On Approximation of Functions by Product Means

H. K. Nigam

Department of Mathematics, Faculty of Engineering & Technology
Mody Institute of Technology and Science (Deemed University),
Laxmangarh, Sikar (Rajasthan), India,
Corresponding addresses
harekrishnan@yahoo.com

Abstract: In this paper, a new theorem on degree of approximation of conjugate of a function $f \in Lip(\xi(t), r)$ using (C,1)(E,q) product summability means of conjugate Fourier series has been established.

Keywords: Degree of approximation, $Lip(\xi(t), r)$ class of function, (C,1) summability, (E,q) summability, (C,1)(E,q) product summability, conjugate Fourier series, Lebesgue integral.

1. Introduction

Alexits [1], Sahney and Goel [12], Chandra [2], Qureshi and Neha [10], Liendler [6] and Rhoades [11] have determined the degree of approximation of a function belonging to $Lip\alpha$ class by Cesáro, Nörlund and generalized Nörlund single summability methods. Working in the same direction Sahney & Rao [13] and Khan [4] have studied the degree of approximation of function belonging to $Lip(\alpha, r)$ class by Nörlund and generalized Nörlund means. Thereafter Qureshi [8,9] discussed the degree of approximation of conjugate of functions belonging to $Lip\alpha$ class and $Lip(\alpha, r)$ class by Nörlund means of conjugate Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function belonging to $Lip(\xi(t), r)$ class by (C,1)(E,q) product summability method. The $Lip(\xi(t), r)$ class is a generalization of $Lip\alpha$ class and $Lip(\alpha, r)$ class. Therefore, in present paper, a theorem on degree of approximation of conjugate of a function $Lip(\xi(t), r)$ class by (C,1)(E,q) product summability means of conjugate Fourier series has been proved.

2. Definitions and Notations

Let f(x) be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of f(x) is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (2.1)$$

with nth partial sums $s_n(f;x)$.

The conjugate series of the Fourier series (2.1) is

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \qquad (2.2)$$

with nth partial sums $\overline{s_n}(f;x)$.

L_r - norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, r \geq 1 \qquad (2.3)$$

and let the degree of approximation be given by (Zygmund [14])

$$E_n(f) = \min_{t_n} \|t_n - f\|_r, \qquad (2.4)$$

where $t_n(x)$ is some n^{th} degree trigonometric polynomial.

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right) \text{ for } 0 < \alpha \leq 1 \qquad (2.5)$$

$f(x) \in Lip(\alpha, r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right),$$

$$0 < \alpha \leq 1, \text{ and } r \geq 1 \qquad (2.6)$$

(definition 5.38 of Mc Fadden [7], 1942).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f(x) \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \qquad (2.7)$$

If $\xi(t) = t^{\alpha}$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ reduces to the class $Lip\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$.

The (C, 1) transform is defined as the n^{th} partial sum of (C, 1) summability

$$t_n = \frac{s_0 + s_1 + s_2 + \dots\dots + s_n}{n+1}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \ \ as \ \ n \to \infty \qquad (2.8)$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable to s by (C,1) method.

If

$$(E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k \to s \ \ as \ \ n \to \infty. \qquad (2.9)$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,q) to the definite number s (Hardy[3]).

The (C,1) transform of the (E,q) transform defines (C,1)(E,q) transform of the partial sum s_n of series $\sum_{n=0}^{\infty} u_n$ and we denote it by $C_n^1 E_n^q$. Thus if

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} E_k^q \to s, \ as \ \ n \to \infty$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{n-\nu}\right] \to s, \ \ as \ \ n \to \infty \qquad (2.10)$$

where E_n^q denotes the (E,q) transform of s_n and C_n^1 denotes (C,1) transform of s_n. Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C,1)(E,q) method or summable (C,1)(E,q) to a definite number s.

