}{2}$ have geometric multiplicity one [19].

Remark 1. According to the OPCL control method, when control matrix H is a Hurwitz matrix, the error system (7) is asymptotically stable and so, $e(t) \to 0$.

3. System Description

In this article, we consider a generalization of the Lorenz system [20] in four dimensional space which was first derived by Swedish physicist Lennart Stenflo to described the low-frequency short-wavelength gravity atmospheric waves [21, 22]. The fractional - order of this system can be described as:

$$\begin{cases} D_*^\alpha x_1(t) = a(x_2 - x_1) + dx_4, \\ D_*^\alpha x_2(t) = x_1(c - x_3) - x_2, \\ D_*^\alpha x_3(t) = x_1 x_2 - bx_3, \\ D_*^\alpha x_4(t) = -x_1 - rx_4, \end{cases} \tag{8}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, d, r are positive constant parameters of the system. Obviously, one of the fixed points of this system is $E_0^* = (0,0,0,0,0)$. Linearization around this fixed point gives the following eigenvalues.

$$\lambda_0 = (-1, -0.7, 3.9, -5.94). \tag{9}$$

Apart from this trivial point, for $ar(c-1) > d$ there are two other fixed points:

$$E_\pm^* = (\pm rw, \pm(r + \frac{d}{a})w, \frac{r}{b}(r + \frac{d}{a})w^2, \mp w),$$

where, $w = (\frac{b(arc - ar - d)}{r^2(ar + d)})^{1/2}$. Choosing $a = 1, b = 0.7,$ $c = 26, d = 1.5, r = 1,$, we have

$$E_\pm^* = (\pm\sqrt{6.58}, \pm 2.5\sqrt{6.58}, 23.5, \mp\sqrt{6.58}). \tag{10}$$

At the equilibrium point $E_\pm^* = (x_1^*, x_2^*, x_3^*, x_4^*)$, the Jacobian matrix for the system is given by

$$J = \begin{pmatrix} -a & a - r & 0 & d \\ c - x_3^* & -1 & -x_1^* & 0 \\ x_2^* & x_1^* & -b & 0 \\ -1 & 0 & 0 & -r \end{pmatrix}, \tag{11}$$

whose eigenvalues computed at equilibrium $E_\pm^*$ are given by

$$\lambda_\pm = (-2, -0.17713 \pm 2.2842425i, -2.05426). \tag{12}$$

According to Theorem 1, a necessary condition for the fractional system (8) to exhibit a chaotic attractor is

$$\min_i \{|\arg(\lambda_i)|\} < \frac{\alpha\pi}{2}. \tag{13}$$

Therefore, system (8) is chaotic whenever $\alpha > 0.96012$. As we can see in Figures 1 and Figure 2, system (8) is chaotic for $\alpha = 0.965$, and it is stable for $\alpha = 0.95$.

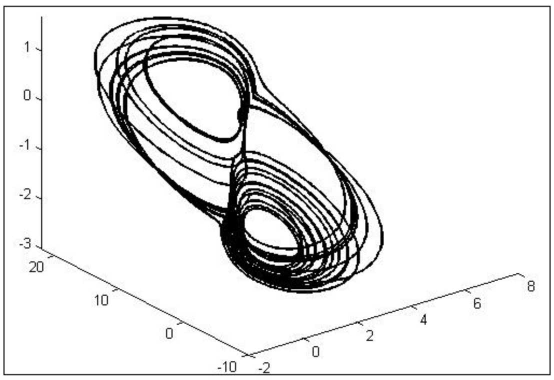

Figure 1. Chaos in system (8) for $\alpha = 0.965$.

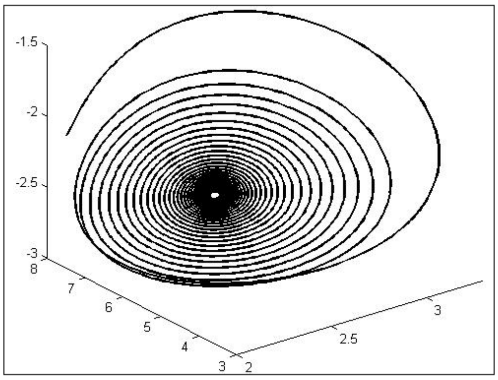

Figure 2. The attractor of Lorenz-Stenflo system in (x_1, x_2, x_4) space for $\alpha = 0.95$.

4. Numerical Example

Consider the fractional - order Lorenz - Stenflo system (8) and fix $\alpha = 0.965$, as derive system. Then constant matrix H for response system can be selected as

$$H = \begin{pmatrix} -a & 0 & 0 & d \\ c & -1 & -p_1 & 0 \\ p_2 & p_1 & -b & 0 \\ -1 & 0 & 0 & -r \end{pmatrix}, \qquad (14)$$

where, $a = 1, b = 0.7, c = 26, d = 1.5, r = 1$ and p_1, p_2 are two parameters that has to be determined for H as a Hurwitz matrix. In this case, the characteristic equation of matrix H is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

at which,

$$a_0 = p_1 p_2 + 2.5 p_1^2 - 16.45, a_1 = 2p_1^2 + p_1 p_2 - 38.55, \quad a_2 = p_1^2 - 19.4, a_3 = 3.7 .$$

Here, the Rout-Hurwitz conditions [23]

$$a_2 a_3 > a_1, \quad a_1 a_2 a_3 > a_0 a_3^2 + a_1^2, \quad a_i > o \quad (i = 1,2,3)$$

satisfy for some values of p_1 and p_2. For example, if we choose $p_1 = 10$, $p_2 = 2$, calculating $\frac{\partial F}{\partial X}$ and H by (11) and (14), respectively, then using OPCL control scheme the response system forms as

$$\begin{cases} D_*^{\alpha} y_1(t) = a(y_2 - y_1) + d\, y_4, \\ D_*^{\alpha} y_2(t) = y_1(c - y_3) - y_2 + x_3(y_1 - x_1) \\ + (x_1 - 10)(y_3 - x_3), \\ D_*^{\alpha} y_3(t) = y_1 y_2 - b\, y_3 + (2 - x_2)(y_1 - x_1) \\ + (10 - x_1)(y_2 - x_2), \\ D_*^{\alpha} y_4(t) = -y_1 - r\, y_4, \end{cases} \qquad (15)$$

and the error system (7) forms as

$$\begin{cases} D_*^{\alpha} e_1(t) = a(e_2 - e_1) + d\, e_4, \\ D_*^{\alpha} e_2(t) = -e_2 - 10e_3, \\ D_*^{\alpha} e_3(t) = 2e_1 + 10e_2 - be_3 e_2, \\ D_*^{\alpha} e_4(t) = -e_1 - r\, e_4. \end{cases}$$

Consequently, by Remark 1, the error vector $e(t)$ converges to zero. As a numerical example, if we choose the initial conditions

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (1.5, 5, 20, -1.5),$$

$$(y_1(0), y_2(0), y_3(0), y_4(0)) = (-1.5, 10, 25, -0.5)$$

and solve the fractional-order systems (8)and (15) by using the improved predictor - corrector algorithm [24], then the numerical results can be shown in Figures 3-6. In this example, if we solve above resultant error system (7), then, as we can see in Figure 7, the error vector will converge to zero, asymptotically. This means that synchronization has been occurred between drive system (8) and response system (15).

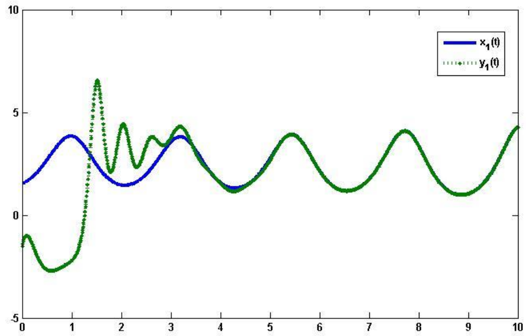

Figure 3. Time series of state variables $x_1(t)$ and $y_1(t)$.