We use the following notations:

$$\psi(t) = f(x+t) + f(x-t)$$

$$\tau \ = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right]$$

$$\overline{K}_n(t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{n-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]$$

3. Main Theorem

We establish the following theorem:

3.1 Theorem:

If $\overline{f}(x)$, conjugate to a 2π-periodic function f belonging to $Lip(\xi(t), r)$ class, then its degree of approximation by (C,1)(E,q) summability means of conjugate series of Fourier series is given by

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}(x)\right\|_r = O\left[(n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{(n+1)}\right)\right] \qquad (3.1)$$

where $\overline{C_n^1 E_n^q}$ denotes $C_n^1 E_n^q$ transform as defined in (2.10), provided

$$(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k} = O(n+1), \qquad (3.2)$$

$\xi(t)$ satisfies the following conditions:

$\frac{\xi(t)}{t}$ be a decreasing sequence, (3.3)

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \qquad (3.4)$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \qquad (3.5)$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1 \le r \le \infty$, conditions (3.4) and (3.5) hold uniformly in x and $\overline{C_n^1 E_n^q}$ is $\overline{\text{(C,1)(E, q)}}$ means of the series (2.2) and

$$\overline{f}(x) = -\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\ dt \qquad (3.6)$$

4. Lemmas

4.1 Lemma 1:

$$\overline{K}_n(t) = O\left(\frac{1}{t}\right) \text{ for } 0 \le t \le \frac{1}{n+1}$$

Proof: For $0 \le t \le \frac{1}{n+1}$, $\sin(t/2) \ge (t/\pi)$ and $|\cos nt| \le 1$

$$|\overline{K}_n(t)| = \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right|$$

$$\le \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{|\sin(t/2)|}$$

$$= \frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}$$

$$=\frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}(1+q)^k$$

$$=\frac{1}{2t(n+1)}\sum_{k=0}^{n}1$$

$$=O\left(\frac{1}{t}\right)$$

4.2 Lemma 2:

For $0\le a\le b\le\infty,\ 0\le t\le\pi$ and any n, we have

$$\overline{K}_n(t)=O\left(\frac{\tau^2}{(n+1)}\right)+O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right)$$

Proof: For $0\le\frac{1}{n+1}\le t\le\pi,\quad \sin(t/2)\ge(t/\pi)$

$$\left|\overline{K}_n(t)\right|=\frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right|$$

$$\le\frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\le\frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|\left|e^{\frac{it}{2}}\right|$$

$$\le\frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\le\frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$+\frac{1}{2t(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right| \quad (4.1)$$

Now considering first term of (3.1)

$$\frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\le\frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right|\left|e^{i\nu t}\right|$$

$$\le\frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right]$$

$$=\frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}1$$

$$=\frac{\tau}{2t(n+1)}$$

$$=O\left(\frac{\tau^2}{(n+1)}\right) \quad (4.2)$$

Now considering second term of (3.1) and using Abel's leema

$$\frac{1}{2t(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\le\frac{1}{2t(n+1)}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}\max_{0\le m\le k}\left|\sum_{\nu=0}^{m}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right|$$

$$\le\frac{1}{2t(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}$$

$$=O\left[\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}\right] \quad (4.3)$$

Combining (4.1), (4.2) and (4.3), we get

$$\overline{K}_n(t)=O\left(\frac{\tau^2}{(n+1)}\right)+O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right) \quad (4.4)$$

5. Proof of the Theorem

Let $\bar{s}_n(f;x)$ denotes, the n[th] partial sum of the series (2.2). Then following Lal [5], we have

$$\bar{s}_n(x)-\bar{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (2.2) the (E,q) transform (E_n^q) of $\bar{s}_n(f;x)$ is given by

$$\overline{E_n^q}-\bar{f}(x)=\frac{1}{2\pi(1+q)^n}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos\left(k+\frac{1}{2}\right)t\right\}dt$$