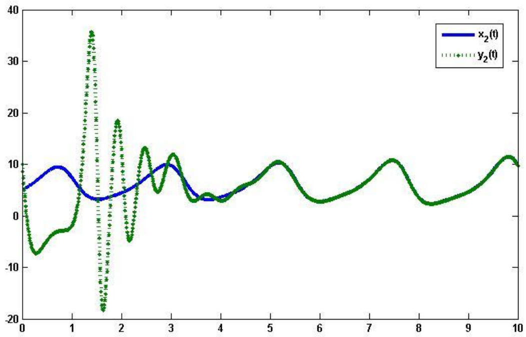

Figure 4. Time series of state variables $x_2(t)$ and $y_2(t)$.

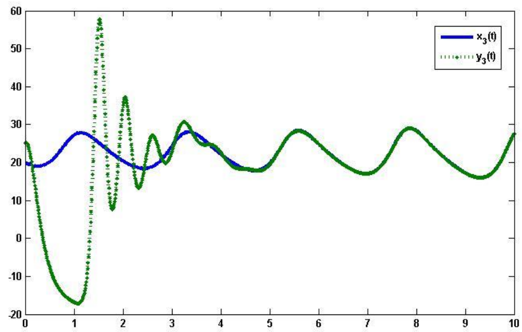

Figure 5. Time series of state variables $x_3(t)$ and $y_3(t)$.

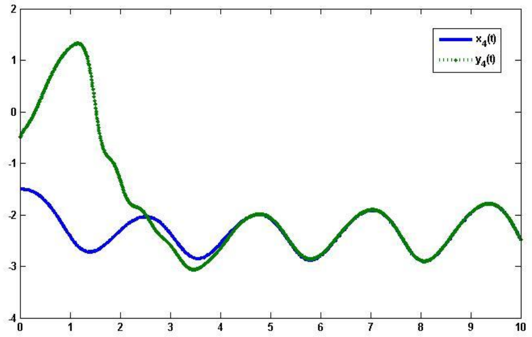

Figure 6. Time series of state variables $x_4(t)$ and $y_4(t)$.

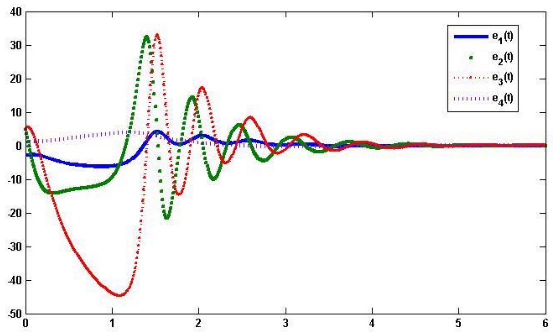

Figure 7. The time evolution of error components between systems (8) and (15).

5. Conclusion

In this article, an OPCL coupling scheme is used to achieve the synchronization between two fractional-order Lorenz-Stenflo systems. Hurwitz matrix has plied an important role in coupling and synchronizing of derive and response systems. Since, if all eigenvalues of Hurwitz matrix have negative real parts, then by Matignon stability theorem the resultant error system will converge to zero as required.

6. Acknowledgements

This publication was made possible by NPRP grant NPRP 6-137-1-026 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

References

[1] C. P. Li, W. H. Deng, D. Xu, "Chaos synchronization of the Chua system with a fractional order", *Physica A: Statistical Mechanics and its Applications*, vol. 360, no. 2, pp. 171-185, 2006.

[2] C. Li, G. Peng, "chaos in Chens system with a fractional order", *Chaos, Solitons and Fractals*., vol. 22, no. 2, pp. 443-450, 2004.

[3] I. Grigorenko, E. Grigorenko., "Chaotic dynamics of the fractional Lorenz system", *Physical review letters*, vol. 91, no. 3, pp. 34101-34105, 2003.

[4] C. Li, G. chen., "chaos and hyperchaos in the fractional-order Rossler equations", *Physica A: Statistical Mechanics and its Applications*, vol. 341, pp. 55-61, 2004.

[5] L. J. Shen, H. K. Chen, J. H. Chen, L. M. Tam, et al, "Chaos in the Newton-Leipnik system with fractional order", *Chaos Solitons Fractal*s, vol. 36, no. 3, pp. 98-103, 2008.

[6] G. Chen, X. Dong., "On feedback control of chaotic continuous-time systems", *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions*, vol. 40, no. 9, pp. 591-601, 1993.

[7] H. Nijmeijer, H. Berghuis., "On Lyapunov control of Duffing equation", *IEEE Transactions on Circuits and Systems I: Fundamental theory and applications*, vol. 42, no. 8, pp. 473-477, 1995.

[8] M. C. Ho, Y. C. Hung., "Synchronization of two different systems by using generalized active control", *Phys. Lett. A*, vol. 301, no. 5-6, pp. 424-428, 2002.

[9] E. W. Bai , K. E. Lonngren., "Synchronization of two Lorenz systems using active control", *Chaos Solitons Fractals*, vol. 8, no. 1, pp. 51-58, 1997.

[10] T. L. Vincent,J. Yu., "Control of a chaotic system", *Dynamic Control*, vol. 1, pp. 35-52, 1991.

[11] H. P. Ren, D. Liu., "Chaos control and anti-control via a fuzzy neural network inverse system method", *Chinese physics letters* vol. 19, no. 8, pp. 1054-1061, 2002.

[12] W. Tan,Y. N. Wang, F. Duan, X. Li., "LMI-based output feedback fuzzy control of chaotic system with uncertainties", *Chinese Physics*, vol. 15, no. 11, pp. 2529-2534, 2006.

[13] E.A. Jackson, I. Grosu., "An open-plus-closed-loop (OPCL) control of complex dynamical system," *Physica D: Nonlinear Phenomena*, vol. 85, no. 1, pp. 1-9, 1995.

[14] I. Grosu., "Robust synchronization of an array of coupled stochastic discrete-time delayed neural networks", *Neural Networks, IEEE Transactions*, vol. 19, no. 11, pp. 1910-1921, 2008.

[15] L.Q. Chen., "An open-plus-closed-loop control for discrete chaos and hyperchaos", *Physics Letters A*, vol. 281, no. 5-6, pp. 327-333, 2001.

[16] Q. K. Han, X. Y. Zhao, B. C. Wen., "Synchronization motions of a two-link mechanism with an improved OPCL method", *Applied Mathematics and Mechanics* vol. 29, no. 12, pp. 1561-1568, 2008.

[17] Y.C. Tian, M. O. Tade, J. Tang., "Nonlinear open-plus-closed-loop (NOPCL) control of dynamic systems", Chaos S*olitons Fractals*, vol. 11, no. 7, pp. 1029-1035, 2000.

[18] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo., "Theory and Applications of Fractional Differential Equations", *vol. 204 of North-Holland Mathematics Studies.* Elsevier Science Limited; 2006.

[19] D. Matignon., "Stability results for fractional differential equations with applications to control processing", in *Computational Engineering in Systems Applications*, vol. 2, pp. 963-968, 1996.

[20] E. N. Lorenz., "Deterministic nonperiodic flow", *Journal of the atmospheric sciences*, vol. 20, no. 2, pp. 130-142, 1963.

[21] L. Stenflo., "Equations describing solitary atmospheric waves", *Phisica Scripta*, vol. 43, pp. 599-600, 1991.

[22] L. Stenflo., "Generalized Lorenz equation for acoustic gravity waves in the atmosphere", *Phisica Scripta*, vol. 53, no. 1, pp. 83-84, 1996.

[23] F. R. Gantmacher., The theory of matrices, New York: Chelsea, 1960.

[24] K. Diethelm, N. Ford, A. Freed., "A predictor - corrector approach for the numerical differential equations", *Nonlinear Dynamics*, vol. 29, no. 1-4, pp. 3-22, 2002.