Now denoting $\overline{(C,1)(E,q)}$ transform of $\bar{s}_n$ by $\left(\overline{C_n^1E_n^q}\right)$, we write

$$\overline{C_n^1E_n^q}-\bar{f}(x)=\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\left\{\frac{1}{(1+q)^k}\right\}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$

$$=\int_0^{\pi}\psi(t)\overline{K}_n(t)dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \psi(t) \overline{K}_n(t) dt$$

$$= I_1 + I_{.2} \quad \text{(say)} \qquad (5.1)$$

Applying Hölder's inequality and the fact that $\psi(t) \in Lip(\xi(t), r)$, condition (3.4), Lemma 1 and second mean value theorem for integrals, we have

$$|I_1| \le \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t|\psi(t)|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t) \left| \overline{K}_n(t) \right|}{t} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)}{t^2} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left[\left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) \left\{ \int_{\in}^{\frac{1}{n+1}} \left(\frac{dt}{t^{2s}} \right) \right\}^{\frac{1}{s}} \right]$$

$$\text{for some } 0 < \in < \frac{1}{n+1}$$

$$= O\left[\left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) \left[\left\{ \frac{t^{-2s+1}}{-2s+1} \right\}_{\in}^{\frac{1}{n+1}} \right]^{\frac{1}{s}} \right]$$

$$= O\left[\left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) (n+1)^{2-\frac{1}{s}} \right]$$

$$= O\left[\xi\left(\frac{1}{n+1}\right) (n+1)^{1-\frac{1}{s}} \right]$$

$$= O\left\{ (n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\} \quad \because \frac{1}{r} + \frac{1}{s} = 1 \qquad (5.2)$$

Now using Leema 2, we have

$$|I_{2.}| = O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{t^2 (n+1)} dt \right] + O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{(n+1)t} (1+q)^{\tau} \sum_{k=\tau}^{n} \frac{1}{(1+q)^k} dt \right]^{\frac{1}{s}}$$

$$= O(I_{2.1}) + O(I_{2.2}) \quad \text{(say)} \qquad (5.3)$$

Using Hölder's inequality, conditions (3.3) and (3.5), we have

$$|I_{.2.1}| \le \left(\frac{1}{n+1}\right) \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{-\delta+2}} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\{(n+1)^{\delta-1}\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-2}} \right\}^s \frac{dy}{y^2} \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta-1} \xi\left(\frac{1}{n+1}\right) \right\} \left[\int_{\frac{1}{\pi}}^{n+1} \frac{dy}{y^{s(\delta-2)+2}} \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta-1} \xi\left(\frac{1}{n+1}\right) \right\} \left[\frac{(n+1)^{s(2-\delta)-1} - \pi^{s(\delta-2)+1}}{s(2-\delta)-1} \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta-1} \xi\left(\frac{1}{n+1}\right) \right\} \left[(n+1)^{(2-\delta)-\frac{1}{s}} \right]$$

$$= O\left\{ \xi\left(\frac{1}{n+1}\right) (n+1)^{1-\frac{1}{s}} \right\}$$

$$= O\left\{ (n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\} \because \frac{1}{r} + \frac{1}{s} = 1 \qquad (5.4)$$

Similarly, using Holders inequality, conditions (3.3) & (3.5) and mean value theorem, we have

$$|I_{2.2}| \le \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{1-\delta}} \frac{1}{(n+1)} (1+q)^{\tau} \sum_{k=\tau}^{n} \frac{1}{(1+q)^k} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{1-\delta}} \right\}^s dt \right]^{\frac{1}{s}} \text{ by (3.2)}$$

$$= O\left\{ (n+1)^{\delta} \right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{1-\delta}} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta} \right\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1}} \right\}^s \frac{dy}{y^2} \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta} \xi\left(\frac{1}{n+1}\right) \right\} \left[\int_{\in_1}^{n+1} \left\{ \frac{dy}{y^{s(\delta-1)+2}} \right\} \right]^{\frac{1}{s}}$$

for some $\frac{1}{\pi} < \in_1 < (n+1)$

$$= O\left\{ (n+1)^{\delta} \xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(1-\delta)-1} - (\in_1)^{s(1-\delta)-1}}{s(1-\delta)-1}\right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta} \xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{1-\delta-\frac{1}{s}}\right]$$

$$= O\left\{ (n+1)^{1-\frac{1}{s}} \xi\left(\frac{1}{n+1}\right)\right\}$$

$$= O\left\{ (n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} \qquad \because \frac{1}{r}+\frac{1}{s}=1 \tag{5.5}$$

Thus, we get

$$\left|\overline{C_n^1 E_n^q} - \overline{f}\right| = O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = \left\{\int_0^{2\pi} \left|\overline{C_n^1 E_n^q} - \overline{f}\right|^r dx\right\}^{\frac{1}{r}}$$

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = \left\{\int_0^{2\pi}\left\{(n+1)^{\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of main theorem.

6. Applications

Following corollaries can be derived from our main theorem:

Corollary 1: If $\xi(t) = t^{\alpha}$ then the degree of approximation of a function $\overline{f}$, conjugate to 2π –periodic function $f \in Lip(\alpha, r),\ \frac{1}{r} < \alpha < 1,$ is given by

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Proof: We have

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = O\left\{\int_0^{2\pi}\left|\overline{C_n^1 E_n^q} - \overline{f}\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} = O\left\{\int_0^{2\pi}\left|\overline{C_n^1 E_n^q} - \overline{f}\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1) = O\left\{\int_0^{2\pi}\left|\overline{C_n^1 E_n^q} - \overline{f}\right|^r dx\right\}^{\frac{1}{r}} . \ O\left\{\frac{1}{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)}\right\}$$ Hence,

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

for if not the right-hand side will be O(1), therefore

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_r = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Corollary 2: If $r \to \infty$ in corollary 1, we have for $0 < \alpha < 1$,

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Corollary 3: If $\xi(t) = t^{\alpha}$ and $q_n = 1$ then the degree of approximation of a function $\overline{f}$ (x), conjugate to 2π -periodic function $f \in \text{Lip}\,(\alpha, r)$, $\frac{1}{r} \le \alpha \le 1$ is given by

$$\left\|\overline{(CE)_n^1} - \overline{f}\right\|_r = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Corollary 4: If $r \to \infty$ in corollary 3, we have for $0 < \alpha < 1$,

$$\left\|\overline{(CE)_n^1} - \overline{f}\right\|_r = O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Remark:

Independent proofs of above corollaries 1 and 3 can be obtained along the same lines of our theorem.

7. Acknowledgement

Author is thankful to his parents for their encouragement and support to this work.

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Pergamon Press, London, 1961.

[2] Prem Chandra, Trigonometric approximation of functions in L_p - norm, *J. Math. Anal. Appl*. 275(2002), no. 1, (13-26).

[3] G. H. Hardy, *Divergent series*, first edition, Oxford University Press, 1949, 70.

[4] H. H. Khan, On degree of approximation of functions belonging to the class Lip(α, p), *Indian J. Pure Appl. Math*. 5(1974), no.2, 132-136.

[5] Shyam Lal, On K^{λ} summability of conjugate series of Fourier series, *Bulletin of Calcutta Math*. Soc., 89 (1997), 97-104.

[6] László Leindler, Trigonometric approximation in L_p - norm, *J. Math. Anal. Appl*. 302(2005).

[7] Leonard McFadden, Absolute Nörlund summability, *Duke Math. J*. 9 (1942), 168-207.

[8] K. Qureshi, On the degree of approximation of a periodic function f by almost Nörlund means, *Tamkang J. Math.* 12(1981), no. 1, 35-38.

[9] K. Qureshi, On the degree of approximation of a function belonging to the c lass Lipα , *Indian J. pure Appl. Math*. 13(1982), no. 8, 560-563.

[10] K. Qureshi, and H. K. Neha, *A class of functions and their degree of approximation*, Ganita, 41(1) (1990) 37.

[11] B.E. Rhoades, On degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series, *Tamkang J. Math*, 34 no. 3, 245-247. (2003).

[12] B. N. Sahney, and D.S. Goel, On the degree of continuous functions Ranchi, *University Math. Jour*., 4 (1973), 50-53.

[13] B.N. Sahney and D.S. Goel, On the degree of continuous functions Ranchi, *University Math. Jour*., 4 (1973), 50-53.

[14] A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, 1959.